Engineering Mathematics - II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 17
Taylor's Series

Welcome back to lectures on Engineering Mathematics 2. So, this is lecture number 17 on Taylor's series.

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So, here we first we will discuss about what are the power series and radius of convergence of those series. And also then we will talk about the Taylor's series and as a special case, this Maclaurin's series. Finally, we will go through some examples where we will construct a Taylor's series.

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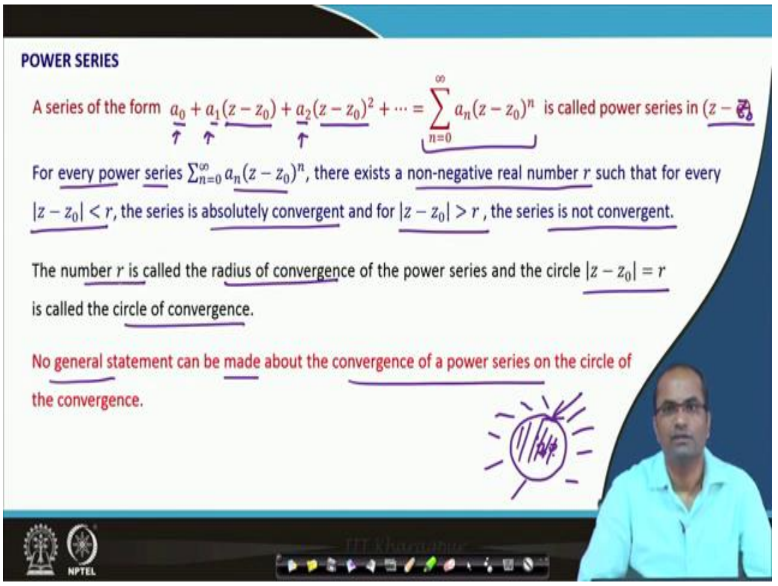

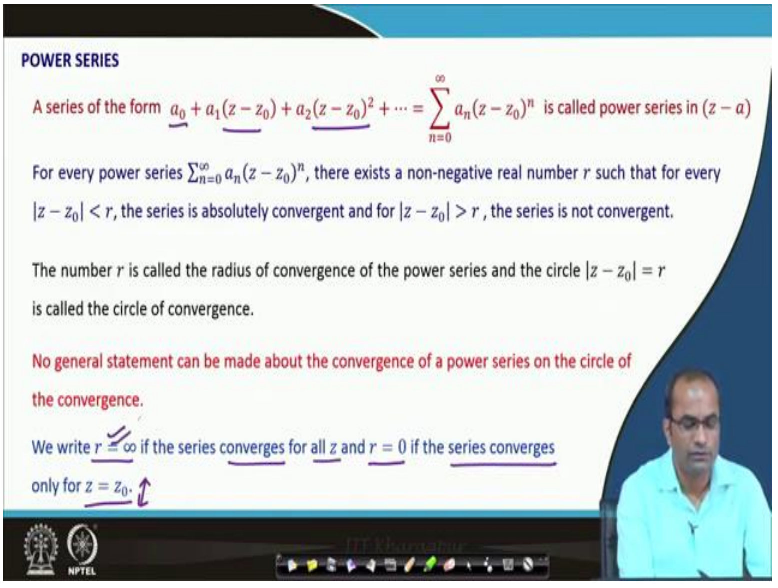

So what is the power series in the connection with the complex variable? So, a series of this form. So we have a 0 than some constant again and z minus z 0, another constant z minus z 0 square or we can write down in a compact form, so a n, z minus z 0 power n, n goes from 0 to infinity and this is called a power series in z minus a or z minus z 0 here.

So, because this is all in powers of z minus z 0 with some constants. So, every term has some power of this z minus z 0, the first term is a constant term. So we can think as a power 0. So this is a power series and every power series, this is the known fact that ever every power series, there exists in non-negative real number r such that every z in this neighborhood z minus z 0 less than r, the series is absolutely convergent. And for z minus z 0 greater than r, so outside the circle, with central z 0 and radius r, the series is divergent, so not convergent.

And this number r is called the radius of convergence of this power series and the circle here z minus z 0 is equal to r is called the radius of convergence. So, here in the context of the complex it is same as what we have learned for the real numbers. And no general statement can be made about the convergence of a power series on the circle of the convergence.

So, here we are talking about that, inside this circle the series will converge and outside the circle with center z 0. So, inside the series will converge and outside, the series will divert and on the circle we have No information that will depend on that particular case where we have to check separately what is happening on this circle. But what we know about from knowing the radius of convergence, that inside the circle with radius r, centered z 0, the series converges, whereas outside the circle the series diverges.

So, we also write that r is infinity, so the radius of convergence is infinity, if the series converges for all z and we say that r is 0, so the radius of convergence is 0. If the series converges only for z equal to z 0 because at that equal to z 0, all these terms will become 0. And definitely that series will converge because that will have only this constant term. So, there is no worry about this, always this will converge.

But in this case, if the series converges only for this z 0, then we call that, that the radius of convergence is 0, but if the series converges for all z, then we call that the radius of convergence is infinity.

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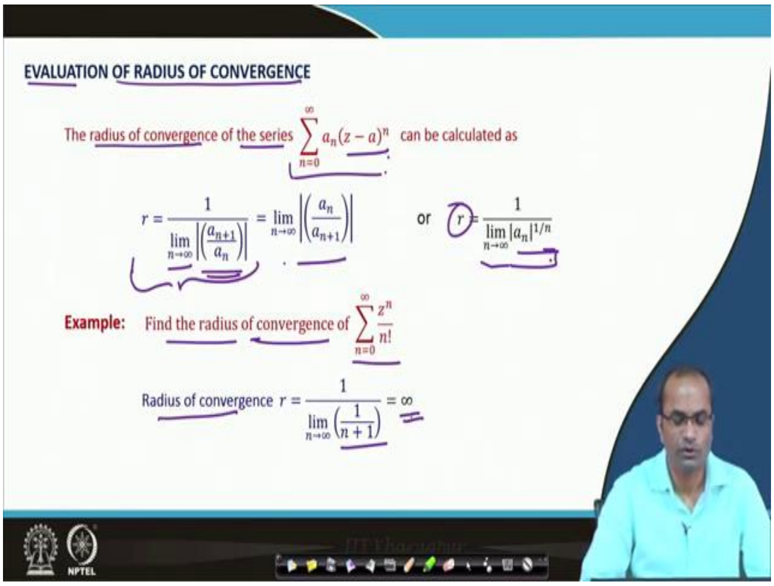

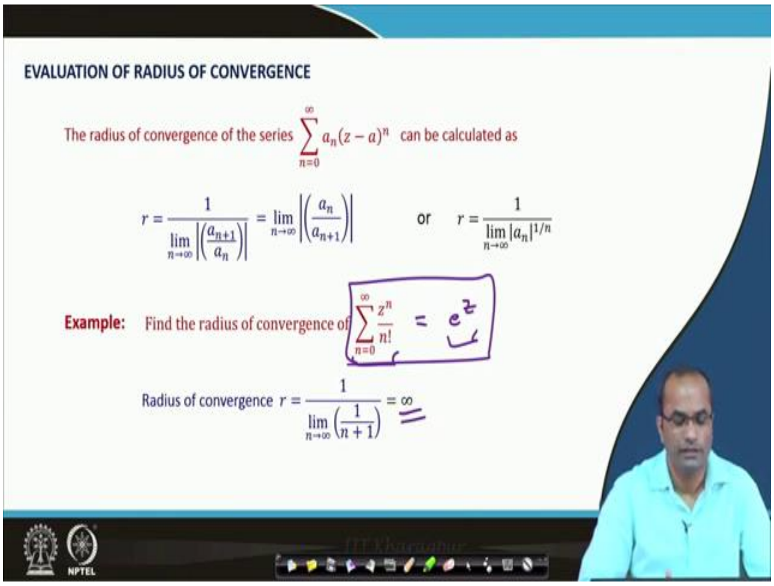

So this was some preparation. And there are the form last developed from last year not going through all these concepts, because actually finally, we will see in Taylor series, we do not need all these calculation of the radius of convergence etcetera, there is another way of knowing that the region of this convergence or the radius of convergence. But for a general power series, the radius of convergence of such series can be calculated either by this so called the ratio formula which is again parallel to what we have for real numbers.

And there is another one, the root test and then here from the root test, we also get this the radius of convergence. So, either this which whichever is convenient for a particular example, either we take this kind of ratio of the coefficients and then take the limit as n approaches to infinity or we take this nth root and then take nth approaches to infinity. In either way, we can get this radius of convergence of a given power series. So, here if you want to find taken the radius of convergence, for instance of this series z power n or factorial n. So we will apply the first formula that the radius of convergence he can get with the help of this ratio and then the limit and we will readily see here n approaches to infinity this will go to 0.

And so we have the radius of convergence for this series is infinity which is also yes if you take a close look at the series, this is nothing but the series of e powers z. And as we know from the real numbers also that e power z this is defined for any this argument z here. So, in the complex context also this e power z is very much defined. So, here this is defined for all z and therefore, this radius of convergence is coming to be infinity. So, we can fix any z this series will converge to just e power z, so here the radius of convergences infinity in this case.

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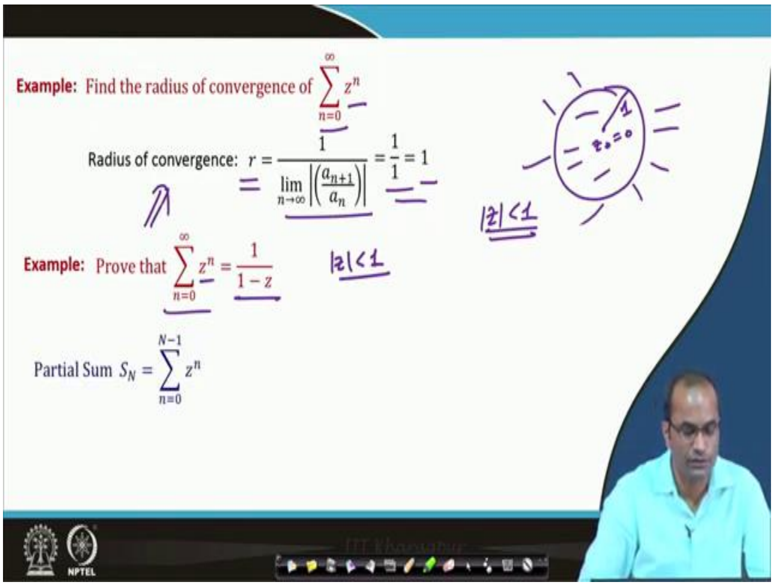

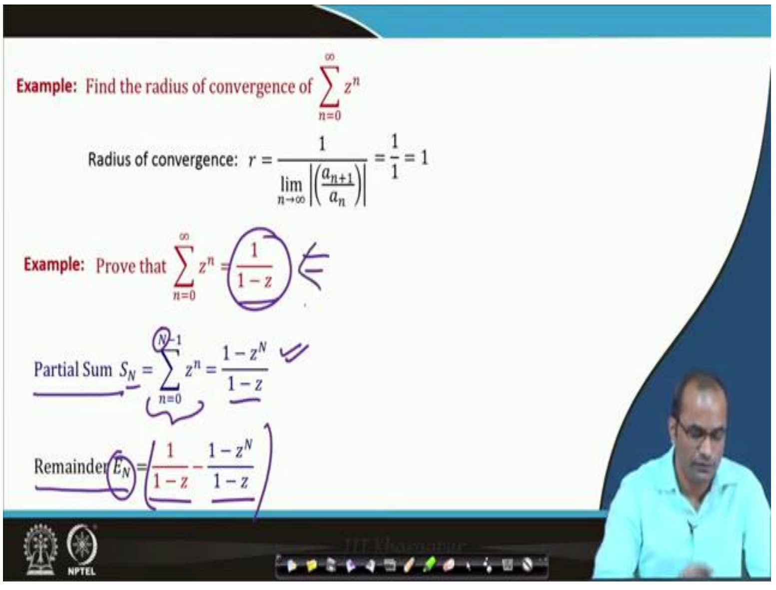

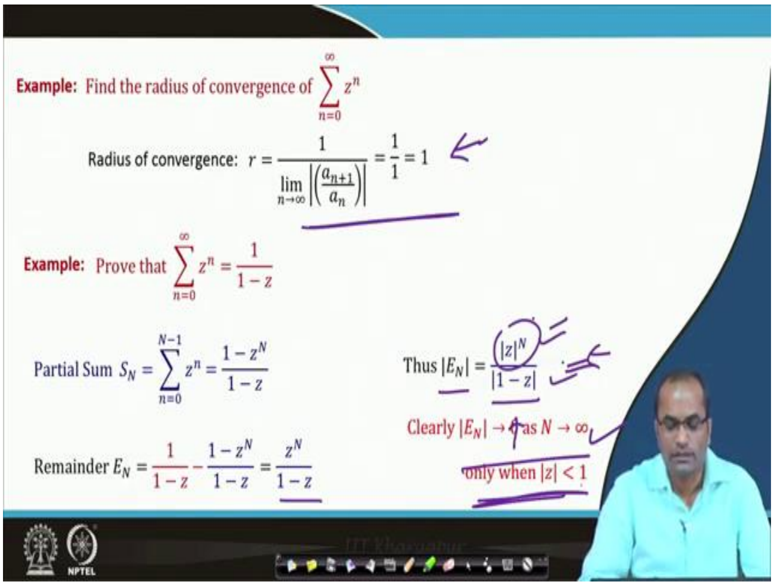

Well, so, here for instance if we want to find the radius of convergence of this series z power and n and approaches from 0 to infinity, some is over n 0 to infinity. So, the radius of convergence again we can apply this formula. So here the coefficients are 1, 1 for each term. So, we get just 1. So that means the radius of convergence for this series is 1.

So if we talk about this point here z 0 is 0. And then this radius of this circle here is 1. So outside the circle, this diverges and inside the circle, this radius, this series converges, this is what we know from the radius of convergence. So, that means for all z this less than 1 this series converges, this is what we know from this direct calculation of radius of convergence. We can indeed prove that this summation z when equal to 0 to infinity is z power n is 1 over 1 minus z which is naturally defined for z less than 1 where we have this is what we have seen already from the radius of convergence.

So, to prove this that the sum is equal to 1 over 1 minus z we can think about taking the partial sum this is 1 way we can prove this. So, we take the partial sum by introducing here the n minus 1. So, first n from we are considering and calling it S N. So, from this geometric series, so it is some of these geometric numbers so, we can just get the formula we already know. So, it is a 1 minus z power n and over this 1 minus z power n. So, here the remainder if we consider that means, this was the actual value 1 over 1 minus z and then we have here 1 minus z power n over 1 minus z from this partial sum.

So, this is the remainder term and we will see that as n approach to infinity this remainder here will go to 0 that means, this is eventually the sum of the given series. So, here we have z power n over this 1 minus z that is just after a simplification of this 1. And then what we observe if we take the absolute value there, so we have this absolute value of z power n and then 1 minus z, the absolute value.

And here we can observe now that if this mod z is less than 1, if z lies inside the circle of center 0 and radius 1, in that case, this will go to 0. But if this the z is greater than 1, then this will not approach to 0. So this error will, will grow indeed. So this itself, indicates here that this radius of convergence is 1 but we had the direct formula for the calculation of the radius of convergence.

And here it shows that this if mode z is less than 1 the E N will go to 0 and as N approaches to infinity, but the condition is that when this modulus of as z is less than 1, so, that is the condition which is again indicating us that this is the radius of convergence in this case is 1.

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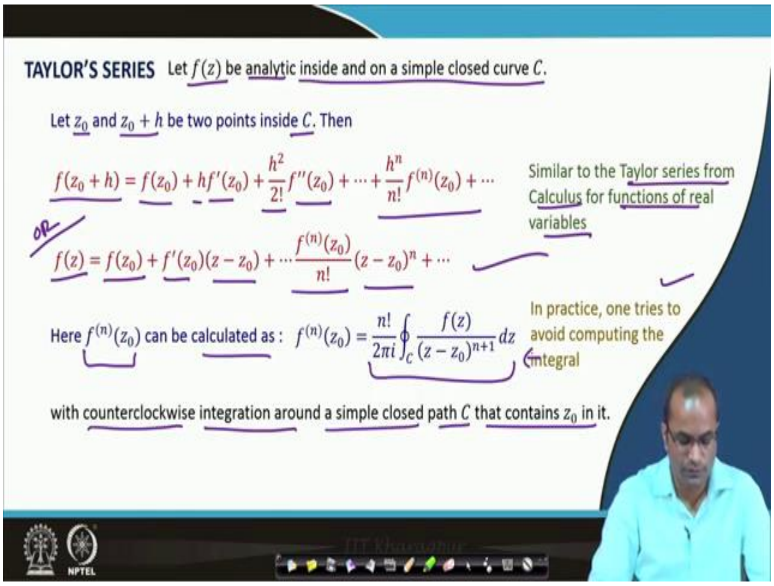

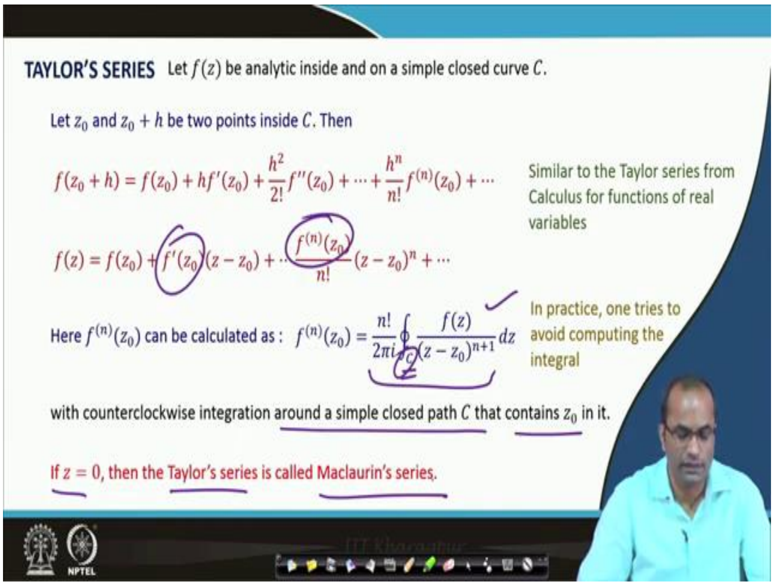

So, coming back to the Taylor's series the main topic of this lecture, we will again relate everything to what we know already for functions of real variable this Taylor series. Here it is a similar structure. So, the proof is rather involved. So, not go through the proof because, but otherwise it is analogous to what we have already studied for the functions of real variables. So, here f z is analytic inside and on a simple closed curve. So, this analyticity plays very important role in this complex analysis.

So, now again this f z is analytic inside and on a simple closed curve C and then we take two points z 0 and z 0 plus h maybe in the neighborhood of this z 0. The two point inside again the C. So, where the function is analytic, then we have this expansion that the value of f at z 0 plus this h, h could be a complex number of course, so this can be written as f z 0 h, the first derivative of f h square by 2, the second derivative of the f prime and so on.

So, this is the data series which we know from the real analysis as well or we can write in this form, so f z is can be written as f z not and f prime z not z minus z not again the nth derivative z minus not power n and we can continue with this expansion. So, this is similar to the Taylor’s series from the calculus which we have studied already in engineering mathematics 1, for functions of real variables.

In this case, the difference is that we know the formula already from the previous lecture that the derivative of this complex functions can be calculated with the help of such integrals which we have just seen in previous lecture. So, nth derivative is given by factorial n 2 pie i and this f z over z minus z 0 power n dz with counterclockwise integration around a simple closed path c that contains z 0 in it. So, in practice, one tries to avoid computing these integral because computing these integrals may not be easier.

So, we will see the practical approach that how we can avoid computing indeed these derivatives itself which are appearing here in these Taylor’s series. So, concerning this C here that can be any curve inside this around the simple closed path that contains z 0. So containing this z not point we can take actually any simple closed path and this integration can be done. So if z equal to 0 the series this Taylor’s series is called the Maclaurin’s series that is another name of this Taylor’s series for z equal to 0.

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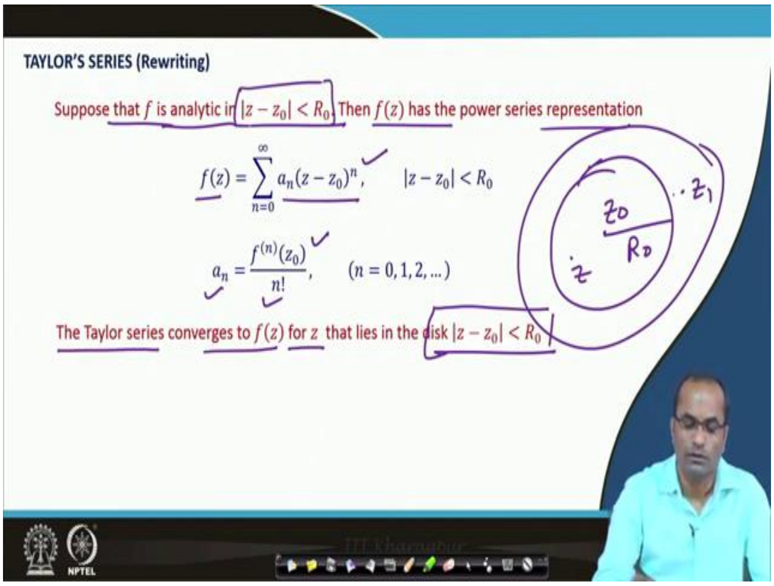

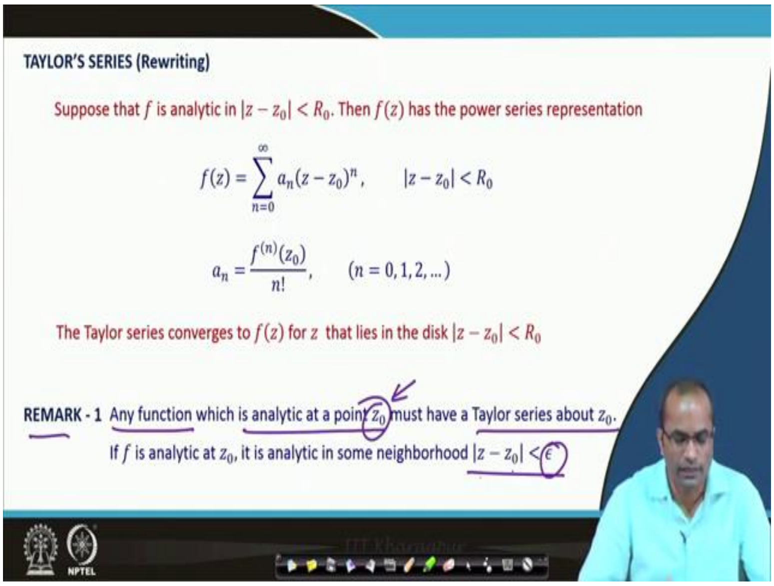

So, we can also rewrite this Taylor's series or in some literature you will find in this way that suppose this f is analytic in this z minus z 0 less than r not, so here we have this disk with radius r not and then f z has the power series representation, so f z we can naturally right in this as a n and z minus z not power n and then a n we can define as the nth order derivative z 0 divided by factorial n. So, it is the same series what we have seen on previous slide, but just written in a slightly different form.

And the Taylor's series converges here to f z for z that lies inside the disk z minus z 0 less than equal to r not. So, what we learn here that, if our function is analytic in this region here with center z not and then we have this radius, which is denoted here by r not. So, as far we can go as long as we are in the region of this analyticity and this Taylor series we will converge for all z we take in this region. So, in the next lecture we will observe that as soon as we find a point here where the if the function is not analytic we cannot be go beyond this, we cannot cross that points. Because the condition here is that f z has to be analytic inside that disk here with radius r not.

So, the whole region should not have any point where the function is not analytic. So, that we will discuss a bit more in detail in the next lecture indeed. Now, some remarks, so the any function which is analytic at a point z 0 must have a Taylor's series about z not. So, that is also important that suppose we know that the function is analytic at a point we have no other information of the region of analyticity.

So, but we can write down its Taylor's series because when we know that the function is analytic at a point z not there must be some neighborhood z minus z not less than epsilon, epsilon could be very very small. But analyticity ensures that there will be a neighborhood where the function will be differentiable or it will be analytic. So, just knowing that the function is analytic at a point z not we can write the Taylor's series however the radius of convergence which is epsilon could be very small depending on how far we can extend this region of analyticity without having a point where the function is not analytic.

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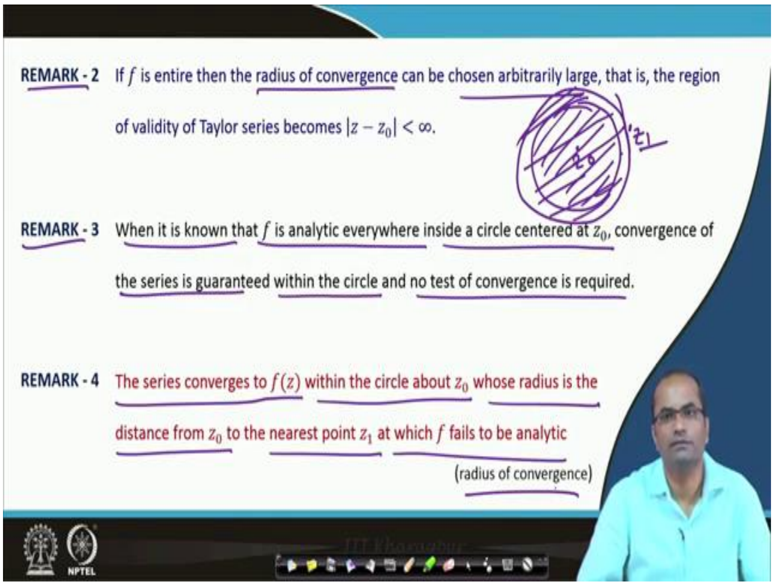

So, the remark two, if f is entire function that means, it is analytic everywhere then the radius of convergence can be chosen, arbitrarily large or meaning that radius of convergence will be infinity in that case if we know that the function is analytic everywhere there is no point where the analyticity breakdown. For instance, we have the exponential z, e power z. So, this exponential z is analytic everywhere and then we can write down its Taylor’s series which must be valid for all z.

So another remark that when it is known that f is analytic everywhere inside a circle centered at z 0, the convergence of the series is guaranteed within the circle and no test of convergence is required. So, indeed we do not have to test the convergence for this series that is already a proven fact that if we are in the region where f z is analytic, then we do not have to prove the convergence of the series. The series will definitely converge as long as this whole region, the disc inside this radius r this the part of the circle inside the radius r.

If the function is in analytic then definitely the series will converge. The series converges to f z or within the circle about z 0 whose radius is the distance that not to the nearest z 1 at which f fails to be analytics. So, this is the point which I try to explain previous slides. So, if we are at this, let say say we have z 0 here. And then we can go as long as we are covering the region of analyticity if your function is analytic in this whole domain, so the series will definitely converge in that domain.

But we should not cross a point for instance, there is a point here is z not where the function is not analytic. So we can at most go to a very close to this z not, but we should not touch that, z 1. So, we should not cross or touch this z 1. And then there is no problem about this convergence of the Taylor’s series where our function is analytic everywhere in this disk. And this is what we can absolutely get the radius of convergence by knowing that how much we can extend this disk so that no such point comes into it where the function become non-analytic.

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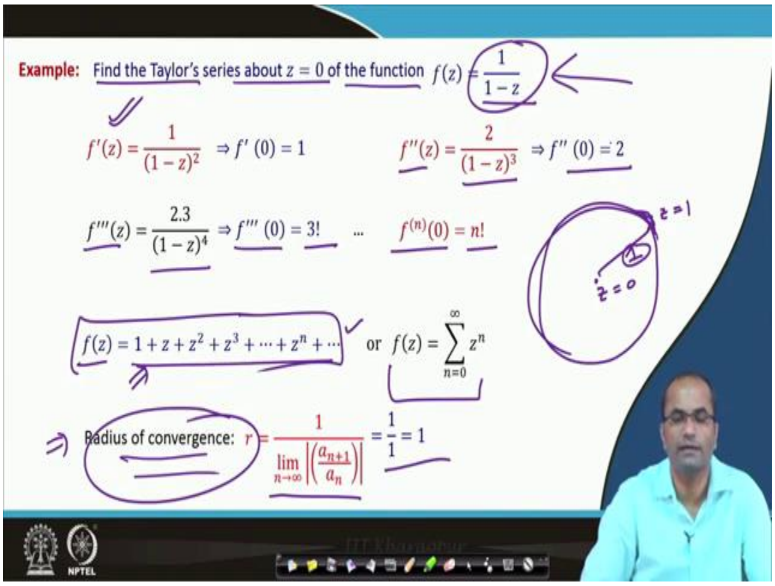

So, find the Taylor series about this z equal to 0 of this function for instance we are taking here 1 over on minus z. So, this function plays a very important role and we will see in a minute that why it is very crucial to expand this function first in Taylor's series and then we will actually utilize this function or we will use this function that series of this function to write the Taylor's series of many other functions. So, to get the series of this function we can Taylor's series we can get the first derivative which is minus, so, the plus indeed so it is 1 minus z.

So you have 1 over 1 minus z square and we can get it 0 the value is 1, we can get the double derivative which is 2 over 1 minus z cube. And again this derivative at 0 will be just 2. So, the third derivative will get 2 into 3 divided by 1 minus z power 4. So, the third derivative at 0 is just the factorial 3, the nth derivative at 0 is factorial n. And this f z, we can write down as 1 plus z plus z square z cube and so on to this z power n.

And here the… or we can write in the compact form that f z is here n 0 to infinity z power n. So, this is a very important series as I mentioned before, that this 1 over 1 minus z which the expansion is 1 plus z plus z square z cube and so on, it is going we are going to use this series for getting many other Taylor series. Coming to the radius of convergence for this series, we can use for instance, this formula and immediately we can get 1.

So, this is one way of getting the radius of such power series or which we have discussed that we are talking about, we expanded expanding this around z equal to 0. So, how long we can go to define this region where the series will converge is the point where exactly the analyticity will break. So, here if we have this z equal to 1 point where the analyticity breaks down, so we can go up to this point, we should not cross or touch this point.

And then this circle here of radius 1 will be, so the 1 will be the radius of convergence from the direct observation of the function regarding this region of analyticity, but the same we can get also by the formula. So, is that now I hope it is clear now that in either the way we can discuss the radius of convergence.

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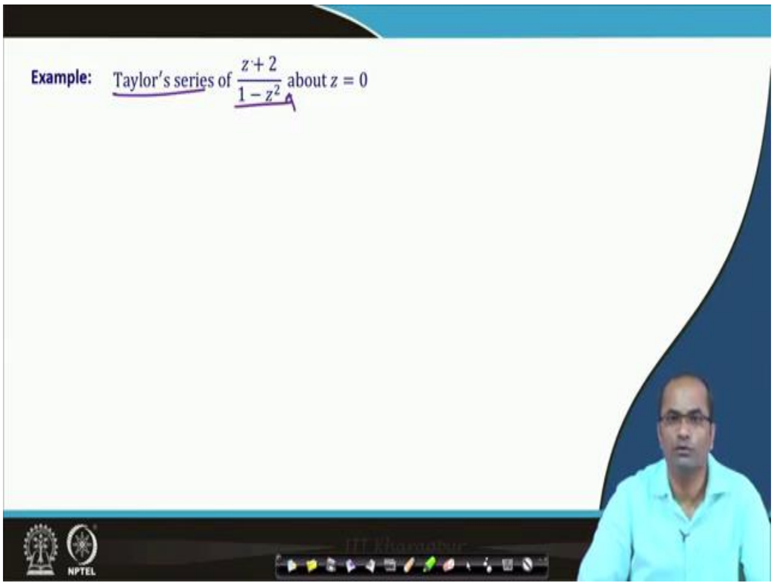

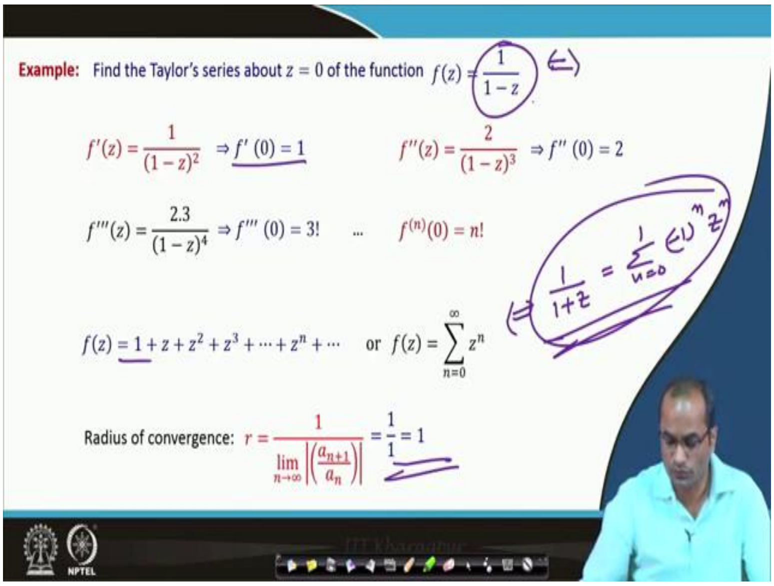

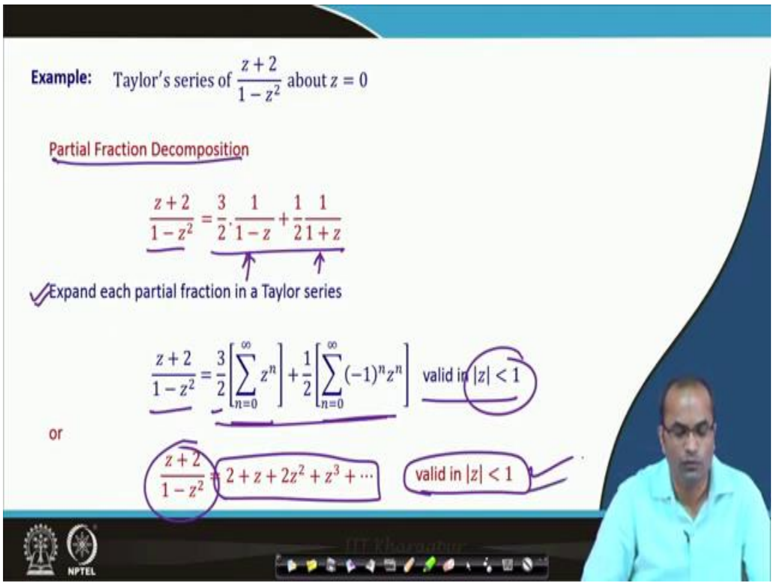

So, for instance, if we take the Taylor's series of this functions z plus 2 over 1 minus z square and about the z equal to 0, we want to get the Taylor's series. So, there are two ways of course, 1 we can use that the standard formula, but there we need to get the first derivative, we need to get the second derivative evaluate them at 0, then third derivative and so on. The process will be complicated if we have a complicated function or even more complicated than this 1.

So, the trick is here that we will try to break our function in terms of the function which we have just studied before that was 1 over 1 minus z. So just to note again here that not only for 1 over 1 plus z which 1 minus z we can also write down for 1 plus z for instance, that will be having the similar structure but the sign will change there. So we will n 0 to 1. And then so here the minus sign will come so we will have minus 1 power n and this z power n. So, we will have the series for we will have the series for the 1 over 1 plus z as well, not only for just 1 minus z.

So, there 1 minus 1 power n will come as an extra term and the radius of convergence will be same again as here it is 1. So, these two, threes the 1 over 1 plus z and 1 over 1 minus z, these two will play a very crucial role and we see in the next slide. So we have this, we want to get the Taylor's series of z plus 2 over z 1 minus the square. So, what we will do? We will use this idea of the partial fractions and that is what we will do in most of the examples. So, this z plus 2 over 1 minus z square, if we break into the partial fractions, we will get these 2 terms where 1 is 1 minus z and another is 1 plus z.

So, now we can expand each of this partial fraction in a Taylor's series. That means, this z plus 2 over 1 minus z square, we can write down 3 by 2 and the Taylor's series of 1 minus z. So, we have these power z power n and for the series of 1 over 1 plus z this is minus 1 power n, z power n. So these 2 series which both of them are valid in this region, z minus 1 that is radius 1 or we can just simplify a bit here. So we will get this what 2 plus z plus 2 z square plus z cube. So this is another series here we got for z plus 2, 1 minus z square which is valid in absolute value z less than 1 .

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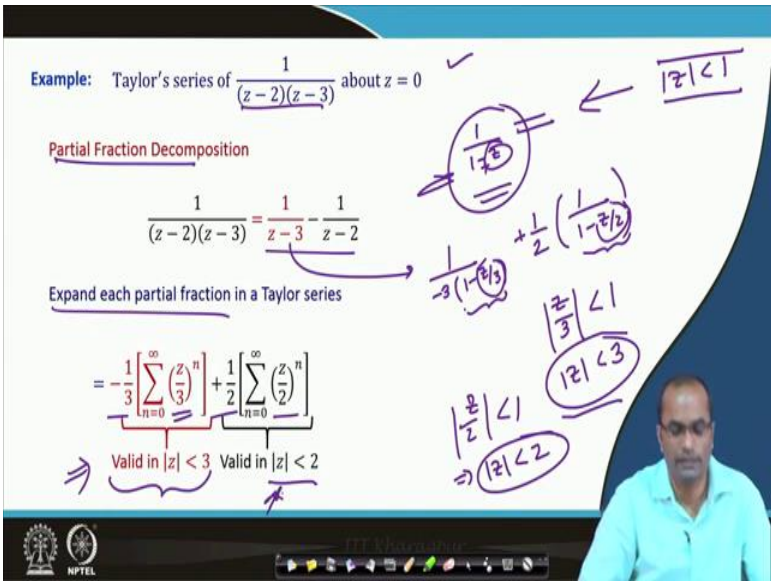

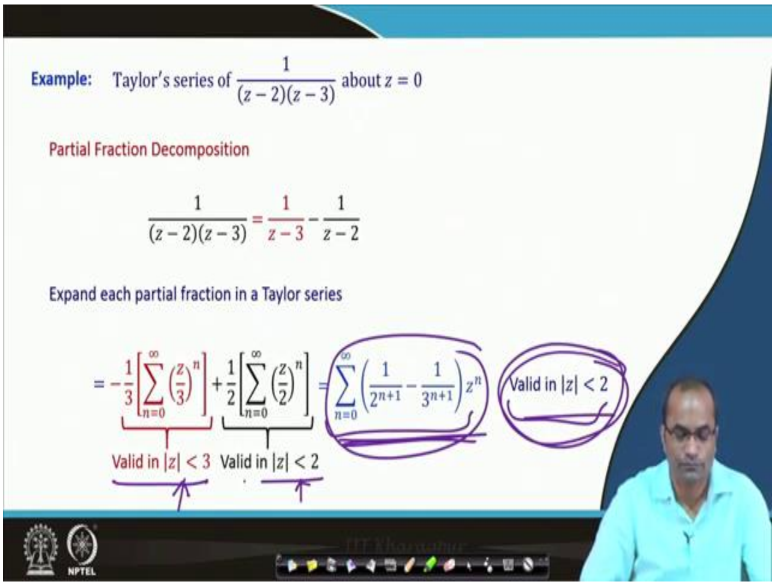

So if we want to get the Taylor's series of 1 over z minus 2 and z minus 3, that is the product there, we will use the same trick again. So, here we can do the partial fractions. So these partial fractions of 1 over z minus 2, z minus 3 will be 1 over z minus 3 and minus 1 over z minus 2. So, but we know the series of 1 minus z or 1 plus we know already the series of 1 minus z or 1 plus z.

So, we will modify these partial fractions in this form so that we get something 1 over plus minus and then some z. So, we can expand this each in the Taylor series because the first 1 we can write down as taking here minus 3 common, minus 3 common and then we have here 1 minus and then z over 3. The second term what we can do? We can take minus 2 common there and we will have again here 1 minus z by 2 from there. So now we have 1 minus the sum constant z here also we have 1 minus, instead of z now, we have z by 2 here, instead of z we have said by 3.

So in this series, we will just replace z by z by 2 and in the second case, z will be replaced by z by 2. So this is exactly the series expansion for 1 over z minus 3. So we have replaced z by z minus 3 in this expansion there. And here z is replaced by z by 2. Otherwise, we have the same series now but the convergence will change. Because in this case, the convergence was when mod z is less than 1, but we do not have this z instead of z we have z by 3 and z by 2. So, accordingly this series will be valid for absolute value z less than 3 because here we have the condition now that the z by 3 should be less than 1 that means the mod z should be less than 3.

So, this will be the region for this first series to get convergence and for the second 1one accordingly we can get from this z by 2 less than 1 we can get that these absolute values z is less than 2. So, from there we can get this convergence for the second series. And what we observe, that we can combine these two and now the validity we have to check because we are talking about that this is our series. The first one was valid when absolute value z less than 3. The second was valid here in the disk radius 2.

So, as combined here both the series are valid in this disk less than 2, radius 2. So, this is the validity now for the whole series because the individual series here, this is valid for the mod z less than 3, this is mode z less than 2. So, as a whole, we have to see the common region which is coming now, the absolute value of z less than 2.

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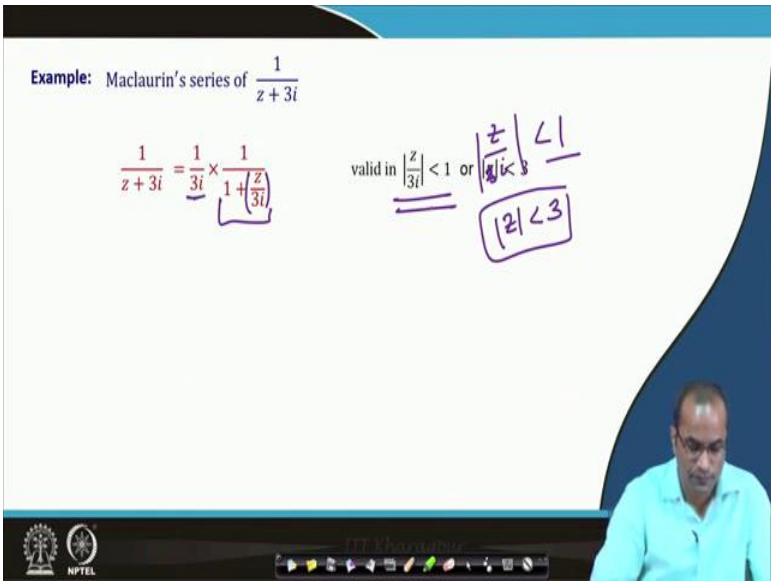

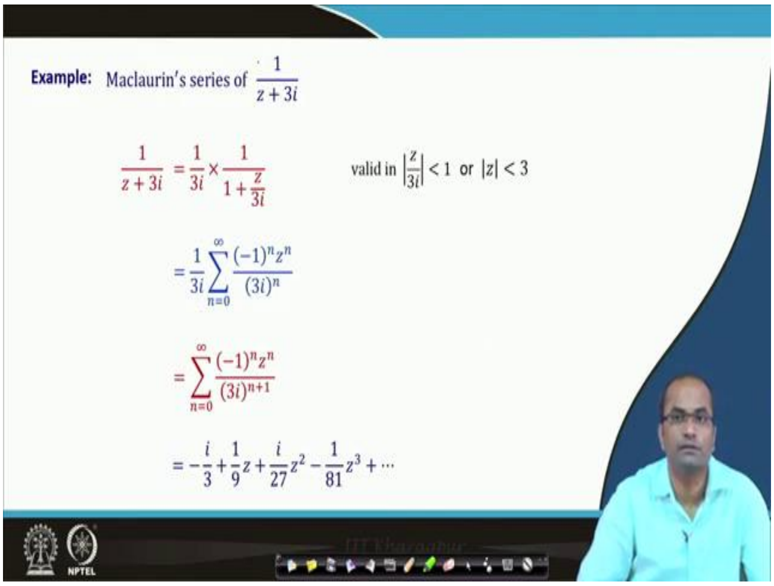

The last example here we will be talking about the Maclaurin's series of 1 over z plus this 3 i. In the same trick, we can use again what we have d1 before, so 1 over z plus this 3 i. So, we take this 3 i common there to have this form 1 minus or 1 plus z by something there. So, if we can expand this again instead of taking z there, we will take z over 3 i and the validity will come from this condition that absolute value z over 3 i should be less than 1 that means this z less than 3.

So, that will be the condition for the validity of the series. And then we can expand it with that standard series. So, minus 1 power n and 3 i power n and z power n. So, which can be just written in this form or we can expand it, we will get such terms out of this series there. So, what we have seen with the help of that two particular, I mean in particular those two series, we can utilize that for the expansion of many other other functions and the more we will discuss, of course, in the next lecture, where we will generalize the idea of this Maclaurin's order, Taylor's series.

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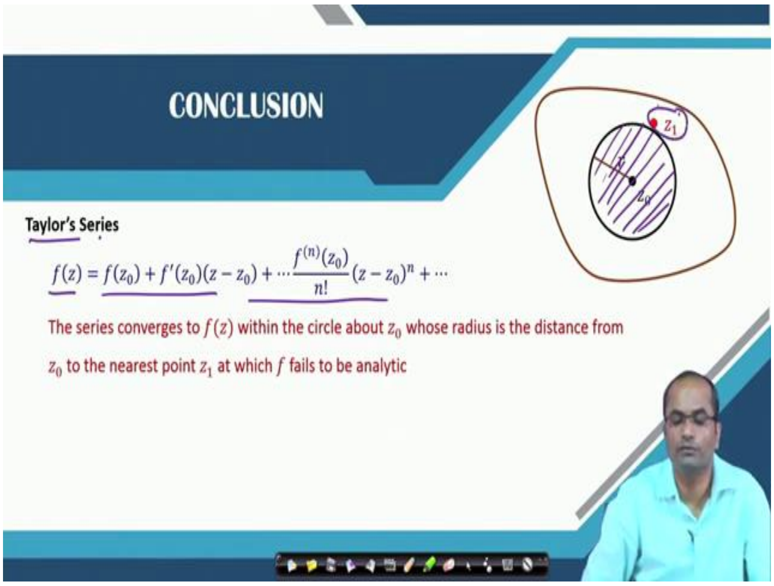

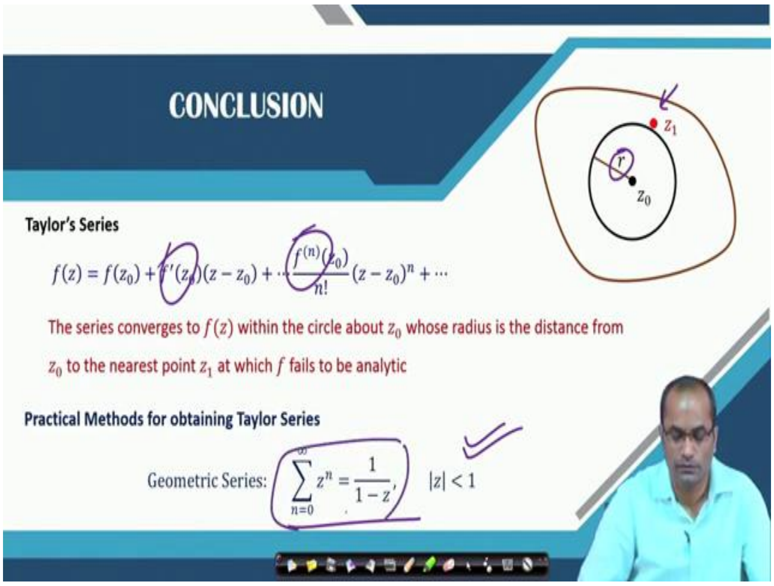

So, these are the references we have used for preparing the lecture. And just to conclude that we have learned now the Taylor's series for f z, the complex functions and that is the similar extension expansion what we have for the real functions as well. And regarding the interesting fact about the convergence, we have seen that the series will converge, we can go to the maximum radius around this z point as long as it does not have any point where the analyticity breakdown for the function z.

So, for instance if we assume that z 1 is the point where the function is not analytic, then we can just go up to this z 1 maximum to have the validity of the Taylor's series or the region of convergence of the the radius of convergence for this such Taylor's series which will be that radius r.

So, and we have seen that practical method for obtaining Taylor series because that is very useful. We do not have to expand and get all these derivatives for many functions and this the expansion of either 1 minus z or 1 plus z was very much useful and we have seen in several examples how we can use this series to expand many other functions. So that is all for this lecture and I thank you for your attention.