Engineering Mathematics-II
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Lecture 16
Cauchy Integral Formula

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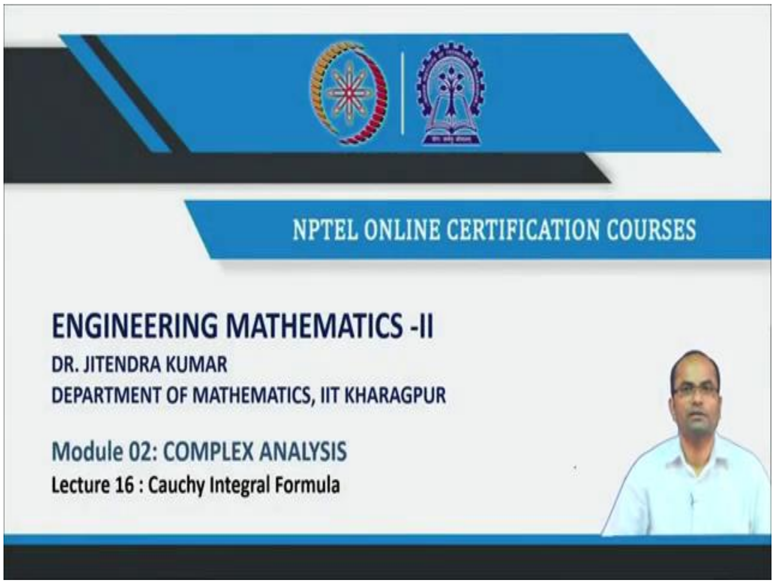

So welcome back to lectures on Engineering Mathematics 2 and this is lecture number 16 on Cauchy Integral Formula.

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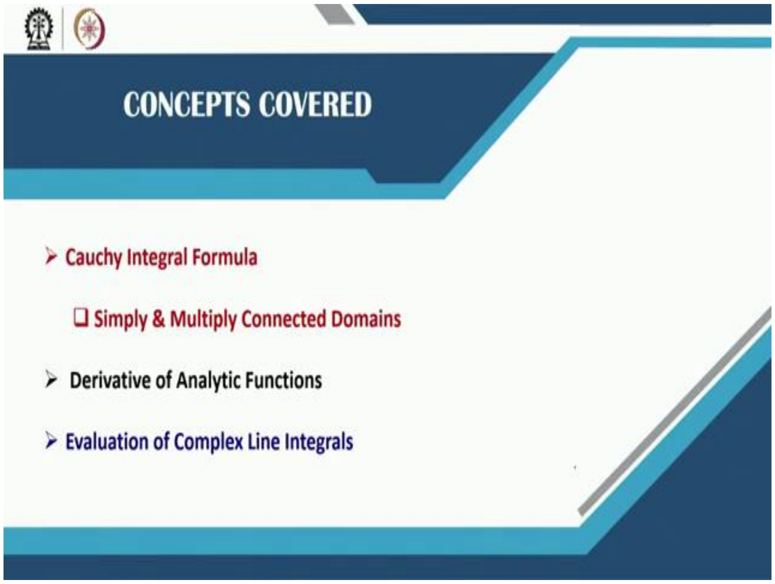

So today we will cover the Cauchy integral formula for simply and multiply connected domains and then we will move to the derivative of analytic functions in terms of the integral we will see that we can find the derivative of an analytic function. And then we will go for some evaluation of complex line integrals with the help of Cauchy integral formula.

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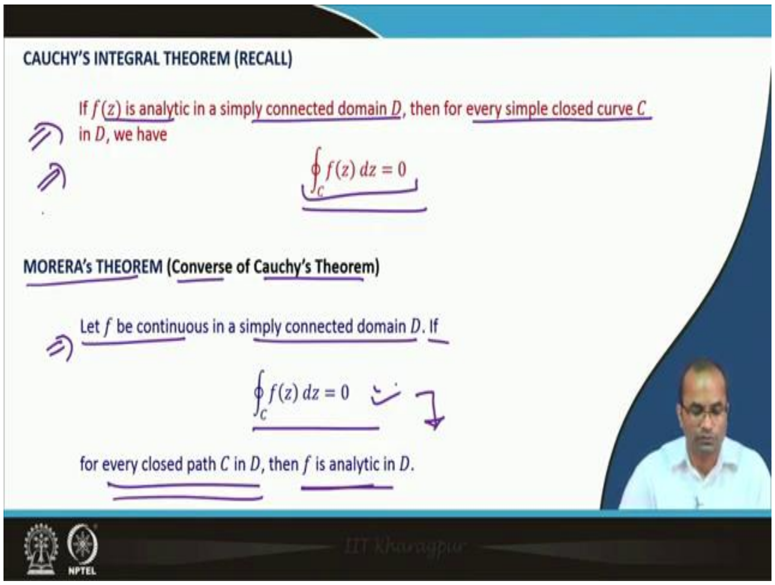

So just to recall in the last lecture we have gone through the Cauchy integral theorem which says that if f z is analytic in a simply connected domain D then every simple close curve C in D we have this closed integral of f z of this analytic function over a closed path is 0 so that was the Cauchy integral theorem and we have gone through several examples or applications based on this theorem for the evaluation of complex line integrals.

So today first let me just tell you that this Cauchy integral theorem there is called the reverse or the converse of this Cauchy integral theorem just to mention we are not going into the detail, we will directly move for Cauchy integral formula. So this Morera theorem says that if f is continuous in a simply connected domain D, so we are not talking about analyticity now here in the Cauchy theorem it was like f is analytic then we have this integral 0, now we are going other way round that if this integral is 0, so we have the assumption that f is continuous in a simply connected domain D.

And if we note that this integral is 0 for every closed path C in D then f is analytic, so in that way this is a converse of this Cauchy theorem because from the integral now you are telling that f is analytic in D whereas in Cauchy theorem we say this f is analytic then this integral is 0 over any closed, a simple closed path C, so this is just to mention that there is a converse of Cauchy theorem as well.

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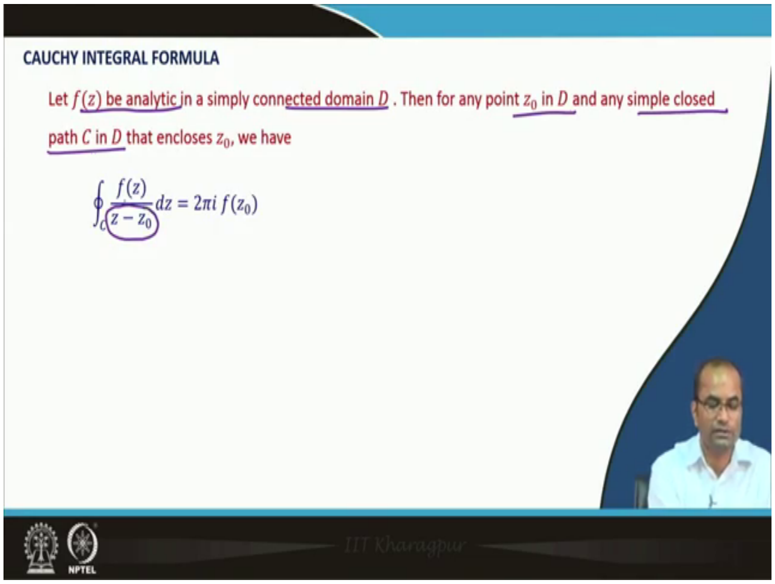

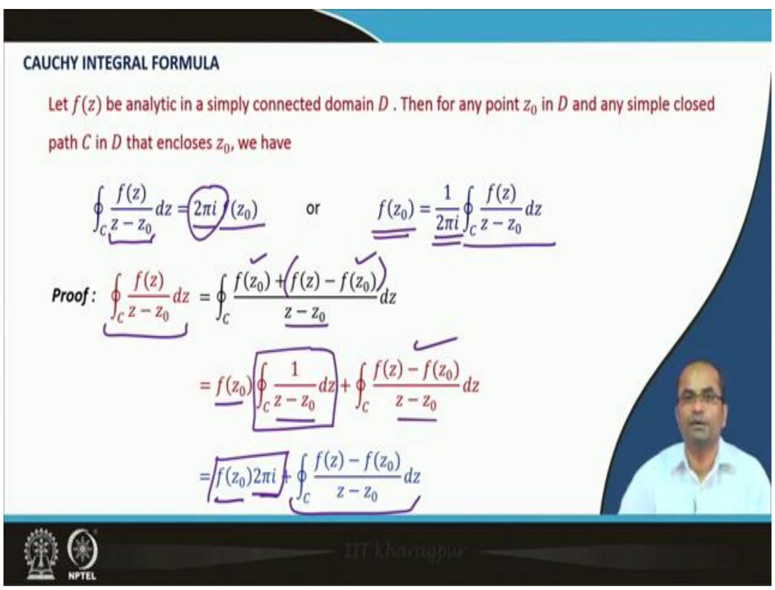

So we will move to now the Cauchy integral formula, so there is a difference we are talking about the Cauchy integral theorem, now we are talking about the Cauchy integral formula. So what this formula says that if f is analytic in a simply connected domain D, so the similar assumptions what we have for Cauchy theorem then for any point z 0 in D and any simple closed path C we take in D that encloses this point z 0.

We have this result that f z over z minus z 0, so this is the extra term here if this is not there then integral over this C fz dz was 0 but we have now z over z 0, so as a integrant so integrant is now here this f, f z over z minus z0 this is no more analytic in that domain because the domain z0 point is there which is enclosed by this curve C. So naturally this integrant is not analytic and we do not have this as equal to 0.

So which was the case earlier when your integrant was analytics, so here the integrant is not analytic because of this term z minus z0 and as a result this integral has the value 2 pi i and the function at z0, so this is what we call the Cauchy integral formula. So it has a various applications for evaluation of such integrals where the integrant is not analytic for instance. So this formula sometimes we also write in this form that f at 0, so this 2 pi i we can bring to the right hand side and then this is a integral over this closed path of this f z over z minus z0 dz.

So going to the quick proof of this we can consider this integral here f z over z minus z0 dz and then we add here this term fz 0 and subtract this fz 0, so again we are with this fz over z minus z0 and then we can write down it as, so fz 0, the first term and here we have 1 over z minus z0 for the second 1, we have this f z minus z0 and then z minus z0, so we have broken this integral into 2 parts and the we know already the result we have seen in previous lectures that the result of this 1 over z minus z0 dz was 2 pi i. So the first integral we have evaluated already its fz 0 and into 2 pi i, the second integral over this closed curve C fz minus fz 0 z minus z0 we will evaluate now.

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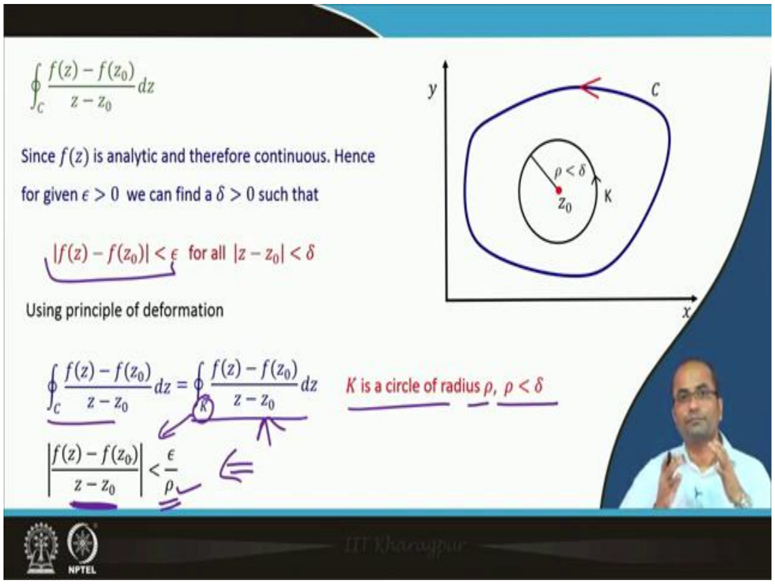

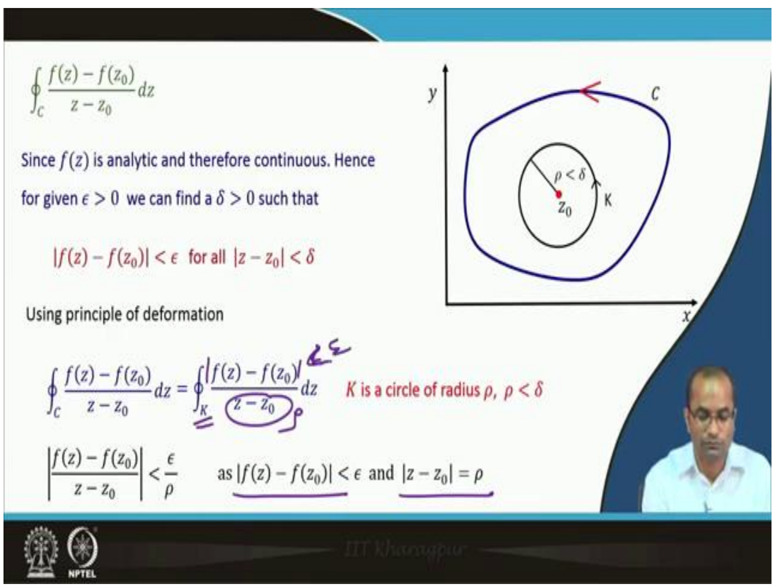

So we have the situation here that this integral we want to evaluate and we consider now the situation around this z0, we take a circle of circle which we are calling it k here of radius rho and the rho is less than delta and what is the delta we will see now this connection here. So fz is analytic and therefore continuous that is ofcourse and hence for given so since this fz is continuous so we can use this definition of continuity for a complex valued function. So here for given epsilon we can always find a delta such that this difference fz minus fz 0 is less than epsilon for any z in this disc here z minus z0 less than delta.

So this epsilon and delta are from the definition of the continuity since fz is continuous so for any given epsilon we can always find a delta such that this relation holds and now we have enclosed this points z0 by a circle whose radius is rho and rho is even less than delta. So for given delta now we can choose a rho where rho is less than delta and we can enclose this z0 by this circle of radius rho.

So using now the principle of deformation, so just to recall what was the deformation that instead of taking this integral over the C we can also do that over a any other curve and then the value will be the same, this is what we have learnt before. So that integral that deformation principle says that this integral fz minus z0 over z minus z0 dz will be equal to fz minus z0 over z minus z0, so that we have already learnt.

So while making a curve there and then we have consider this as a simply connected domain where the function, the integrant is analytic and then we have got this result which says that the integral over this closed curve C, the value of this integral will be equal to any other path we can take inside this domain and the value will be equal so that part was already discussed, so what we have observe now here that this integral is equal to this integral.

And this is going to be a simpler integral because we are talking about this k which is a circle there. So k is a circle of radius rho and the rho we have taken less than delta. So now we will make use of this inequality that fz minus fz 0 over z minus z0. So on this circle k this z minus z0 the absolute value of this z minus z0 the modulus of z minus z0 is rho, so this z minus z0 is replace by rho and then from the continuity of fz we have already this relation that fz minus fz 0 is less than epsilon.

So we have this inequality here that epsilon we can bound this integrant of this integral by epsilon over rho. And now so this is already discussed that this is epsilon and we are exactly on the boundary of this case so that can be replaced just by rho and here the upper one we have just bounded it by epsilon.

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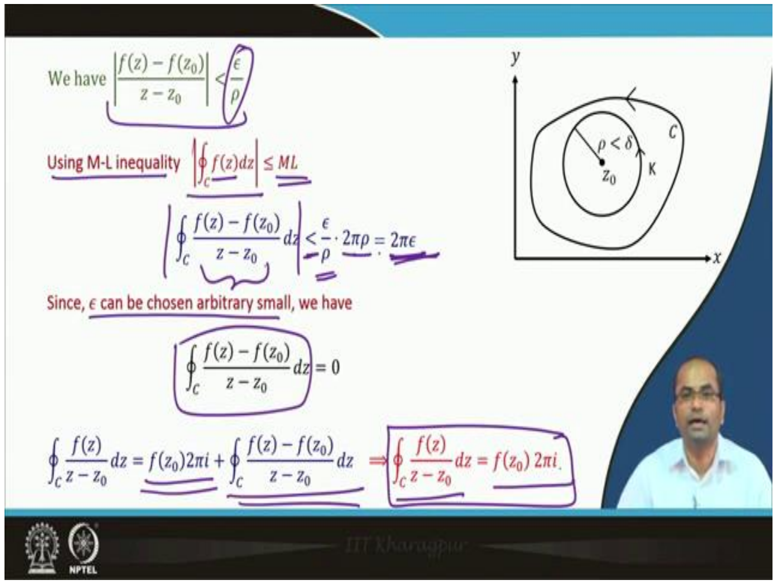

So with this bound we can observe that we will use this M-L inequality which was also discussed in previous lectures. So this fz the absolute value of this integral is bounded by M into the L, L is the length of the curve and M is the upper bound for this integrant, so we have already the upper one for the integrant that means this we can bound by this epsilon over rho that is the bound for this integrant and then 2 pi rho that is the circumference of this circle k, so 2 pi and the rho was radius. So the arc length of the curve that is 2 pi rho.

So if this a simplified this is coming 2 pi and epsilon, so now the point is that this integral the value of this integral, the absolute value of this integral is bounded by 2 pi epsilon and we should note that this epsilon was arbitrary, so from the continuity of fz we have said that for a given epsilon, so epsilon can be chosen as small as possible arbitrarily small, so what this says that if this can be chosen arbitrary small the value of this integral is going to be 0 because the value of this integral is bounded by 2 pi epsilon.

And we are telling that epsilon can be chosen arbitrarily small, this means that the value of the integral has to be 0 otherwise it cannot be less than this 2 pi epsilon and epsilon is arbitrary number. So now from this breakup of the integral the first part already was evaluated which was 2 pi i fz 0 the second we have seen that this is going to be 0, so we have this result which is the Cauchy integral theorem, that is integral value is equal to 2 pi i and fz 0.

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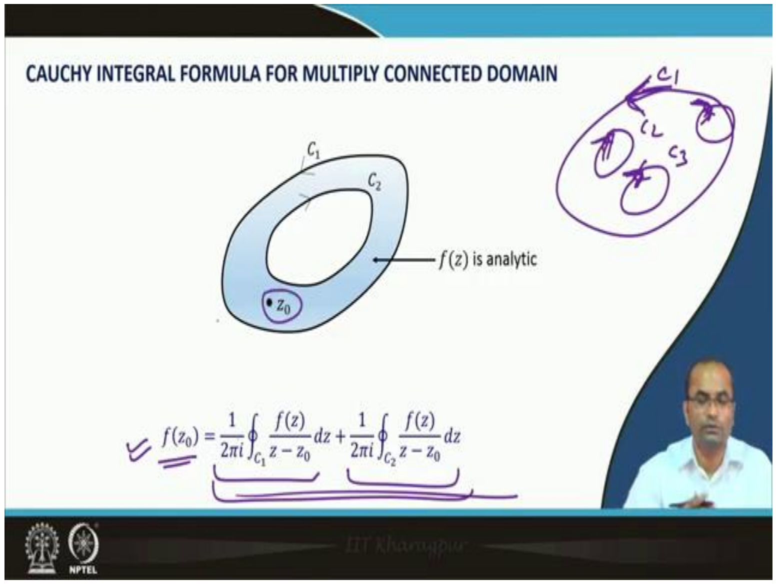

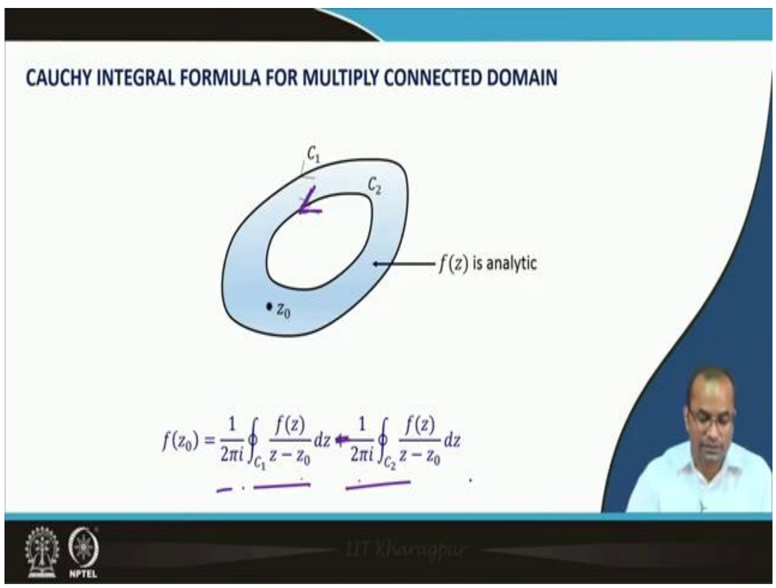

There is an extension for this for multiply connected domain because so far we have seen that the result is valid for simply connected domain, now if we have the multiply connected domain now the outer curve here is traced anticlockwise and the inner one we have taken the clockwise direction. So in that situation the results says that fz0 is equal to the first integral over the C1 and plus over the C2, we can have a many such holes there, so like this like C1 then we have here another curve C2 the hole in this, then we have C3 so then we have C4 and so on.

So we can have many such, so the for the inner circle we should have the clockwise direction to happen to for this result which is a sum of all these curve integrals, so the outer integrals should be in the anticlockwise direction and then if the inner one are in the clockwise directions then we can have we can simply add these integrals and the value will be equal to fz 0. So having this I should make one more remark that what will happen if we take for instance this also in the anticlockwise direction.

If we take C2 in anticlockwise direction then this plus sign will become minus sign and we will have this fz 0 equal to this and this minus 1 over 2 pi integral over C2, so that is also possible. So if we are taking the clockwise direction these two will be added otherwise we can also subtract these results.

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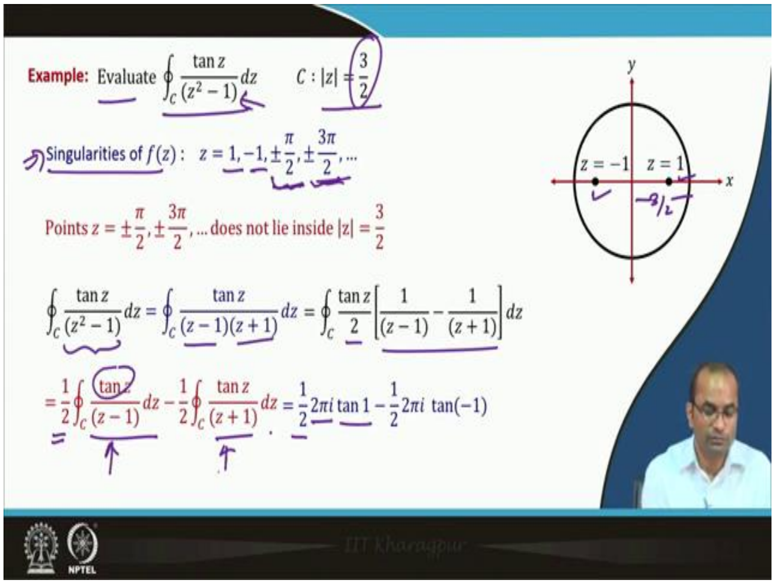

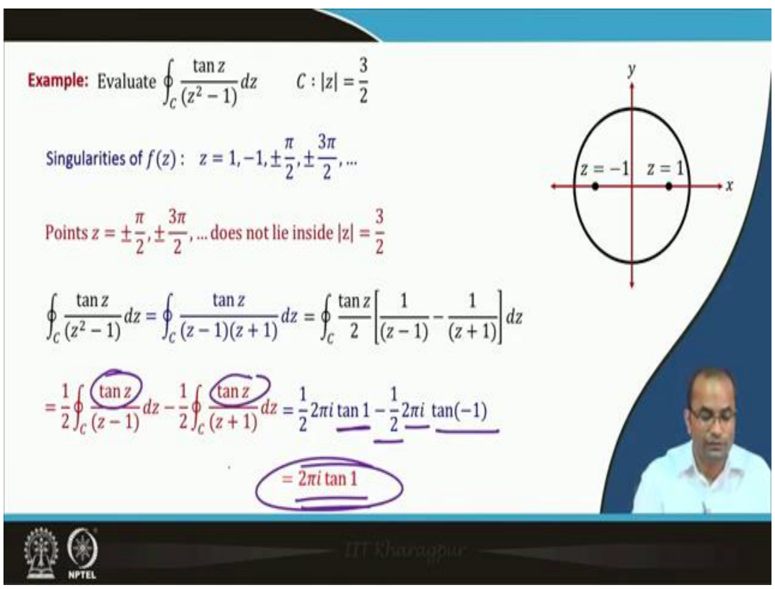

Okay, well so we can now move for the evaluations. So we evaluate these integral tan z over z square minus 1 and then we have here z is the absolute value z is 3 by 2. So in that case we have a integrant which is matching with this integral formula, so in the denominator we have this z plus 1, z minus 1 and we should look now first the singularities of this fz the integrant fz is the integrant, so naturally the z is equal to 1, z is equal to minus 1 we have the problem in the integrant because this will go to 0 in either situation and then this pi by 2, 3 pi by 2 and so on the tan function will be become infinity.

So at all these points the function is not differentiable or function is not analytic at those points so we call these points as singular points of this fz of the integrant, so because we want to identify that where the function is not differentiable or not analytic otherwise if the function is analytic everywhere its straightaway we can use the Cauchy theorem and the value will be 0. So now the situation is that this pi by 2 and 3 pi by 2 because here the radius of this circle here is 3 by 2.

So only these two points lie inside that is z is equal to minus 1 and z is equal to plus 1.the all other points like pi by 2 which is pi is more than 3, so 3 by 2 that is outside the domain of concern outside this C, so here this pi by 2, 3 Pi by 2 they are not at all in the domain, so not of our concern but here z is equal to minus 1 and z is equal to plus 1 these two points are inside the circle C and therefore they will be considered now as the point of where the function is not analytic.

So here now we have tan z over z square minus 1 we can write z minus 1 and z plus 1 and then we can have this partial fraction of the two, 1 over z minus 1, 1 over z plus 1 and the half the first integral is tan z over minus 1 the second integral tan z over z plus 1, so now we have exactly in the form of this Cauchy integral formula where we can directly apply the result, this is like fz over z minus 1 and this 1 is exactly the point inside the circle here and z equal to minus 1 is also inside the circle.

So here we have the half as already there than 2 pi i because of the formula and tan has to be evaluated at 1, so this tan has to be this fz has to be evaluated at 1. Similarly for the second part we have minus half 2 pi i and tan here must be evaluated at minus 1, so this is the final result that 2 pi i and the tan 1 this is the answer of this integral which we have evaluated using the Cauchy integral formula.

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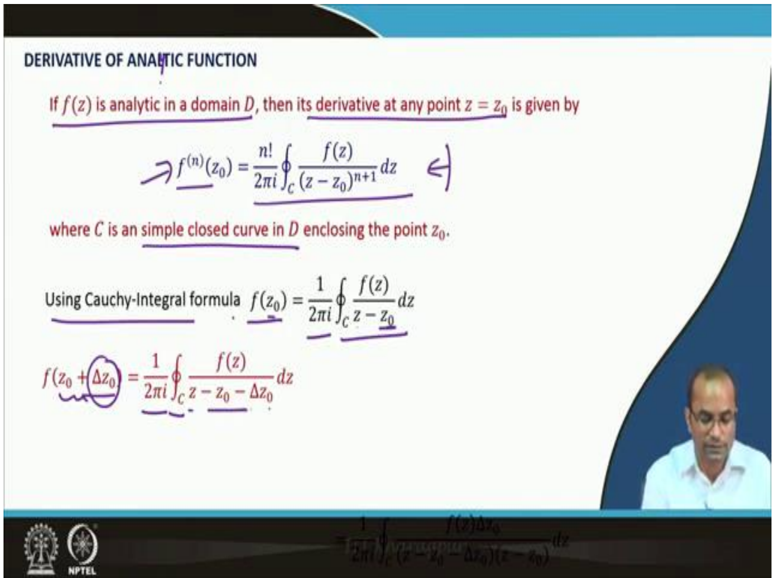

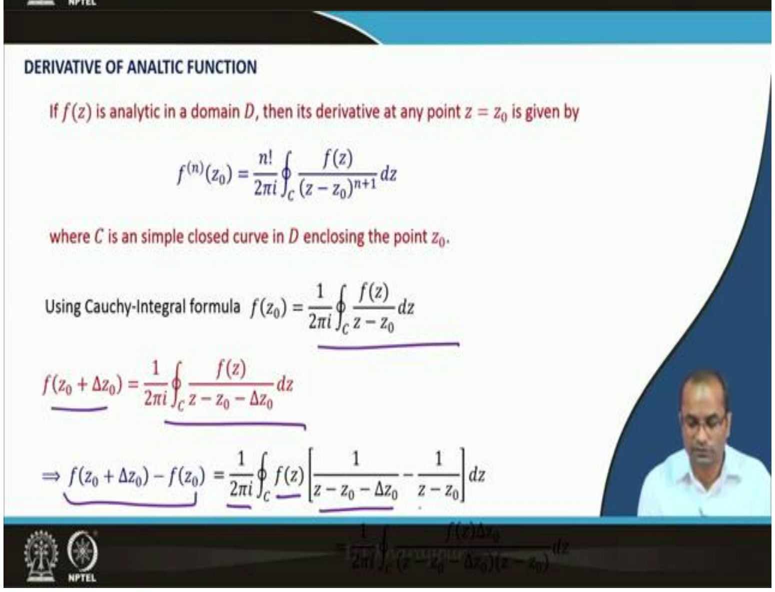

Well, so there is another one, that derivative of analytic function, the derivative of analytic functions the formula is quite similar to what we have seen as a Cauchy integral formula. So if fz is analytic in a domain D then its derivative at any point z equal to z0 is given by this formula here that f the nth derivative at z0 we can compute by this line integral factorial n over 2 pi i and we have fz over z minus z0 power this n plus 1 and dz.

So that is has a nice applications and a very interesting fact that the derivative of this function which is analytic we can compute with the help of such integrals, such a curve integral. So here the C is an simple close curve in D enclosing the point z0 so that is the only condition we can take any C which encloses this z0 point and it is this domain D where your function is analytic that is the only condition we have.

So using this Cauchy integral formula we can realize the above result, so the Cauchy integral formula says that fz 0 is 1 over 2 pi i and the integral fz over z minus z0 dz. From here we can derive its first derivative and then it can be generalized, so for the first derivative let us just make an increment here fz 0 plus delta z and then we will apply the fundamental theorem of differentiability.

So fz 0 plus delta z we have made this increment delta z naught, so 1 over 2 pi i the integral fz over this z is replaced with z naught plus delta z naught. So here instead of z naught we will now make z naught plus delta z naught, so z minus and z naught is replaced by z naught plus delta z naught.

So this is the result now f z naught plus delta z naught and we can look at this difference now f z naught plus delta z minus this f z naught by this integral we can have now 1 over 2 pi i fz and this 1 over the first term z minus z0 minus delta z naught and then the second one 1 over z minus z naught.

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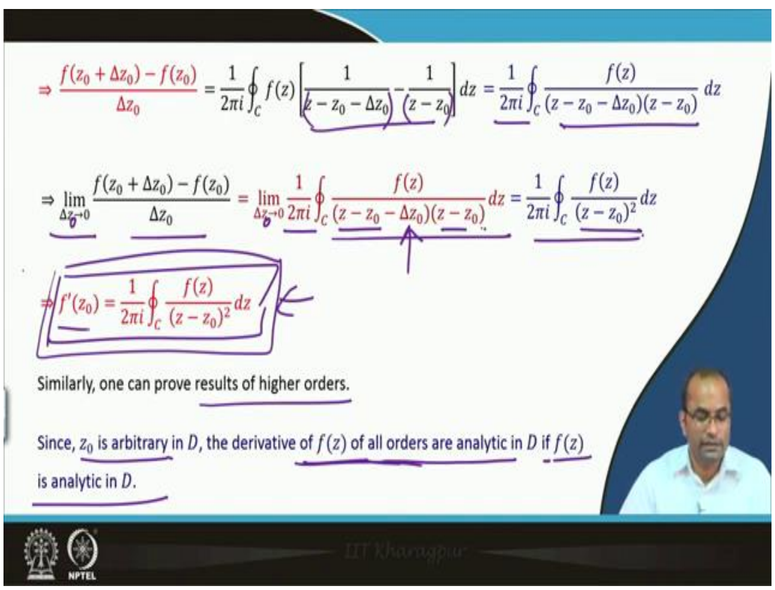

So we can simplify this now, these two can be simplified to have this f z divided by this the product of the 2 taking this LCM and then we can have just 1 there, so 1 over 2 pi i and then we got this fz over z minus z naught minus delta z naught and this product of this one. So now if we take the limit as limit delta z approaches to 0 delta z naught approaches to 0 what will happen to this and that is exactly precisely the derivative of this fz at the z naught.

So taking this limit what we will now observe here, so we have 1 over 2 pi and this integral and we want to take this limit delta z approaches to 0, so what will happen when delta z approaches to 0? This is exactly becoming z minus z0 and z minus z0 whole square we are getting and now this everything is free form delta z naught. So we got this formula for the derivative that f prime at z naught is 1 over 2 pi i and fz over z minus z naught whole square.

Well, so this is just for the first derivative and then we can go for in a similar way we can approach for the higher order derivative which we are not doing now but we have now the idea that how this derivative is written in terms of the integral using Cauchy integral formula. So since this z0 is arbitrary we have chosen just an arbitrary point z0 the derivative of fz for other orders are analytic in D. So what we have seen if fz is analytic we can get this f prime also at any point there in the domain.

So all order derivatives of this fz will be also analytic in D if fz is analytic in D, so that is a very interesting result in this connection of the complex variables that if this fz is analytic in D then all its derivatives will be also analytic in that domain D which is readily prove form this result where we have seen that the derivatives can be computed easily once we know that this fz is analytic.

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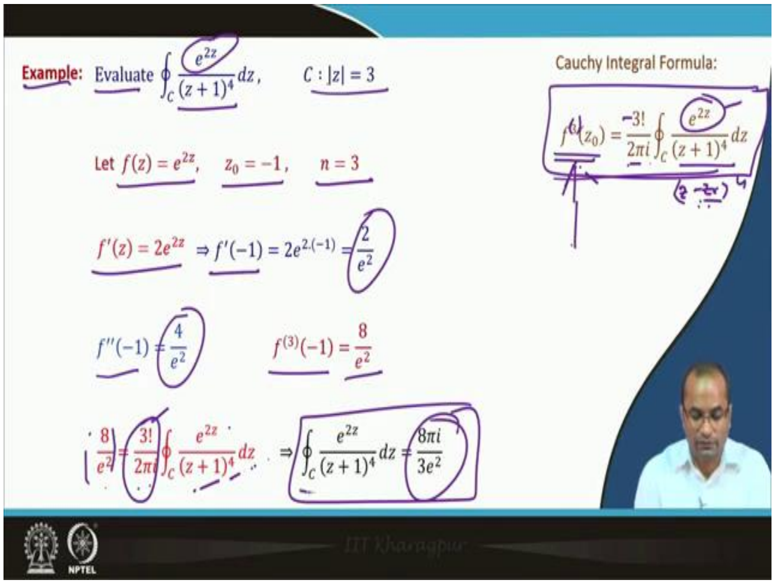

So we will now go through this example, we will evaluate here e power 2z over z plus 1 power 4 dz and here the curve is absolute this modulus z equal to 3, so it is a circle center 0 and radius 3. Cauchy integral formula says that the third derivative we can evaluate by this factorial 3 2 pi i e power 2z and z plus 1, z plus 1 power 4.

So this is exactly z plus 1 power 4 here and this is our fz. So with this formula we see connection to this integral and now we can proceed with this taking this fz as 2 power z and this third derivative and we evaluate at this z naught point, so we will get exactly the result of this. So here this z minus z0 a power 4, so z0 is like minus 1 and we can evaluate the derivatives. So fz is e power 2z, z0 is minus 1, n is 3, first derivative is 2 times e power 2z.

Its derivative at minus 1 we can evaluate that is 2 over e square then the second derivative we can evaluate that is 4 over e square and then the third derivative we can evaluate at 8 over e square, so the third derivative we need for the value of this integral here and this third derivative then we can put there in the integral, so this is the third derivative and equal to factorial 3 2 pi i power z, z minus z naught power 4, so this is the value of the integral then.

This factorial 3 over 2 pi i we can bring to the left hand side, so we are getting the value of the integral as 8 pi i over 3 e square, so this was a direct application of the Cauchy integral formula where we can get the value of such a integral.

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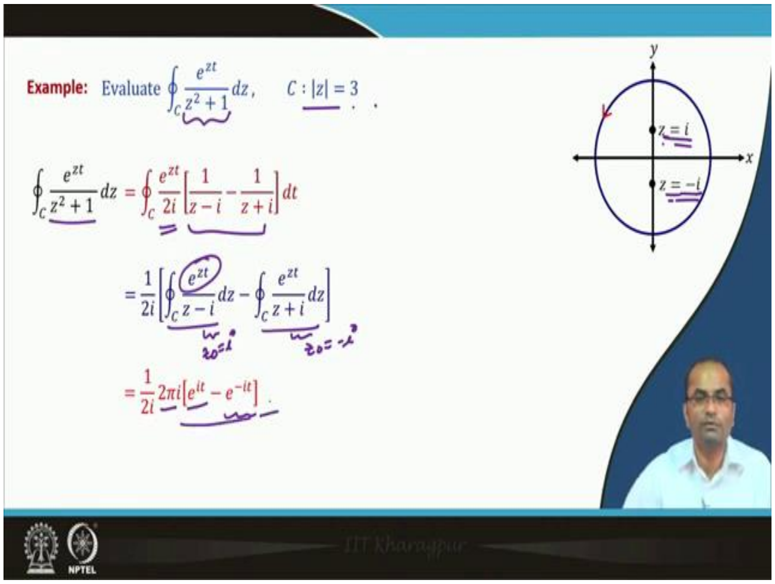

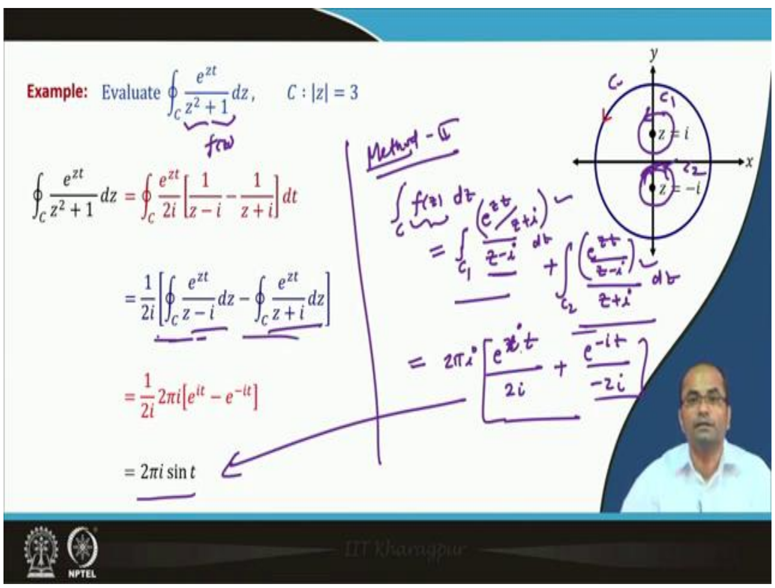

Another example, where we will evaluate e power z t, z square plus 1 dz and here the circle is given by absolute value z modulus z equal to 3, so again we have a circle center 0 and radius 3. So if we see the singularity here z square plus 1 we have z plus 1 and z equal to i and z is equal to minus i, so at these two points the function is problematic, so we will proceed now writing this as a partial fraction as we have done in one of the earlier example.

So e power z t then we have 2i and then this is the partial fraction 2i will be there when we write down in terms of z square plus 1, so that will cancel out. So this is the partial fraction we have for 1 over z square plus 1. So now we have these 2 integrals over z minus i and z plus i, in both we can use the Cauchy integral formula. Here the z naught is i and here z naught is minus i, so z naught is minus i, so in both we can use the Cauchy integral formula and that says that 2 pi i and the value of the function e power z t and that means z is i here and the here z is minus i.

So we got this formula which can be also written in terms of the sign function, so 2 pi i and then e power i t minus e power minus i t over 2i we can write down as sin t. So this was the one approach for instance we have proceed where we have applied this formula the integral formula to get this 2 pi i and sin t. The another approach could be that we can think of as we can take two more circles that enclosed these 2 points, z is equal to iz minus i, so let us call it as C1 and the another curve here we can call it as C2.

So these two curves C1 and C2 and the outer one was the given one the C1, so we have drawn 2 more circles C1 and C2 enclosing these 2 problematic points and then we can write down using this extension for the multiply connected domain that this C the given integral the fz I am talking about the fz is the whole integrant, so this is whole integrant here fz at present. So I can write down that this integral over C will be integral over the C1 and plus C2 and doing this setting now.

So over C1 what I will do, when I do over C1 I will take a z minus i and the rest I will write down e power z t divided by z plus i, because now this numerator is analytic and here z minus i is exactly fitting into the formula of the Cauchy integral. So plus when I do the integral over the C2 what I will do now e power z t and I will take here z minus i as my function which is analytic and here I will do this z plus i and then I have here dz.

So this is another approach, so this method 2 which where we do not have to use for example the partial fractions, we can just write down the given integrant in this form, e power z t over z plus i and z minus i and in this for the second curve we can use z plus i. So this is the for multiply connected domain we seen the integral formula and that idea we have applied here, so then in the first case what we have? This is has to be evaluated, so 2 pi i and the numerator will be evaluated e power z, so z is i now, so i t and then divide by z is i, so i i 2in and then again here 2 pi i will come.

So e power minus i t and minus 2i, so this again can be just written as 2 pi i and sin t, so this is another approach where we can avoid doing this partial fractions, so directly we can compute this partial fraction without computing partial fractions we can use the Cauchy integral formula.

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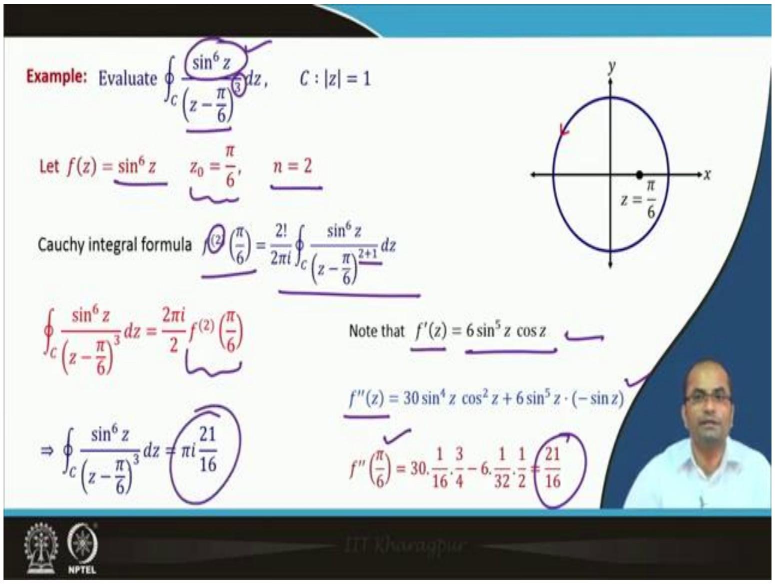

So another example where we have sin 6z and z minus pi power 3, so here we can use, so here sin z is sin 6z and we will apply exactly the Cauchy integral formula which is the extension to the derivatives. So z0 here pi by 6 and n we will take 2, so that here we are getting exactly 3 and plus 3, so this was the Cauchy integral formula, here 2 plus 1 if it is 2 here the derivative is second derivative.

So this is fitting exactly to the given integral we can compute with this computation of the second order derivative. So f prime z is already with sin 6z we have computed here and we can do this computations once again for the second derivative and we can evaluate this at pi by 6, so the value is coming 21 by 16. So here if we write 21 by 16 this is the value of the given integral using this derivative formula of the Cauchy integral.

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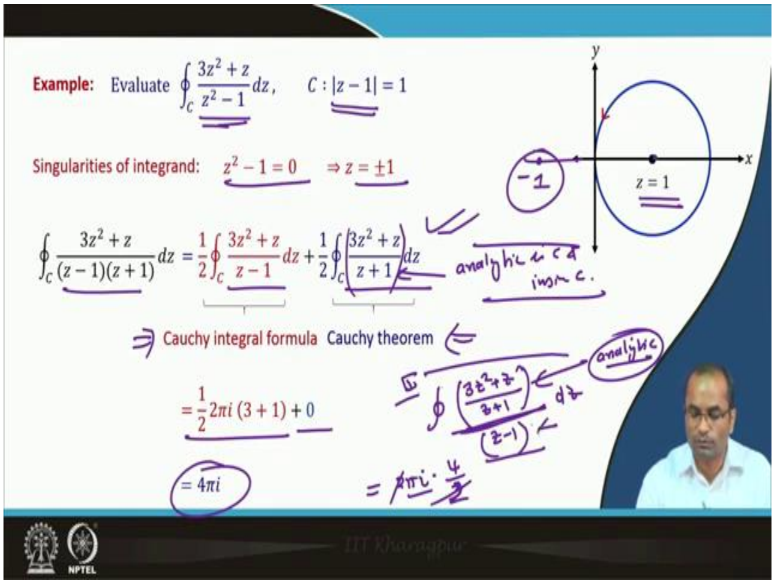

So the last example where we have again the similar situation the z square minus 1 is coming and this is with center at 1 0 and then the radius is 1. So singularities again here we have z plus minus 1, so the minus 1 is outside this circle at center 1 and then the radius 1. So we have only 1 singularity which is coming inside this, z is equal to minus 1 is outside so that is not in the picture. So what we can do, the first approach could be that we can break again into the partial fractions and then the each integral can be evaluated.

Indeed the second integral here where we have z plus 1, so the integrant here is analytic because z is equal to minus 1 is outside the domain, so here this is analytic, analytic in C and inside C, on C and inside C. So this is analytic and the Cauchy integral theorem can be used here to and the first one we have use the Cauchy integral formula and second one we can use the Cauchy theorem, so which says that this plus this 0 and the total answer we have 2 pi i.

Or another approach it could be that the given integral, so the second approach could be that the given integral we can write as 3 z square plus z over this z plus 1 all together and then we can replace z minus 1 and then dz. If we consider this integral now, so this will be our function hence divided by z minus 1 because this is now analytic in our domain, so no problem and then we can apply exactly the formula 2 pi i and this has to be evaluated over z is equal to 1.

So here we have the 4 over and then 1 that means 2 2, so this 2 2 gets cancel and we have 4 pi i, so this could be the another approach where we do not have to do this partial fraction unnecessarily, we can just simply rewrite our integrant so that the numerator everything is analytic and this z is equal to 1 was the problematic point, so we have divided by the z minus 1 and we can evaluate this.

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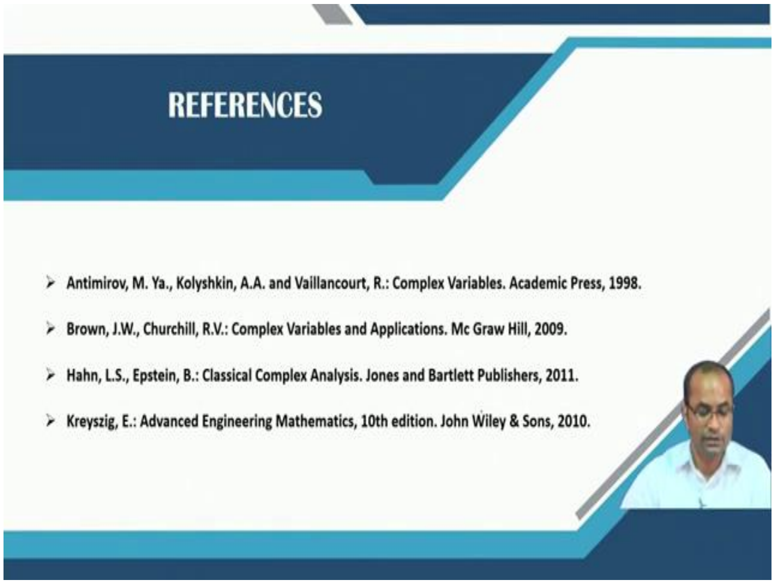

So here we have the references, which are used for preparing the lecture.

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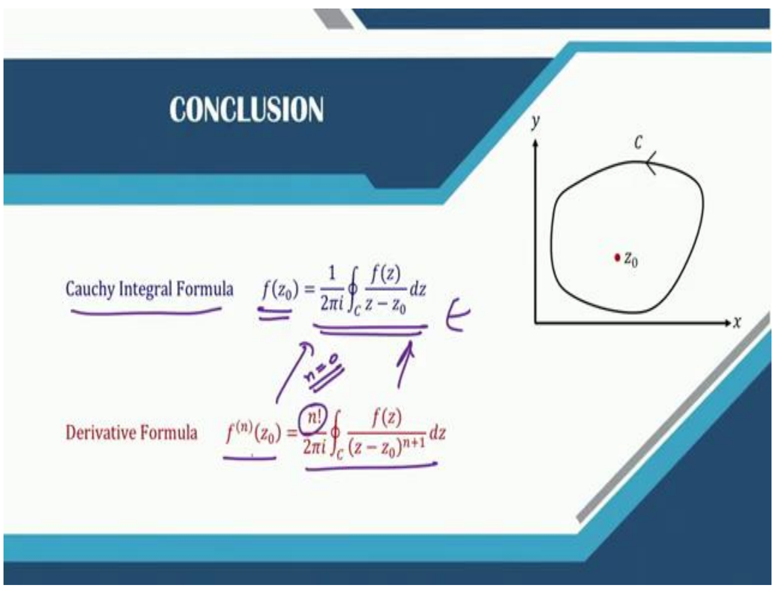

Just to conclude now, so we have learnt about the Cauchy integral formula which is a very useful for evaluating the integrals and we have also seen the derivative formula which is exactly the kind of extension of this or we can say this is particular case let us say when we put n equal to 0 because putting n equal to 0, n factorial 0 if we take 1 we are exactly getting this formula.

So this is a particular case of this more general derivative formula and here we call it Cauchy integral formula, not the Cauchy integral theorem, the Cauchy integral theorem was when the integral was 0 and the integrant was analytic. So with this.