Engineering Mathematics - 2
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Lecture 15
Cauchy Integral Theorem

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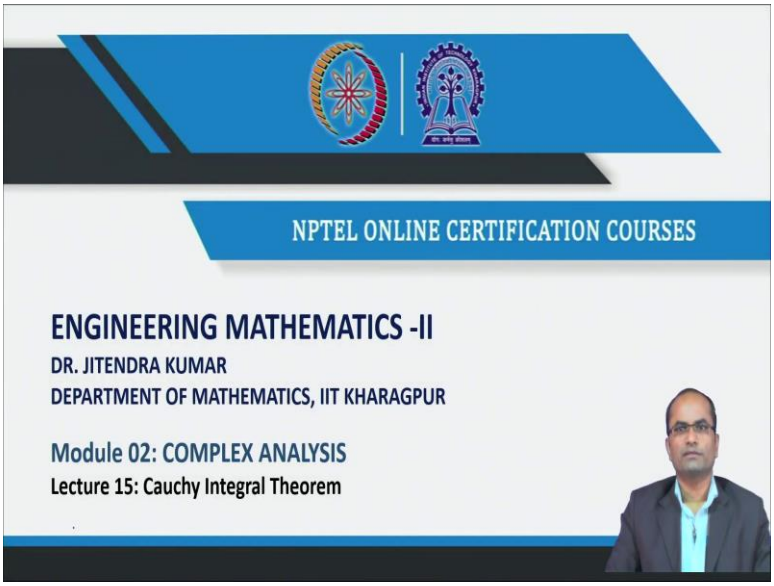

So, welcome back to lectures on engineering mathematics 2. So, this is lecture number 15 on Cauchy integral theorem.

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So, today we will cover this Cauchy integral theorem for simply connected domain and also its extension or generalization for a multiply connected domains.

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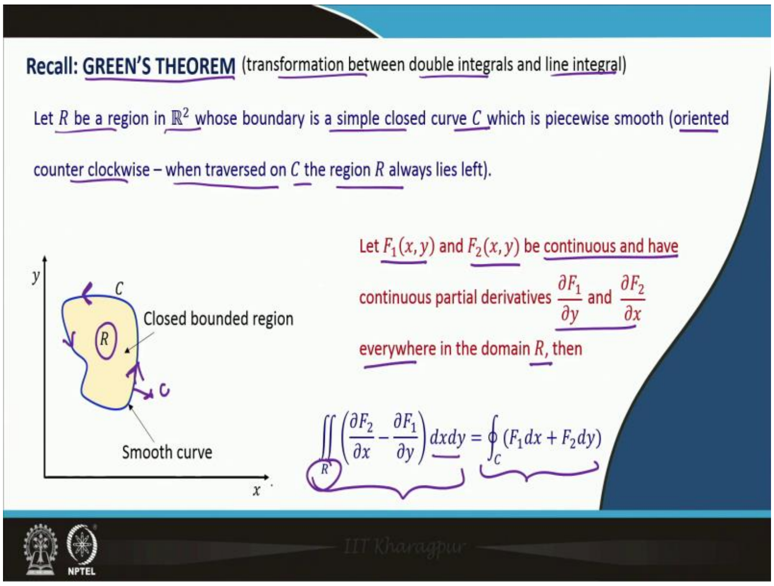

So, since you will be using this Green's theorem, so, just to recall that we have already discussed this Green's theorem, which was the transformation between double integral and lining integral. So, let this R be a region in R 2 plane whose boundary is a simple closed curve C, which is peacewise smooth. So, all these terms were discussed already in vector calculus.

And this curve is oriented counterclockwise, so meaning when we traversed on the C, the region are always lies to the left. So, this is the situation. So, if we are to traversing here on this curve and the region this are remains to the left. So, this is the closed bounded region and we are talking about this this smooth curve C.

Then this theorem says that, if this F 1 and F 2 the components of the the F be continuous and these partial derivatives exist everywhere and they have, they are also continuous in the domain R. Then this integral Del F 1 over Del x minus del F 1 over Del y dx dy. So, this is the area integral will be equal to this line integral F 1 dx plus F 2 dy. So, this line integral is done over this curve C, which encloses the domain R and here this is over the region R. So, this is Green's theorem will be used in today's lecture. So, this was just to recall from vector calculus.

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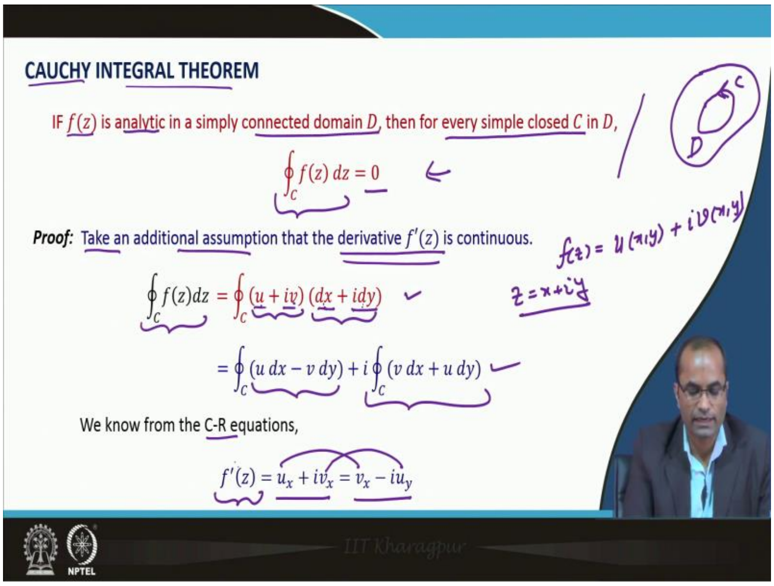

So, for the Cauchy integral theorem what it says? That if this f z is analytic in a simply connected domain, then so we have some simply connected domain here. Then every simple closed curve C in D, so we take any simple closed curve C in this region D. And the theorem says that the integral of f z of the function f z over this closed curve C will be 0. So, we have basically also seen such integral in the previous lecture when we discuss that if f z is analytic in a domain D, then it has its anti-derivative or primitive.

So, that the value of this integral does not depend on the path but depends on the initial and the final point. And if we are talking about the closed curve that means, the initial point and the final point, both are same and in that case naturally this integral will become 0. So, what we will look here, we will go through the detailed proof for how this result we are arriving, so that we can get more insight to this integral to this result. So, the proof, we will take an additional assumption which is not stated in this statement here that the derivative f prime z is continuous.

Later on we will see that we do not need actually this assumption. So, without this assumption also the same result will hold. So, for the time being we will assume that the derivative f prime z is continuous. So, then this integral, the close integral of f z over C, we can write down so, the F is u plus iv, this is what always we take this assumption. So, the u, f z is a complex function which can be written as u, xy plus iv, xy. So, this is u plus iv in short and then dx, so the z is x plus iy as complex number, so the d z is dx plus iy. So, in this expanded form we have written.

And then we can just multiply the 2 here. So, v with dx and this iv with idy will become minus vdy, so this is the real part. So, v udx and then minus vdy and the rest with this i, so vdx and then udy, so this is the result of the multiplication. And we know from the CR equation, so the CR equations will be uC. So f z is analytic, they it satisfies the CR required equations Cauchy Riemann equations.

So, from the CR equation we know in the proof of the CR equations, we discussed this point that the derivative of this f prime is nothing but ux plus ivx or vx minus iuy. So, these are the Cauchy Riemann equations, one can just think here ux is vx and vx is minus uy. So, this result for the f prime that means, the if we have the ux and vx we can get actually the derivative of f z that was the idea which was discussed earlier.

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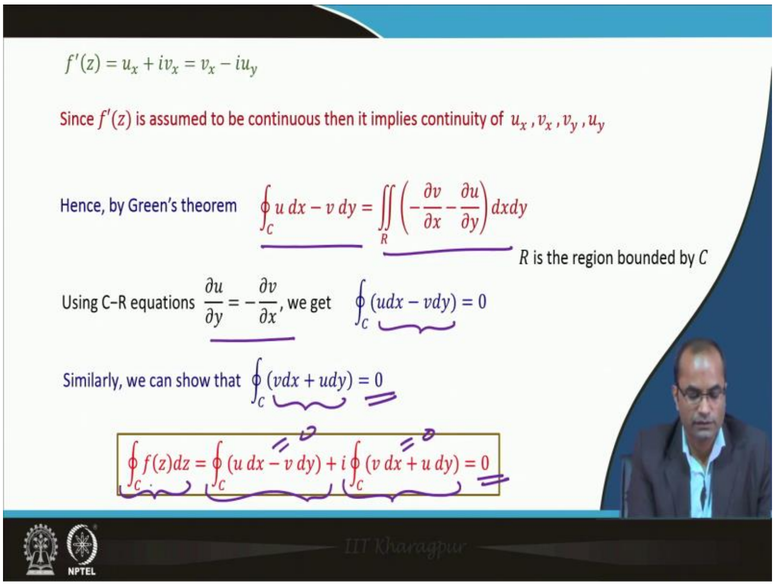

So, we have this result that f prime z is equal to ux plus ivx and which is equal to vx minus iuy and since f z said is assumed to be continuous that means, the right hand side has to be continuous or its component has to be continuous, that means, ux, vx, u, vy and uy all these partial derivatives must be continuous because the left hand side, this f prime z is continuous. So, by the Green's theorem, now we will apply Green's theorem here, which was just discussed in the first slide.

So, the Green's theorem says that udx and vdy. So, here we have this minus v for F 2 and then F 1 we have u in that setting. So, this will be this F 2 over the derivative of F 2 with respect to x. So, the F 2 here was this minus and the F 1 was u. So, F 2 with respect to x minus F one with respect to y. So, Del u over Del xdx. So, this curve integral this curve integral can be written as this area integral using the Green's theorem and R is the region bounded by this curve C. So, the same setting what we have for the Green's theorem.

Now, the CR equation says that uy is equal to minus vx. So, here we have uy. So, if we replace this by minus this Del v over Del x and there was a minus sign already, so this becomes plus and there was a minus here. So, this gets cancel, and what we see? We see that the udx minus vdy is equal to 0. So, one part of the integral which we have for the curve integral of f z has become 0. Similarly, the other part, there were 2 parts there, so, vdx plus udy, also we can use the similar idea what we have here, the Green's theorem and then CR equations and we can show that this is also 0.

And therefore, this curve integral of f z which was written in this form udx minus vdy and this imaginary component vdx plus udy. So, both of them are 0. So, therefore, this is 0 here. So, this integral f z, dz over the C equal to 0 if f z is analytic in a domain or on C and the domain and closed by C, then we have this result that the line integral of f z is actually 0.

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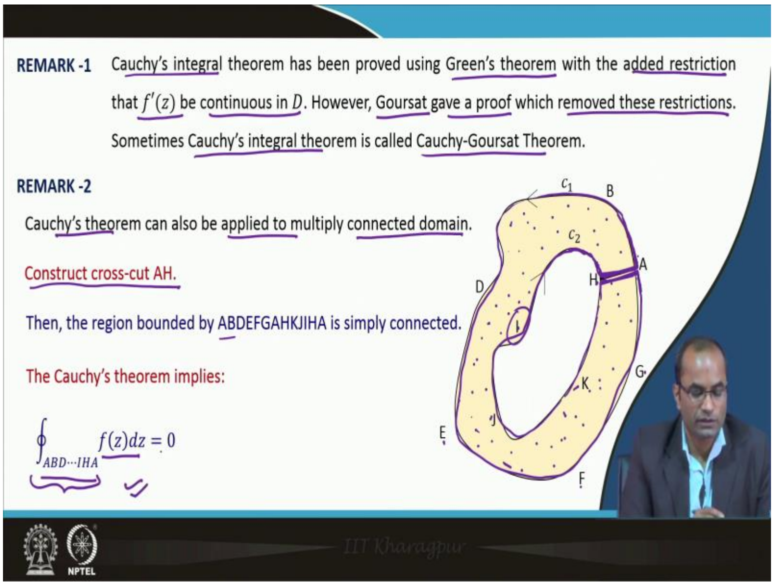

So, there are some remarks now, which will get us a little more into this Cauchy theorem. So, the Cauchy integral theorem has been proved using the Green's theorem, we have used the Green's theorem and we need the continuity of those ux, vx etcetera. So, here we have this added restriction that f prime z be continuous in D. However, the scientists here the Gaurset gave a proof which removed these restrictions and sometimes this Cauchy integral theorem is also called the Cauchy Gaurset theorem.

So, here we do not need actually that restriction, which we have us there that f prime z is continuous in D, we do not need that assumption without that assumption also, we can just prove that result that means, f set is analytic that is enough. Another remark that the Cauchy theorem can also be applied to multiply connected domain. So, far we have taken D as the simply connected domain but now we are talking about that we can also apply Cauchy theorem for multiply connected domains and there is a simple trick which again get back to us for simply connected domains.

So, the idea is that we put a, we construct a cross cut AH, here we have used the cut here AH in this domain. So, this was our domain now, so there is a there is a hole here, which is this part is not the part of the domain. Hence, the F is not analytic in the whole domain because this part this is not a simply connected domain it is a multiply connected domain. So, this white part is not a part of the domain. So, this the outside curve C 1 for instance does not contain the whole region which were the function Fs is defined or Fs analytic.

So, here this is the domain wherever, for instance, the function is given that it is analytic in this domain and then we want to apply the Cauchy theorem. So, directly we cannot apply because of this multiply connected domain. So, what we have done? We have used a cut here and the idea is that this region bounded by now, this one, so here we start now. So, A then we go to this B and along this here, here, here and going all the way to F, G and then again back to A then we go to this H and then we go take a round here K, J, I and C 2, curve which is described by this inner curve, and then we again getting back to A.

So, now this here starting from A and then getting back to again with this A. So, think about this region now. This region has become simply connected domain. Which region? ABDEFG AHKJI and H and A again. So, this has become a simply connected domain and our functions for instance analytic in this domain then we can apply the Cauchy integral theorem.

That means, the Cauchy integral theorem says that this integral over this closed curve so what is our closed curve now? ABDEFGAHK and then J then I here and then again HA. So, this is the closed curve and the f z by the Cauchy theorem, it should be 0. So, this is the point here which is easy to get into this simply connected domain and then we can apply the Cauchy theorem. So, this is the result of the Cauchy theorem, we will now expand it a bit.

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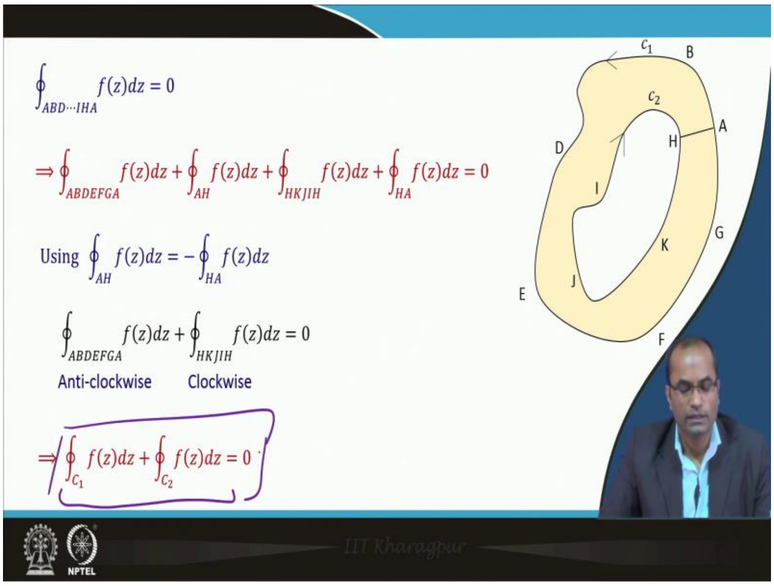

So, this AB up to back to A this closed curve which we have constructed here, we can break into the first curve AB to again A, so ABDEFGA. So, the first the curve C 1 basically, then the AH the inner one here because we get back to H, AH and then the the inner one HKJ and then I and then again H, so this closed curve and then HA again get back to A. So, this whole closed curve is, has been broken into these four parts.

The one and this is number 2, then 3 and then 4. So, the first, this curve which is 1, then we have here, the 2 and then the third one the inner, the C 2 curve and then again the fourth one this HA. So, and this whole should be equal to 0 and we know the property of the line integral, this is AH integral and this is HA integral. So, only the direction is reversed. So, there will be minus sign and as a result, these 2 will get cancel. So, these 2 will get cancelled and then we have the results. So, the first one we have EB to A.

So, A, I am going to B in this direction. So, anti clockwise direction AB to again get back to A that is the curve C 1 there. Then we have the inner one HK, so here we are going in this direction, so HKI and JI and H again. So, this is in the clockwise direction, so and this two, the sum is equal to 0 which the notation itself we have taken in this format that the C 1 the first curve is the C 1 the outer one, the second is the C 2 which is traverse in the clockwise direction, the C 1 is traverse in the anti-clockwise direction, the sum of the 2 here that is what the Cauchy theorem now.

The generalization of the Cauchy theorem for multiply connected domain says that the sum of all these but the outer ones should be in an anti-clockwise direction and the inner one is the clockwise direction this is the the idea and the sum is equal to 0 for all these integral. This is the Cauchy integral theorem, the extension of Cauchy integral theorem for multiply connected domains.

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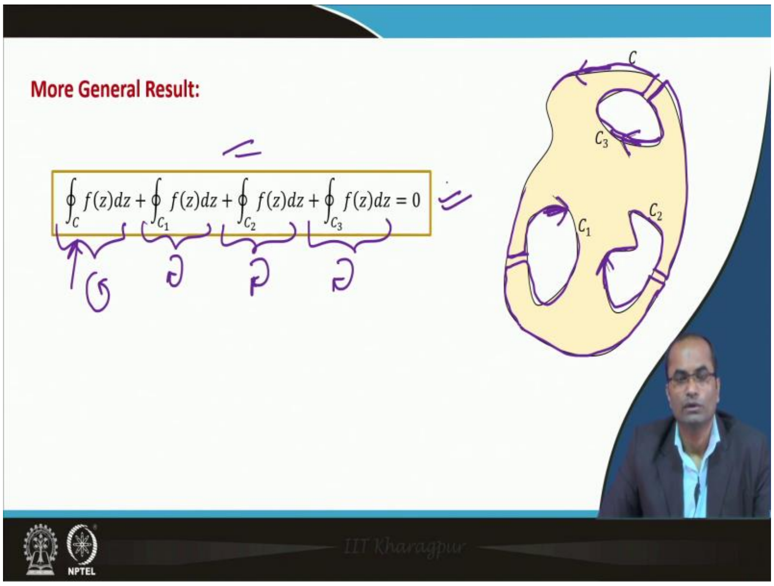

We can generalize this result for many such holes or multiply connected domain. So, here we have considered these three for instance inside. The outer curve C is traversing here the clockwise direction, all other are in anti clockwise direction. So, here you have C 1, then we have the C 2 and then we have here the C 3. So, all the C 1, C 2, C 3 are traverse in clockwise direction, whereas, the outer one is traverse in anti clockwise direction.

So, the similar extension what we have done in each case we have to take a cut here and so that we can open this and make a simply connected domain. So, we will come from this direction, then go from this along this and come back again here. So, going back to the second, C 2, so in and then we will traverse across this, C 2 and come out of this, then it will go to the third one. Go in and then traversed clockwise direction and then go out and then again this one.

So, if we take such a such a cut here, so, the whole domain here will become a simply connected domain and then we can apply again the Cauchy integral theorem and as a result he will get that the integral over C plus integral lower C 1 plus C 2 plus C 3 equal to 0 and the C was traversed the outer one the most outer one is traversed anti clockwise direction and then these are in the clockwise direction. So, that is the difference which we have to keep in mind, then only the sum will be equal to 0.

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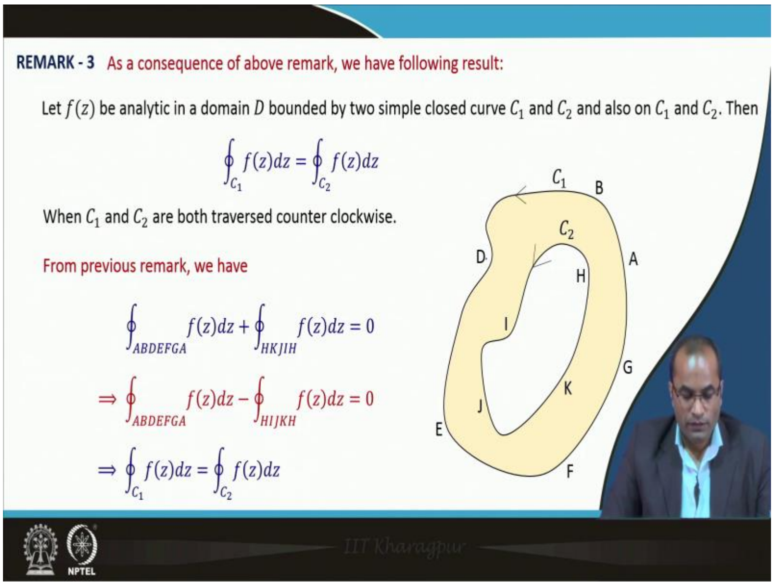

So, well going next to the remark three. So, another important remark, so as a consequence of this above remark, so the Cauchy integral theorem for multiply connected domain, what we have? We have the following result let f z be analytic in a domain D bounded by 2 simple closed curves C 1 and C 2 and also on C 1 and C 2. So, we have f z a function f z which is analytic in this domain here bounded by these 2 curves and also on this curve. So, then what the result we have? That the integral over C 1, so now, we should note that the both the curves are traverse in anti clockwise direction.

So, same direction unlike the previous remark. But now we have that the integral over C 1 is equal to integral over C 2 the integral value whether we take over the C 1 or we take over the C 2, it is equal. And C 1 and C 2 are both diverse counterclockwise direction. So, using the above remark we can easily observe this result because from the previous remark what we have? We have this result that the sum of the 2. So, the first one is anti-clockwise direction. So, we are moving AB, DF and GA back to A again and then from the, for the inner one we have HK, JI and H again.

So, this is clockwise direction and the sum has to be 0 this is from the just from the previous remark we have and now we have changed the direction of this inner one to make a HIJKH. So, exactly in the anti-clockwise direction which in our notation now, in our setting this has become the integral over C 2. So, this is integral over C 1 and this is integral over C 2. So, we have the 2 integral because of this minus sign and equal to 0 that this integral over C 1 is equal to the integral over C 2. So, the value of both the integrals are 0 whether we take on C 1 or we take on C 2 it actually does not matter.

So, this is a very useful result that once we know that our function is analytic in some domain then we can choose any path and do the integration and we know that this integration will be equal to the integration carried over C 1 or integration carried over C 2. So we can choose any path there as per our convenience and perform the integral because we know, because of this because of this result that the value of the integral will be the same as long as we are in this domain where your function is analytic.

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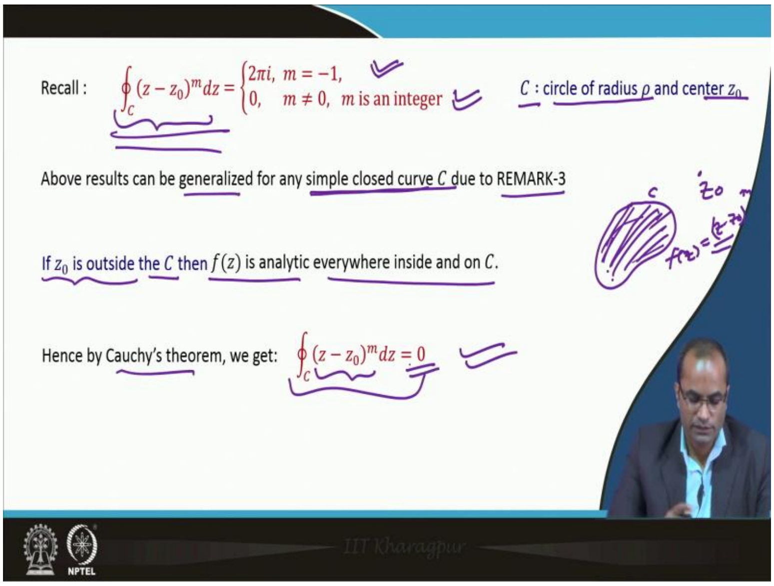

So, now moving to the next, there was an important result in the previous lecture where we have seen that the curve integral z minus z 0 power m dz was the value was 2 pi i in the line integral we have learned this when m was minus 1 the value was 2 pi i. And otherwise whatever integer we take m the value was 0 and C was a circle of radius rho and center z 0. So, in the previous lecture, we have derived this result for this circle of radius rho and center z 0.

Now, what we will do? We will generalize this result for more general curves not just for the circle of radius, rho and center z 0. So, the result can be generalized for any simply simple closed curve not necessarily the circle of radius rho and center, z naught but we can generalize this for any simple close curve C and this is all because of the previous remark.

So, if z 0 is outside the curve C, so we are talking about this curve C and letting this z outside for instance. Then there is no issue because f z is analytic everywhere inside and on C, so we have some this closed curve C here and suppose z 0 is setting there. And then we are talking about this f z which is z minus z 0 power m. So, this f z is analytic anyway in this C and inside C on C inside C, so, we can apply the Cauchy integral theorem for example, and we will get that value 0.

So, by Cauchy theorem we get that this value will be 0. So, we are generalizing this result for any value of z 0 and for any kind of the simple closed curve. So, if z 0 is outside there is no question, there is no discussion, this integrant is analytic on C and inside C and then this value will be 0 directly from the Cauchy integral theorem.

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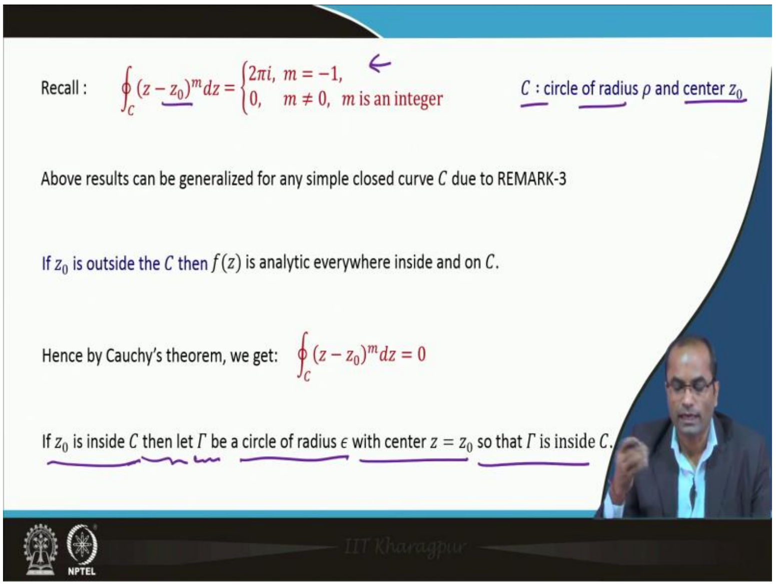

Now, if z 0 is inside C that is the problem which we have observed last time and when we talk about that the C was the circle of radius, rho and the center z 0, so z 0 is naturally inside C. And as a result for m is equal to minus 1, we got the value 2 pi i. So, if z 0 is inside C then what we will do? We let another this gamma, a circle of radius epsilon with center z 0. So, that this the circle lie inside the curve, the simple closed curve C.

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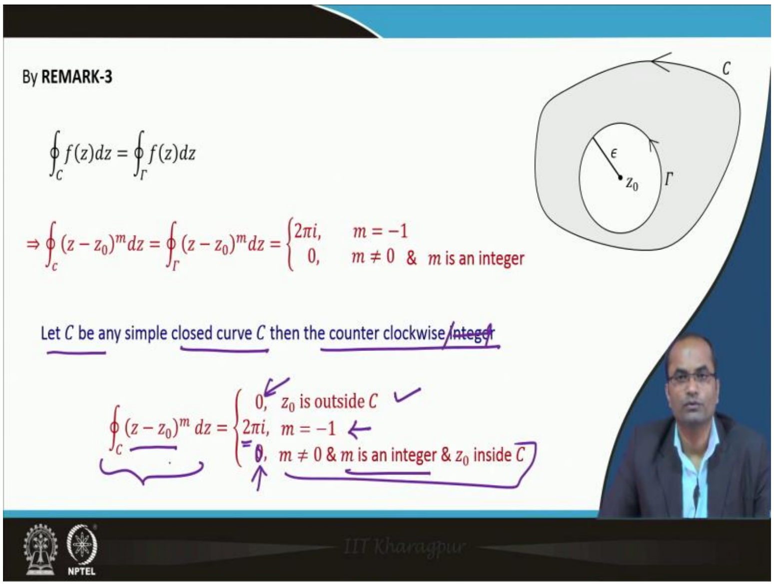

So, this is the situation, this is the simple closed curve here and this was z 0 which is inside this. So, enclosing this z 0 we have drawn a circle of radius some rho or epsilon. In this case we have taken this epsilon radius and then our remark says that the curve integral over the C is equal to the curve integral over this gamma because our f z is analytic everywhere at least in this domain it is analytic.

It is analytic except this z is equal to 0 for some m. So, that remarks says that whether we integrate over this curve C or we integrate over this gamma the value will be same. So, this remark we have used here that the value of this f z over C will be equal to this integral over this gamma. And the integral over gamma that is the interesting part, the integral over gamma we know from the previous lecture. So, that means, this integral over C over a very general curve is equal to this integral over that circle where the result we know that it is 2 pi i when m is minus 1 and 0 otherwise, that result we know already.

So, this same result we got for any curve here not just for circle, but for any curve. So, if C is any simple closed curve C. Then the counter clockwise integer counterclockwise direction what we will get? This result that C, this close curve C, z minus z 0 power this m dz, is anyway 0 and z 0 sets outside this C or 2 pi i when m is equal to minus 1 that is the only change where the value is different and 0. So m is minus 1, otherwise this whether z 0 is inside or z 0 is outside, so this value is 0.

For any other m here m is integer. So, this the value is 0, only for m is equal to minus 1, the value is 2 pi of this integral z minus z 0 power m over a simple closed curve C.

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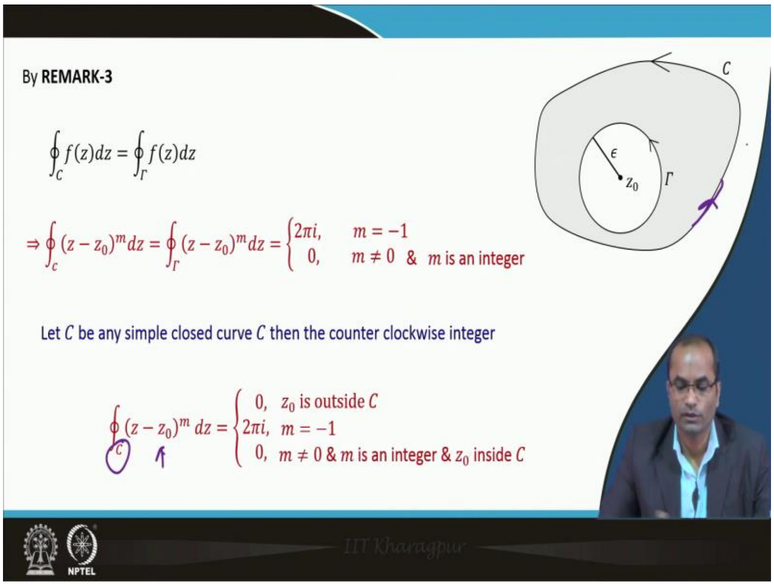

So, this result we have generalized which in the previous lecture we had proved when the C was just a circle with center z 0. But now we have observed that the C can be any curve any closed curve, simple closed curve and this result is still true.

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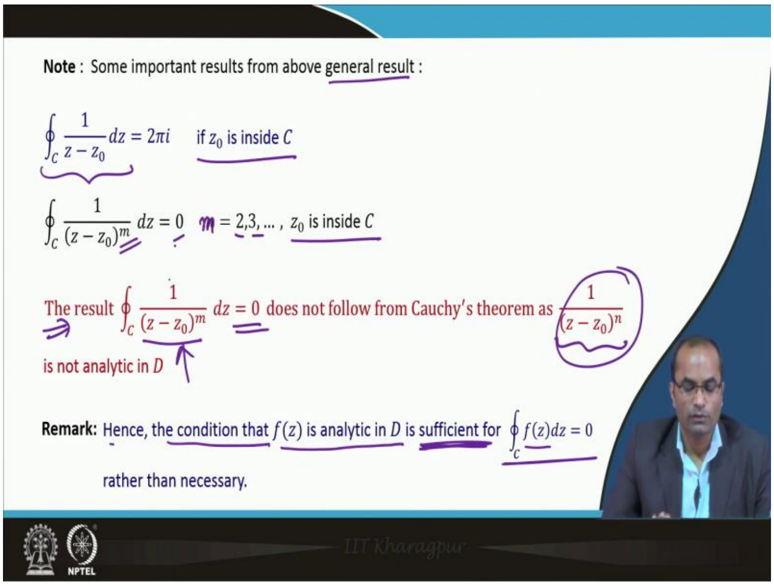

So, just to conclude with this from this above result what we have the value this one over z minus z 0 dz is 2 pi i for m is equal to minus 1 and if z 0 obviously lies inside and for m any other value of m that means, here m can be 2, 3 or z 0 is insight C. In that case, this value is going to be 0 and this result, just this should be noted that, that this result equal to 0 does not follow from the Cauchy theorem because this here is not analytic in D, so this we should not confuse that the value of this integral is 0 of with this integrant.

And it follows from the Cauchy theorem, so that is not the case because Cauchy theorem we can apply when your integrant is analytic in the whole domain or the curve C and the region which is bounded by this enclosed by the C. But this is not the case here because z minus z 0 is not analytic in D, but it is still the value of this in some integral can be 0, but this is not because of the Cauchy theorem.

The Cauchy theorem, those conditions are sufficient condition the Cauchy theorem says this f z is analytic then definitely it is going to be 0. But does not mean that in other cases when f z is not analytic integral cannot be 0, it does not say so. So, the integral can still be 0, but that is not because of the Cauchy theorem. So, which remark here we have made has the condition of condition that f z is analytic in D is sufficient.

So, these are the sufficient condition to have this result, which says that the integral over this curve C equal to 0 rather than necessary. So, these conditions are not necessary, they are sufficient conditions. So, we can have some function for example, it is here one over z minus 0 power m where the integral can be 0, but this is not because of the Cauchy theorem.

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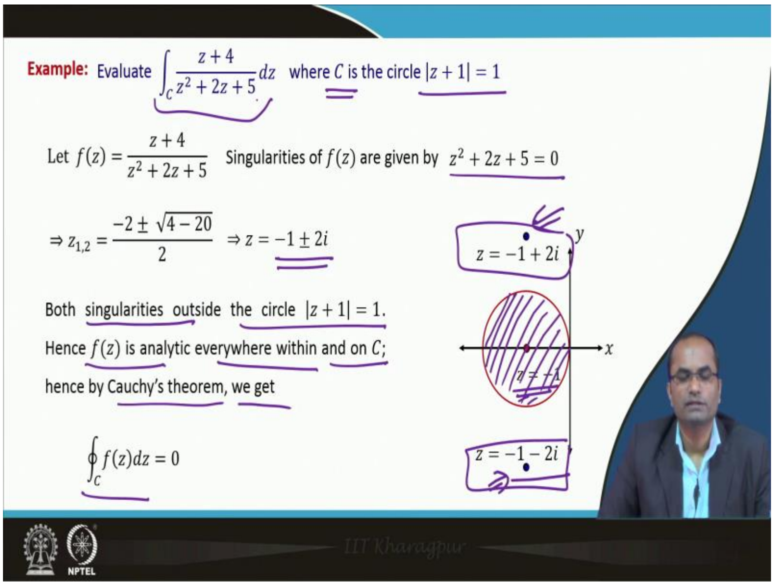

Well, so, just an example to illustrate again that how we can apply this Cauchy theorem. So, integrant has given here z plus 4 z square plus 2 z plus 5. And the C is a circle given by this z plus 1 equal to 1. So, this let f z equal to z plus 4 over z square. So, this is the integrant here and we will talk that where this denominator is 0 which we are calling here the singularities of this f z. So, we will discuss the similarities part in some lectures later on, but at this moment the singularities are those points where this f z is not analytics.

So, other than those points where this denominator becomes 0, the function is naturally analytic. So, we will look for that where this is equal to 0 the denominator is 0 that means z square plus 2 z plus 5 equal to 0. Those are the points where your function is not analytic. So, these are the points which minus 1 and plus minus 2 y. So, at these points, these 2 points the function, this f z, the integrant is not analytic. Now, we will see that the C where we are integrating it is a circle with this center minus 1 0 and then the radius 1.

So, this is the situation here the centralized at this z equal to minus 1 and these points, the 2 points which were the function has is not analytic that is minus 1 and minus 2 y and then minus 1 and 2 y. So, these are the 2 points where the function is not analytic and they lie outside completely outside the circle the C. So, as long as the circle C and this region inside or enclosed by the C is concerned the f z is analytic.

So, there is no problem to apply the Cauchy theorem for the given integrant. Because here the Cauchy theorem can be applied because these points where the function is not analytic, they are outside this domain so, they are not anywhere concerned of this department, this domain here. So, the Singularities both the Singularity these points we called singularities will be discussed more in detail later outside the circle.

Hence f z is analytic everywhere within and on C. So, the Cauchy theorem what we get? That this integral has to be 0. So, the given integral the value is 0, just by the Cauchy theorem we have easily concluded. And there will be much many more examples, which we will see later on with the application of the Cauchy theorem, we can simply simplify them. So, here the value of this integral is 0.

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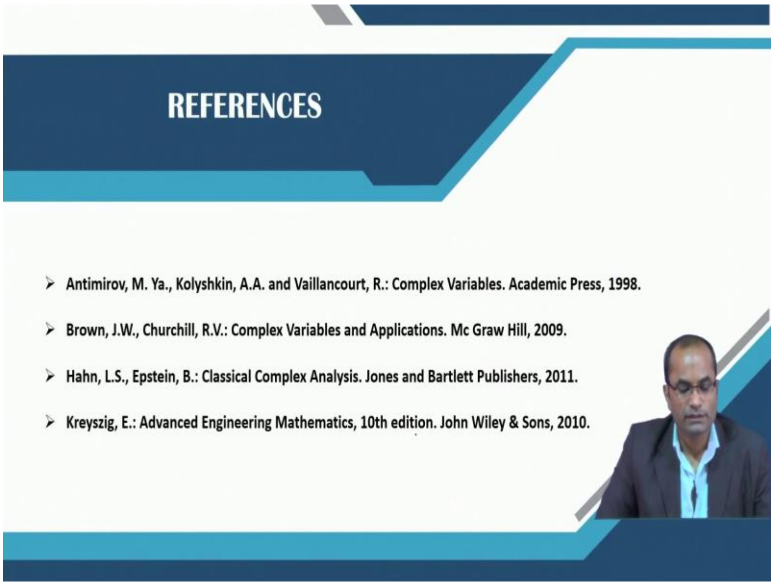

So, these are the references which we have used for preparing the lecture.

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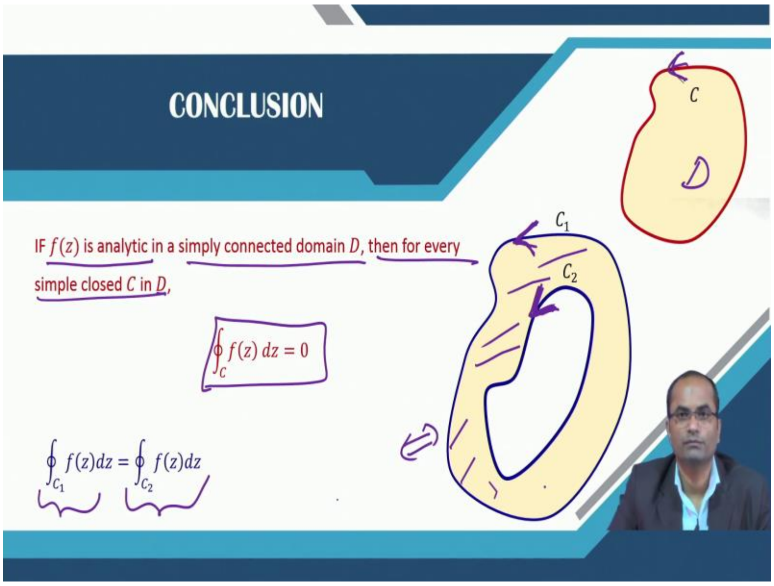

And just to conclude again that we have discussed the Cauchy theorem. So, if f z is analytic in a simply connected domain here D, then every simply closed curve C, so if we take any curve which is you have taken already here. So, any simply closed curve C the value of this integral is going to be 0 and we have also extended this for multiply connected domain for instance.

So, if we take this multiply connected domain, then that result says, so here the orientation is anticlockwise for both the directs, for both the curves. So, then this value of this f z over the

C 1 will be equal to the value of this integral over C 2 this is what we have discussed and generalize one more result based on this from the previous lecture. So, that is all for this lecture. And I thank you very much for your attention.