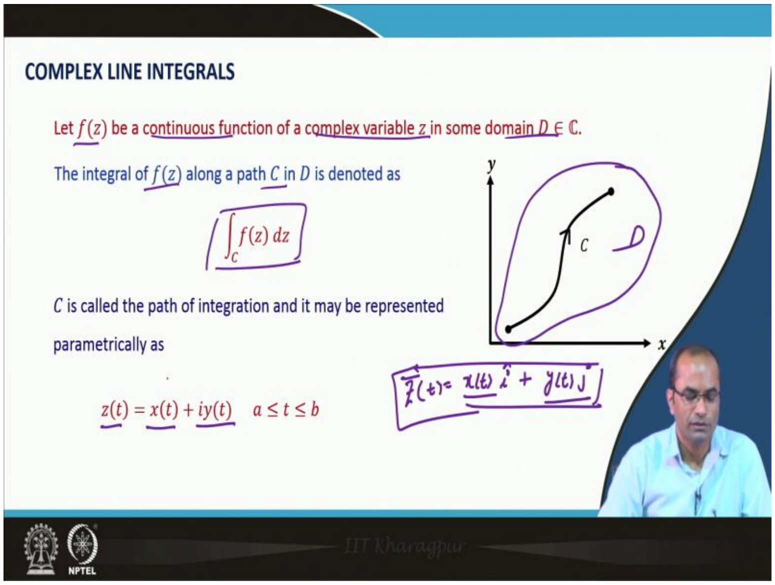
Engineering Mathematics - 2
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 14
Line Integrals

So welcome back to lectures on Engineering Mathematics 2 and today we will talk about the Line Integrals.

(Refer Slide Time: 00:22)

So, here what are the complex line integrals and what are the properties of complex line integrals and then how to evaluate such line integrals? These we will cover today in this lecture.

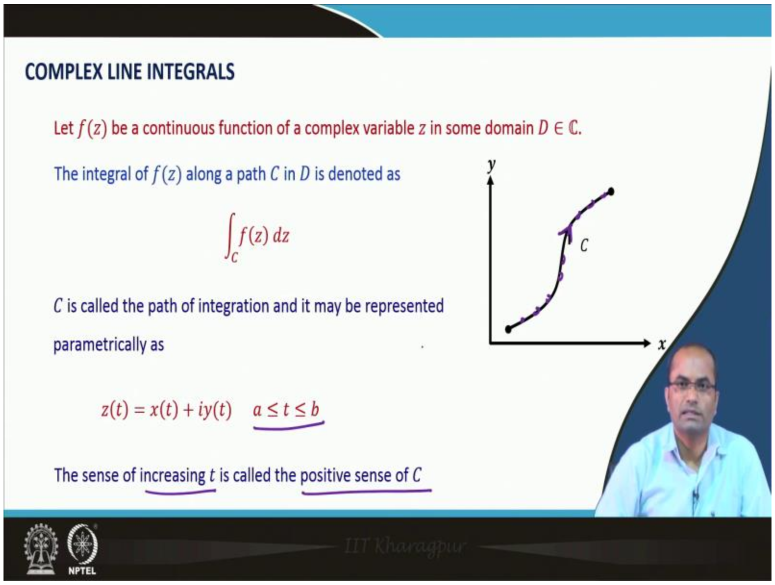
(Refer Slide Time: 00:37)



So we have already discuss this line integral in the context of vectors and very extensionally we have derived results and discussed all terminologies coming for the line integrals. So, here since there is no new idea which is coming up for the line integrals in this context of the complex, we will go through some of the concepts quickly and come to the evaluation part where we will observe again how to evaluate the such a complex line integrals. So, here just to go quickly that let f z we be a continuous function of a complex variable z in some domain D then the integral of f z along a path c, so we have a path c in some domain here, D is denoted by this notation.

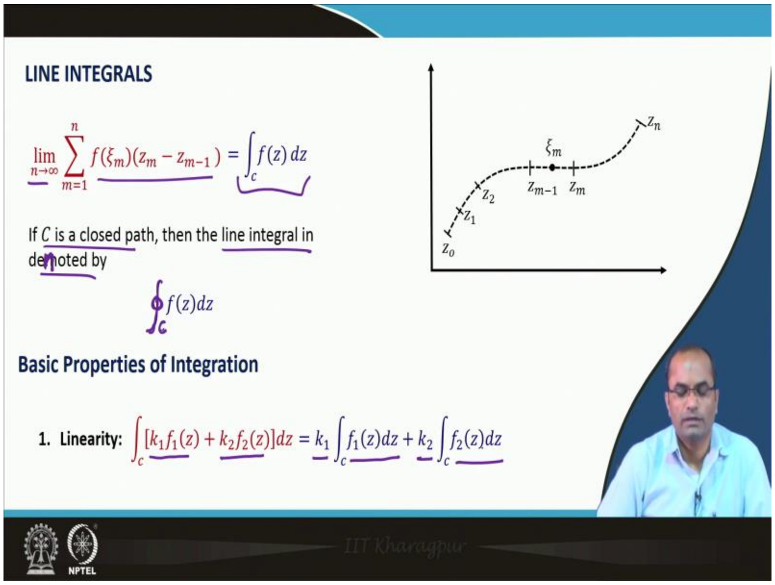
So, integral over C of f z dz, a similar notation we had earlier in the context of a vector. So the C is called the path of integration and it may be represented parametrically similar to again for the vector context as zt where xt plus i y t remember there we used to have this like Z t is equal to x t the unit vector in the direction of x axis and then y t in the z, in the direction of y. So, we were defining the vector quantity by this xt yt, the 2 components in this context we have this in terms of the imaginary or the real part, so we denote similar to the earlier one but the notation we will follow from the complex context.

(Refer Slide Time: 2:33)



So, z t is equal to x t plus i y t and t varies from a to b and again similar sense of increase in t is called the positive sense of C. So, as we move along this t here, so we must be moving on this curve and thus, increasing values of t, the direction we are moving on, this curve is called the positive sense, positive direction of C.

(Refer Slide Time: 03:08)



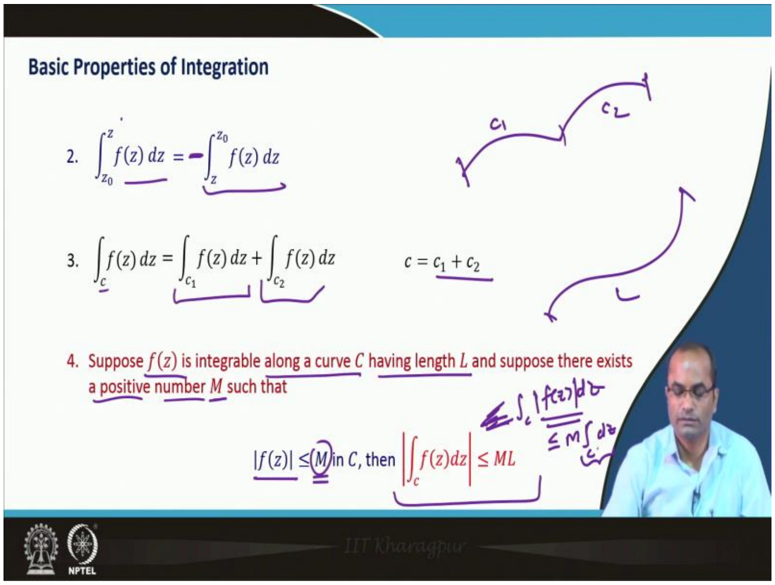
Well, so coming to the line integrals. So, the idea is the same for all integrals whether it is a real or complex, so for instance from z 0 to z n we want to integrate along this given curve C and our function is let us say f z which we are integrating. So we will break this curve into small curves or small sectors and suppose in this sector here from z m minus 1 to z m we have a point here which is denoted by zi m. So, how this integral, what is the physical meaning and how do we define? It will be clear now, so this is a summation here, over all these small sectors, m goes from 1 to n, let us say these are the n sectors and F the value of the function evaluated at this zi m and this difference here zm and zm minus 1.

And if we sum this and take the limit n approaches to infinity that means this number of sectors which we have divided that is approaching to infinity now. So, as we approaches to infinity this width will go to 0 and eventually the error which we were introducing just by subtracting zm to this zm minus 1 will got to 0 and we are eventually integrating over this given curve. So, when we take the limit of this summation, this is what we call the integral.

Integral of this F over this C with respect to Z, so this is the physical interpretation and if c is a closed path then this line integral is denoted, will be denoted by this symbol here, the line integral over this closed path C fz dz and coming to the basic properties, they are very much similar to the real integrals. So, what we have, the first property is the so called linearity. So the linearity means that if we have this integrant here k1f1 z and this k2 f2 z, we are integrating over z, so the linearity says that we can have sum of the 2 integrals.

So, the k1 because it was a constant, f1 integrated over dz, k2 as another constant there and this integral over f2, so this is the linearity property common for all integrals, so here also for line integrals this is valid.

(Refer Slide Time: 05:54)

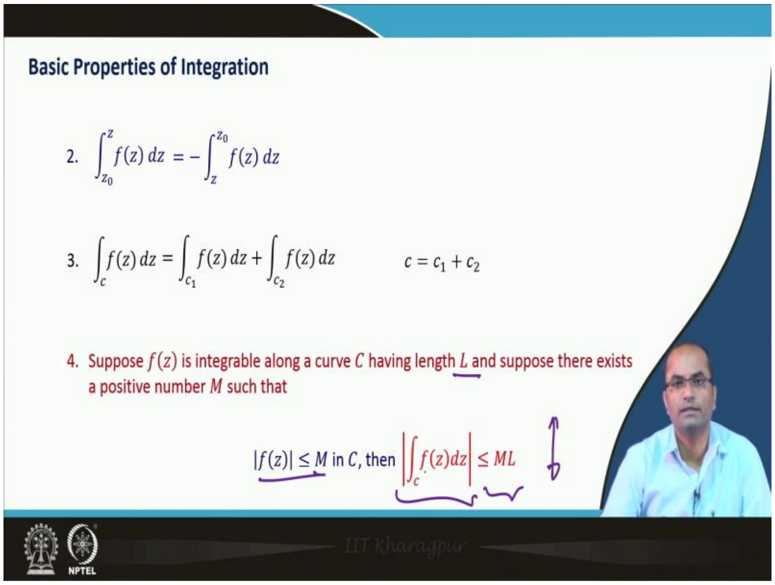


Another properties for instance here the direction change, so if we are going here from z0 to zf, z dz and now in the right hand side we are moving from z to z0, so we have changed the direction, the orientation, so the minus sign will just appear in front of the integral. Another property that we have a curve C here which we can think of as sum of 2 curves C1 and C2, so we have 2 curves for instance, this is C1 and then we have here C2 curve.

So, we can break this whole curve into 2 parts, so the first integral over C1 and then integral over C2. Another interesting inequality so called this also m l inequality is well known, so this f z is integral along a curve C whose length is L, so we have a curve here whose length is L and suppose that there is positive number m so such that, that this value of this, the absolute value of this f z is always less than equal to m upon this curve on the given curve.

Then what we have, this inequality here, the absolute value of this curve integral f z dz over this C we can bound by m into L. So this integral is nothing but we can also write like f z and d z and this f z absolute value of f z, so this will be less than equal to, this integral will be bounded by this integral and this f z is bounded by this m, so we have this bounded by m and this curve integral this d z and this is nothing but the length of the curve. So, if we just integrate over the curve C, we will get the length so that is m l.

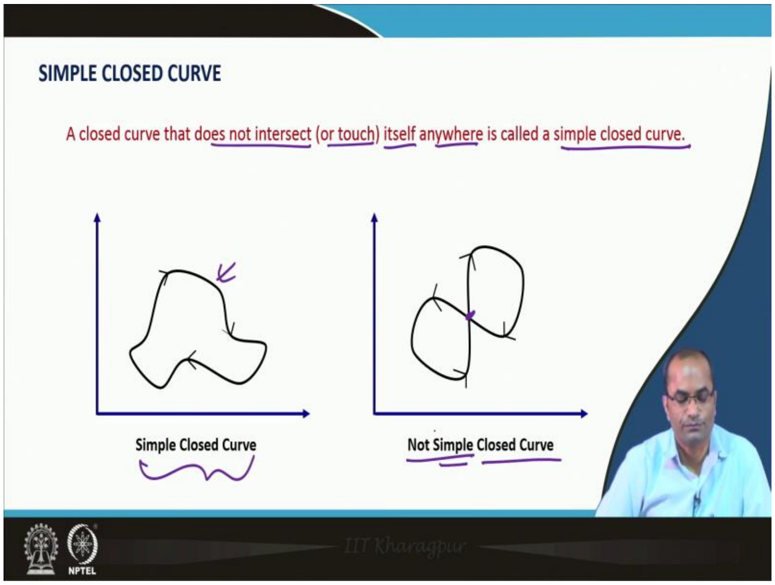
(Refer Slide Time: 7:55)



So this is what we have here, that this integral is bounded by the m l, m is the upper bond for the function, the absolute value of the function and then l is the length of the curve. So, this is also valid here for the curve integral a similar inequality we do have for real integrals as well.

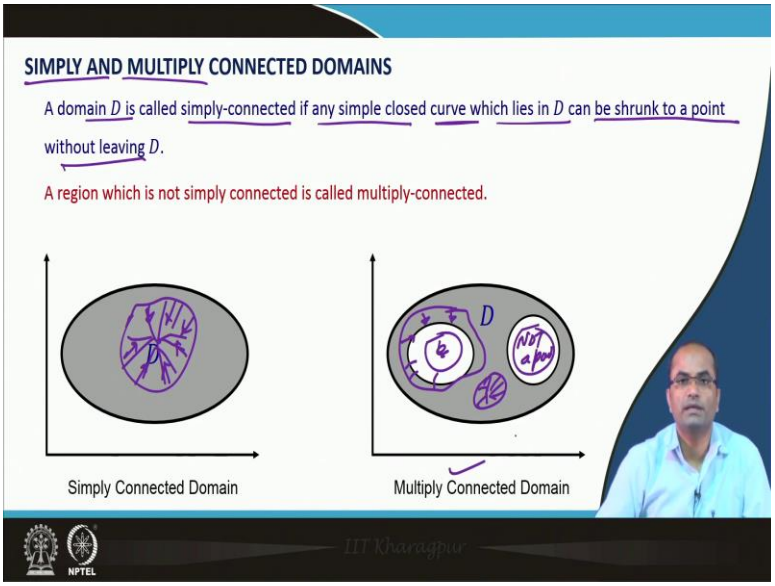
Okay, so there are various other properties of the integral we are not going through all of them. Now we have come into these some terminology, the simple close curve, so all these terminologies we have discussed already in the vector calculus. So, there also we have talked about the line integrals, so all these terms which I am just revising again we have already discussed in previous lectures.

(Refer Slide Time: 08:44)



So, a closed curve that does not intersect or not touch itself anywhere is called the simple close curve and for instance this one. This curve is not touching itself or crossing itself so this is a simple close curve, close means the starting and the last point are the same. So, it is closed and for instance this ones intersecting at this point here, so this is not a closed curve sorry, sorry it is a closed curve but it is not a simple closed curve, so it is not simple though it is a closed curve.

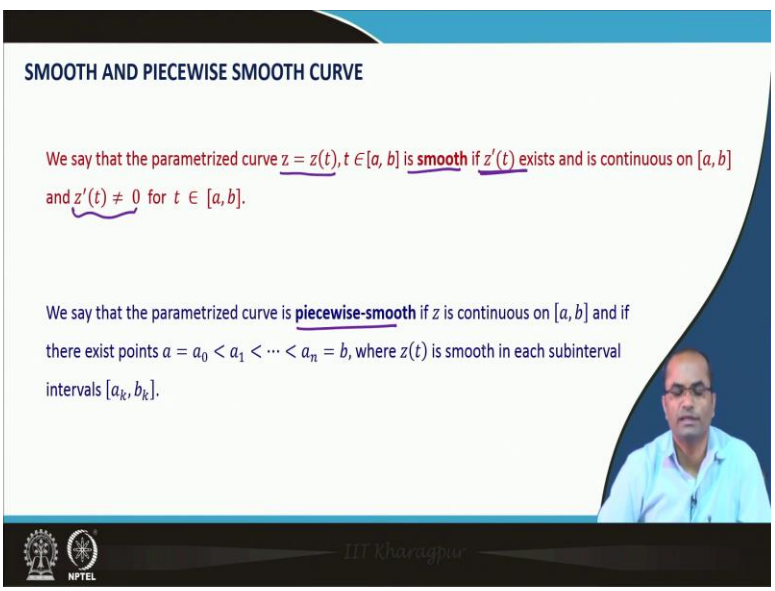
(Refer Slide Time: 09:22)



Okay, so here we will be talking about the simply and multiply connected domain which was already discussed in detail. So, domain D is called simply connected, if any simple closed curve which lies in D can be shrunk to a point without leaving the domain D. That means if we talk about this one, this is the domain D, we can take any simple closed curve here and we can shrunk this or this curve can be shrunk to a point here without leaving, I mean continuously shrunk here to this point without leaving the domain.

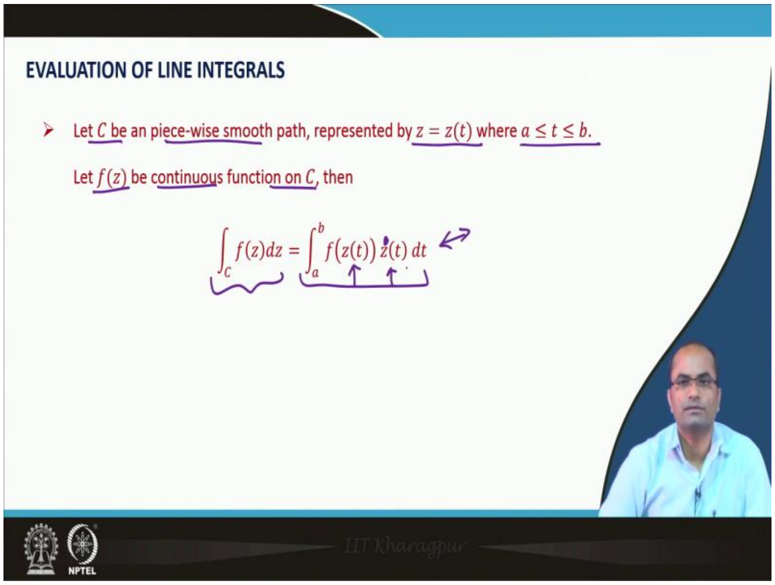
So, this is what we call the simply connected domain and for instance here all curves cannot have this property. If you take this one okay, this can be shrunk to a point without leaving the domain but if I take for instance this curve here and if I try to shrunk it to one point this is not possible because this is not a part of this or this is not a part of domains. So, this here and these holes are not part of the domain, so we cannot shrunk such a curve to a point continuously without leaving the domain. So, this is not simply connected domain or this such domains we call multiply connected domain, so these are the two types of regions domains you will be considering now.

(Refer Slide Time: 11:00)



Smooth and piecewise smooth curves we had enough discussion there, just to remind you so if you have a parameterized curve z is equal to z t is smooth when this z prime t exist and it is continuous and there was one more condition which we have well operated there that this derivative should not be equal to 0 for any t, so we call such a curve is a smooth curve and there was a term piecewise smooth, so piecewise smooth means the curve is smooth in pieces. So, if we have a several pieces let us say of this continuous curve then each piece has this property of smoothness and as a whole we call this piecewise smooth curve.

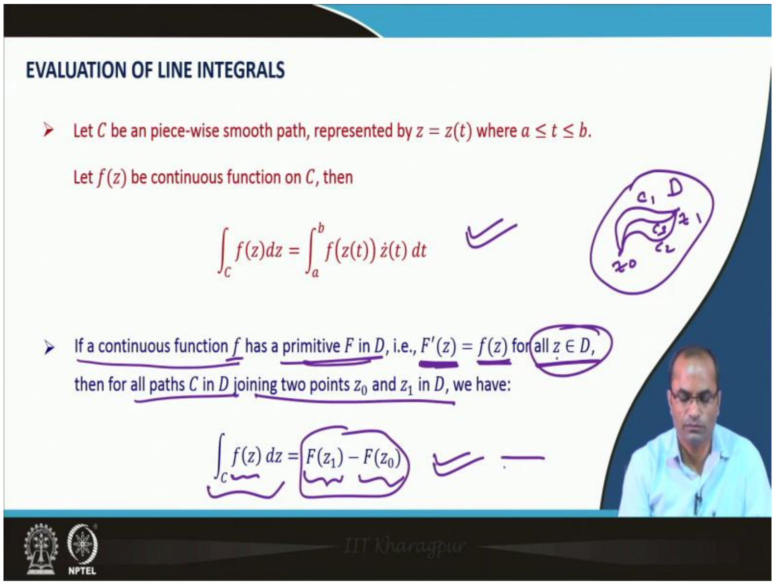
(Refer Slide Time: 11:46)



Okay, now coming to the evaluation of the line integrals. So, there are basically at least 2 ways we can evaluate this integrals or we will discuss 2 ways to evaluate the line integrals. The first one let us see piecewise smooth path, smooth curve which is represented by this z equal to z t where t lies between a and b, so if this f z is continuous function on c then we have this line integrals c f z d z can be evaluated just by this integral here. The f we substitute this parameterized function here z t for a z and then its derivative here z dot so this is the derivative.

So. then we can evaluate this line integral, so I am not discussing much into the details of this because we have already discussed such curve integrals or line integrals in vector calculus, there also we had the same formula for the evaluation and which was also look into that how to, how do we get such a formula and so on all other details one can find in the lectures on which we have discussed on vector calculus and there was a section where line integrals were discussed.

(Refer Slide Time: 13:20)

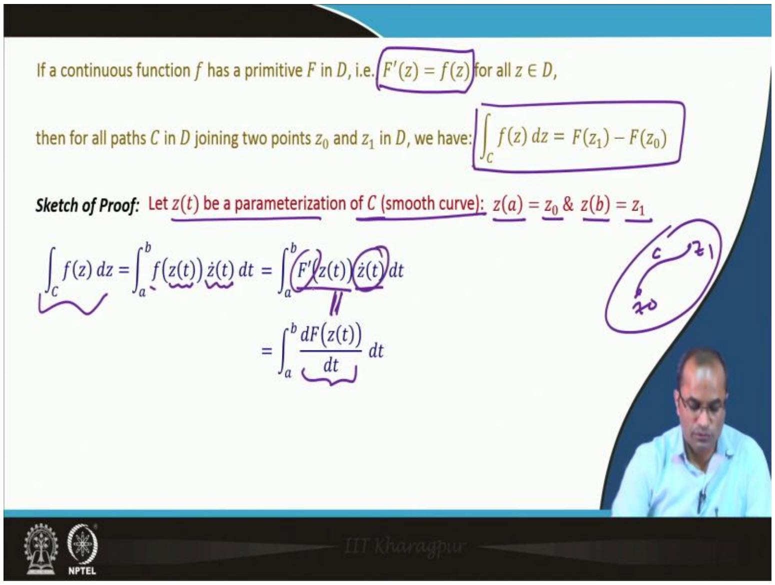


So, here coming to this, so this is the one approach which we can obviously use for evaluating these line integrals. The another would be that if a continuous function f has a primitive, so primitive means the anti-derivative, so if it has a primitive f and D that means if there is a function whose derivative is exactly this given function f z for all z in D, so this is also important that in the domain in the whole domain we have this primitive and then for all path.

So, any path we can take C, any path C in this domain D which joins for example these 2 points so we have a domain D where the function f has primitive then between these z0 and z1 point, these are the part of these points belongs to this D we can take any path here, any path we can take so this is like C1 path, C2 path or any other path C3 path, so we can take any path and then we have this value of this line integral just f z1 minus f z0. So, this is path independence which we have discussed there.

So, here this value does not depend on the path which path you follow from to go from z naught to z1 because your function was having this primitive big F here, so whose derivative is the given integrant here f so this value of this integral is just f z1 minus f z0, so we can take a look how to get this formula, this value using the above definition of the integral.

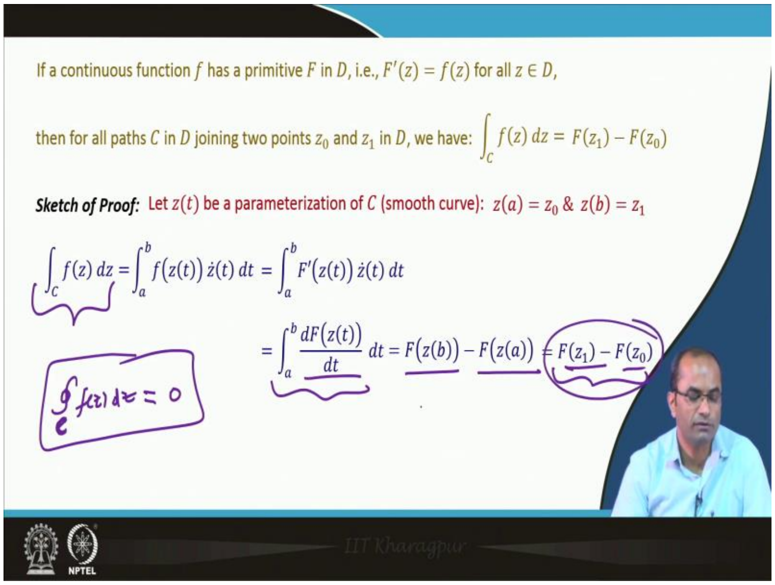
(Refer Slide Time: 15:12)



So how get this one, this is what we can just closely I mean very briefly look into it. So if z, let us say this z t is the parameterization of the smooth curve, so we are restricting to our smooth curves but this can be done for piecewise smooth curves for instance. So this z at a is the initial point, z0 and z evaluated at b is the z1 point we have, the 2 points they are in the domain z0 and then z1 we have a parameterized equations for the curve any curve you can take here so the parameterized equation for some curve is given by the z t.

So this integral line integral f z, d z we can write down f and then we substitute the z t that is the formula we have discussed first and its derivative. So f is given as f prime so here because it has a primitive so f prime z t and then the derivative of this z t dt so this we can write the chain rule, so this is coming exactly from the chain rule, so the derivative of this f z t with respect to t is the derivative of this f with respect to z and the derivative of z with respect to t.

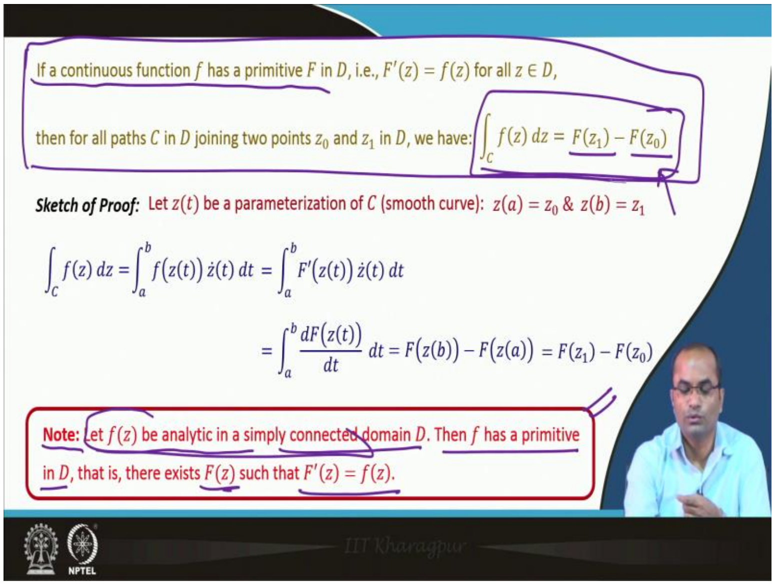
(Refer Slide Time: 16:40)



So this we can write down here and then so this is the derivative and then we are talking about the integrals, so this is just f z evaluated at b minus f z evaluated at a and this z b and z a were exactly z1 and z0. So this integral or this curve integral when there is a primitive existence or the primitive in the whole domain D then any curve we can take the value will depend on the initial and the final point not on the curve itself.

As a consequence of this if we have the closed curve for instance, so if we are talking about the close curve there, if our integral is over the close curve and all this conditions are fulfilled that this f has a primitive, so in the whole domain which included by this and closed by this curve C then this is going to be 0 because initial and the final points are the same there so f z1 minus f z0 this will become 0, so that is a consequence of this result.

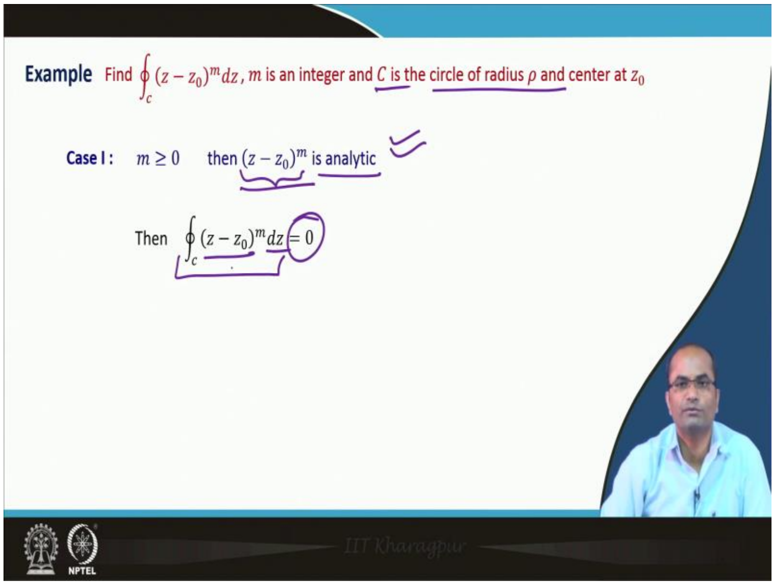
(Refer Slide Time: 17:48)



There is another important result which naturally we will not prove because that is a bit involve. So, let this f z be analytic in a simply connected domain, so if we have a function f z which is analytic in simply connected domain D, then F has a primitive in D that means there exists this f z such that f prime z is equal to f z. So, that is also very important result which this above result can be used now for analytic function. So, if we have f z analytic function instead of saying this if f is a continuous and which has primitive.

So, we can say that if f z is analytic in a simply connected domain D then this formula which we have discussed above that the value depends only on the initial and the final point is also valid. So, not only that the function has to be continuous and it has primitive those conditions are not explicitly required, so analyticity of f z in a domain D is also enough to use this above formula that because this says that if f z is analytic it has a primitive and if it has a primitive we can use this direct formula for the integration. So, we will also utilize this, now in our evaluation of line integrals.

(Refer Slide Time: 19:21)



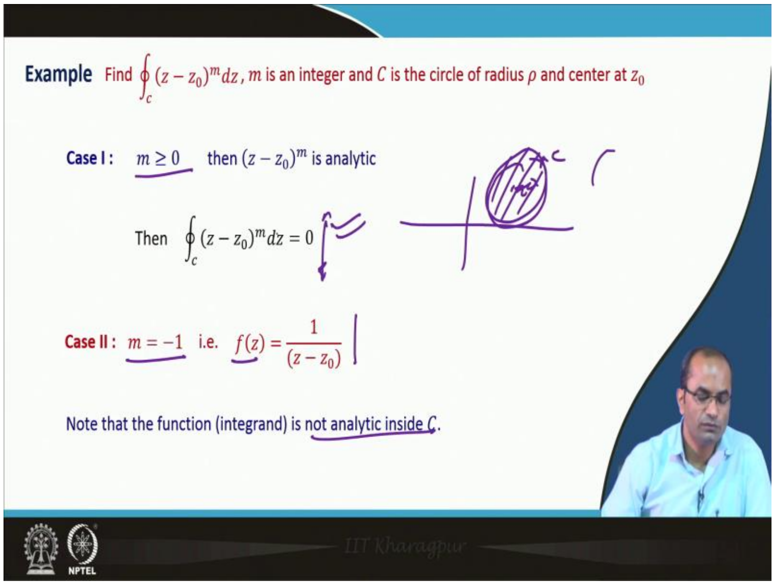
Well, so this is one example which we will discuss here for the evaluation. So we, because this is also important example which we will use its value later on, we will not every time evaluate this, this is standard example. So, we want to evaluate this closed, over the closed curve C, this the integrant is z minus z0 power m dz and m is an integer and c is the circle of radius rho and center at z0. So, we have the center z0 and the radius is rho for the circle where we are integrating this function.

So, the case 1 we will consider that m is greater than 0 because it depends on the different values of m what will be the value of this integral. So, if m is 0 that means the integrant is 1, so naturally it is simple case and m is positive also. So, what is interesting here that z minus z0 power m when m is a non-negative number, so m is greater than equal to 0, this is analytic because there is no point where the function has any problem for getting its derivative.

So, this function is analytic in on the C and domain contend enclosed by C, so this function is analytic and we have just learned before that once the function is analytic and we can easily use the formula there this in terms of the primitive f z1 minus fz naught the initial and the final point but here the we are talking about c which is a circle, so it is a closed curve. So, hence on this z minus z0 power m this value will become 0.

So, z minus z0 power and d z this is analytic here on this closed curve. So the value is 0. So without any we can use obviously the first formula for we can parameterized it and then we can use this formula naturally we will get 0 but we know this is analytic so we do not have to go through all these parameterization and using that formula here.

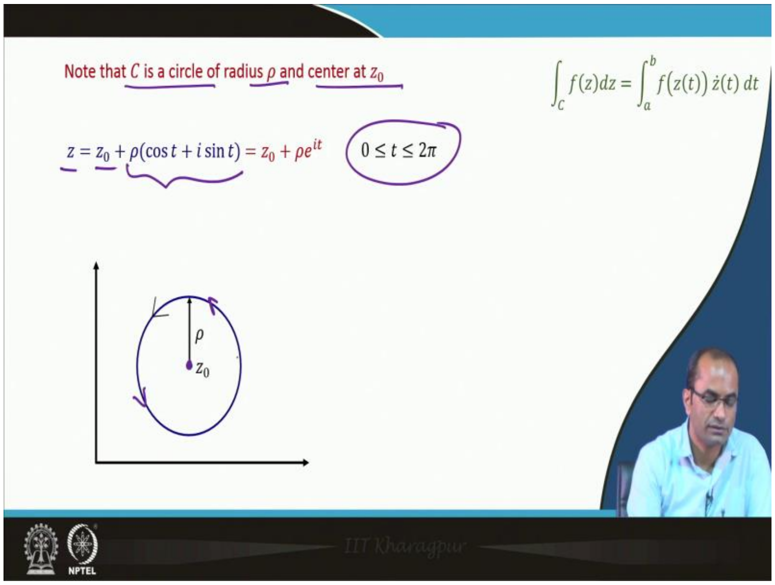
(Refer Slide Time: 21:49)



So, this value is 0 in this case when m is greater than or equal to 0 this is the 1 result we have. The second case we will consider when m is equal to minus 1. So if m is equal to minus 1 as a function is 1 over z minus z0. So in this case this is not analytic in this domain here which contains this z0 point and this curve C. So, this function is not analytic because at z equal to z0 there is a problem, the function is not define indeed here.

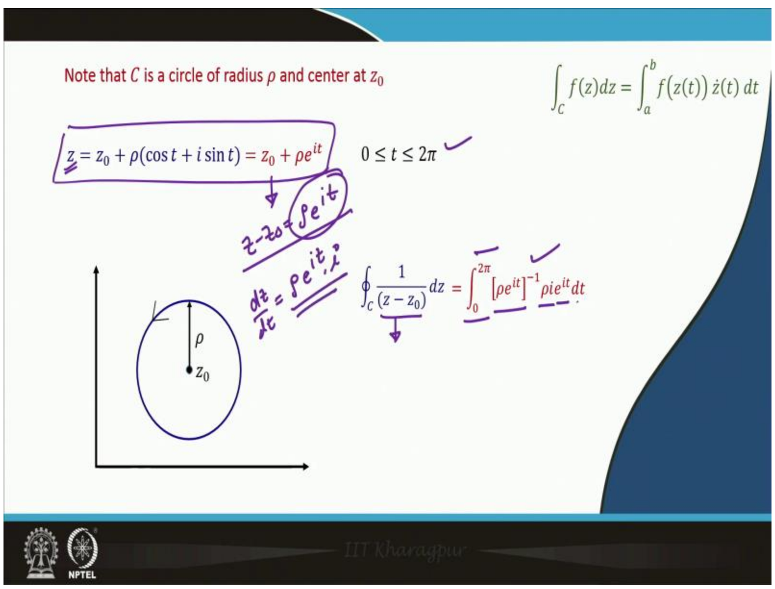
So, we cannot say that the function is analytic in this whole region and closed by this curve C. So, in this case since it is not analytic we cannot use the above result which says that over the closed curve the value will be 0. So we have to evaluate this, it may be 0 may not be 0, so we need to evaluate using the definition, using he formula integral.

(Refer Slide Time: 22:54)



So if c is a circle of radius rho and center z0 we have to parameterize this, so we have the center z0 and the radius rho. We can parameterized this easily, so z is equal to z naught and plus this rho times rho is the radius here, cos t plus i sin t, so as we move along this t form 0 to 2 pi you will be moving here in this direction on this curve.

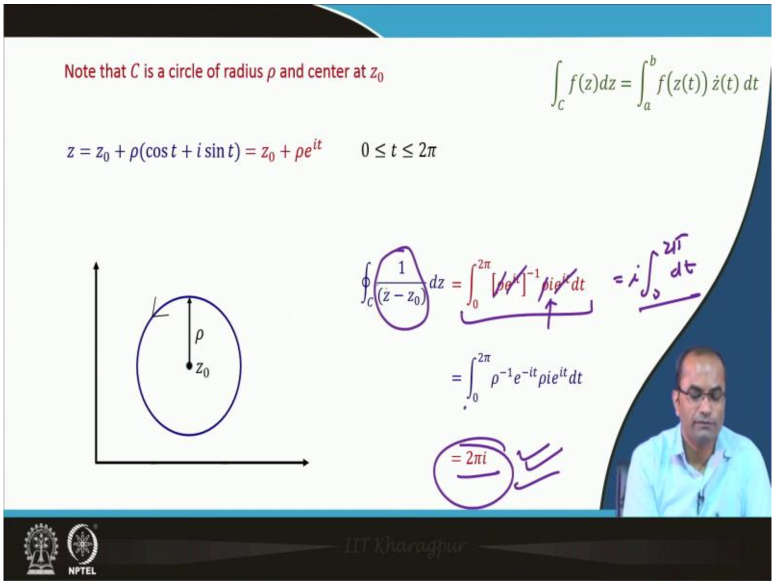
(Refer Slide Time: 23:24)



So we need to parameterized this, so this is the parametric equation of this curve, z is equal to z naught plus rho e power i t and t varies here from 0 to 2 pi and in the anti-clock direction we have the positive direction of this movement on the circle. So, having the parametric equation it is easy now to evaluate the integral this 1 over z minus z0 we have to substitute this z here and in terms of the parameter t, so that means the t is 0 to 2 pi and then we have rho e power i t that is z minus z0.

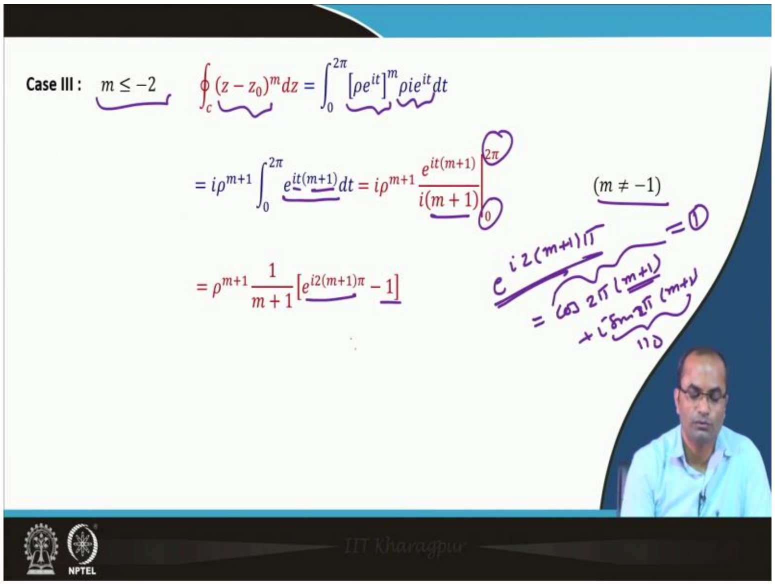
So form this equation from this parametric equation we have basically z minus z0 is equal to rho e i t, so we can just substitute for z minus z0 rho e I t this power minus 1 and then we need for this the derivative of this, the derivative of this z. So, that means this t z over d t we can get, so this is z naught the constants rho e power i t and then 1 i t will also come. So, rho into I and e power i t and the we have dt.

(Refer Slide Time: 24:42)



So this is the formula for the integration which we have used here and just to note that this row with minus 1 this will get cancel e power i t will also get cancel. So over this dt and we have just the 2 pi and i because this i will remain in the integrant and then we have the integral so i and then 0 to 2 pi and dt, so this is 2 pi i. S the value of this integral over this given curve is 2 pi i it is not 0, so we have seen because the function this was not analytics. So over this closed curve so we have the value 2 pi, so this case we have covered when this m was equal to minus 1.

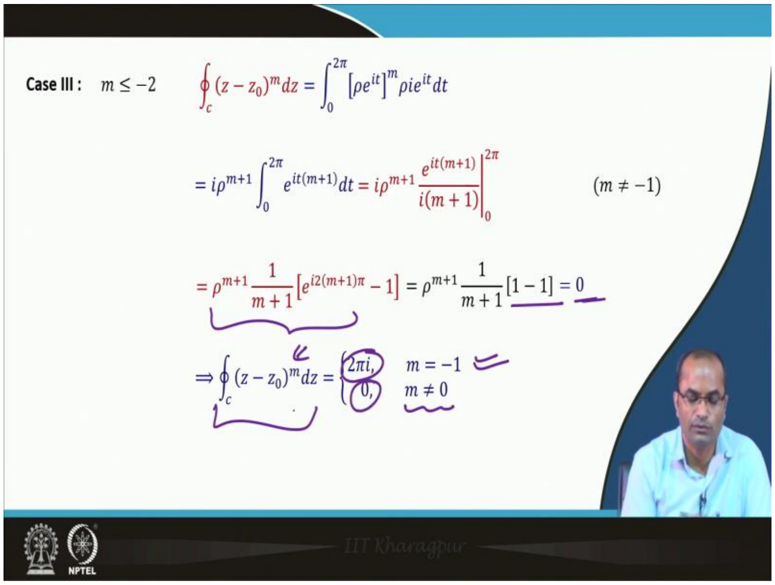
(Refer Slide Time: 25:33)



And when mm is less than or equal to minus 2, we will also see that what value it will take, so we consider this integral again we will substitute this parametric equation, so similar to the previous one then we have here e power i t and m plus 1 dt and then we can integrate this so naturally m not equal to minus 1 that case we have already discussed before, so we have this integral here and then we can substitute the 2 pi for t and then 0 with minus there so we have this integral.

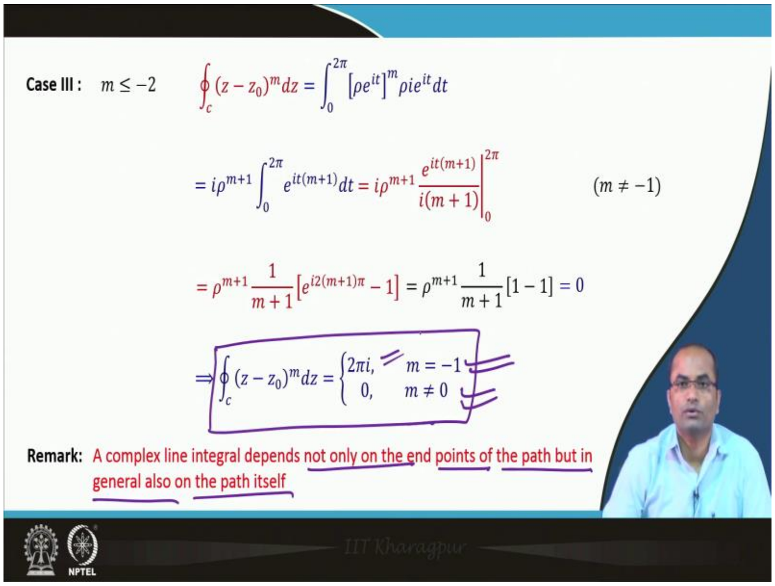
So, note here we have i and 2 m plus 1 pi that means the cos 2 pi with this m plus 1 and plus this i sin 2 pi the product with m plus 1. So this sin 2 pi is going to be 0, the cos also 2 pi with number here the integer this is going to be a 1. So, this value of this is going to be 1 and then minus 1.

(Refer Slide Time: 26:44)



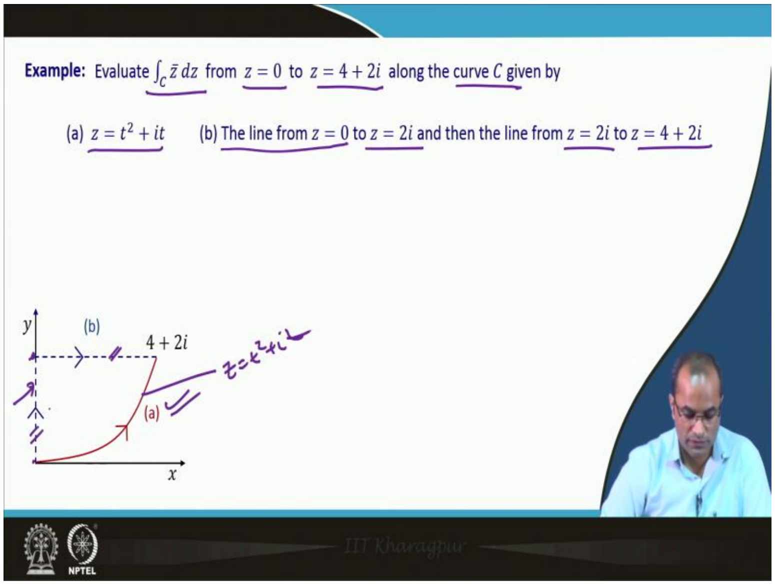
So we will get 1 minus 1 which is 0 there. So the value of this integral hen m is less than equal to minus 2 is coming to be 0. So what do we get? This line integral for all values of m if we write down here, so m is equal to minus 1 the value was 2 pi i and m is not equal to 0 the value is 0.

(Refer Slide Time: 27:10)



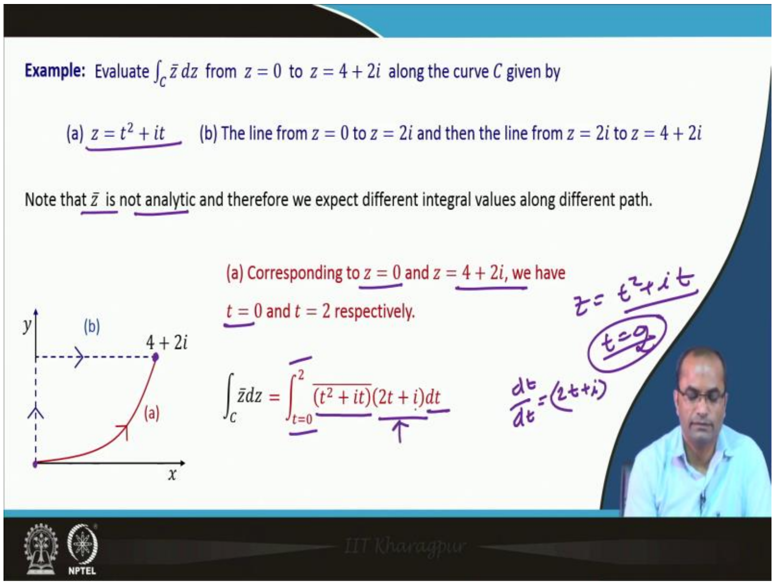
So we have 0 and 2 pi this is a very standard integral which this value we are going to use again only if m is not equal to minus 1 the value is 2 pi other than this for any value of m the value is 0. So, a complex integral depends not only on the end points of the path but in general also on the path itself. So as we have seen here the, it was a closed curve but we have a still the value there, so it does not just depend on the end points but it also depends on the path as well which we will observe again.

(Refer Slide Time: 27:47)



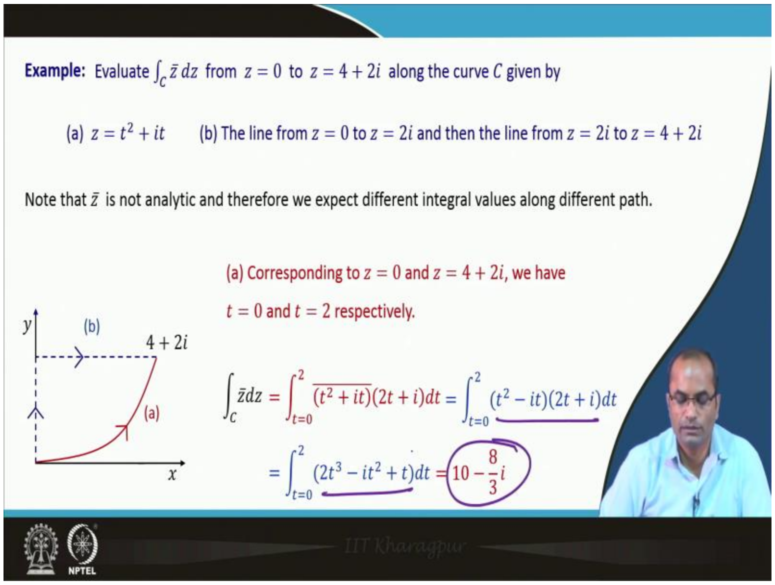
In the next example that this evaluation of this z conjugate is done form z is equal to 0 to 4 plus 2 i along the curve C which is given by this parabolic curves z is equal to t square plus i t and in the second case we will take z equal to 0 to z equal to 2 i and then from 2i to 4 plus 2i. So, let us take in the picture we have the 2 paths here the one is here, so along this path t square plus this i t the second path we are first moving from 0 to this point 2i and then from 2i to we are going from 4 plus 2i. So, variation in x along this and variation in y along this one.

(Refer Slide Time: 28:38)



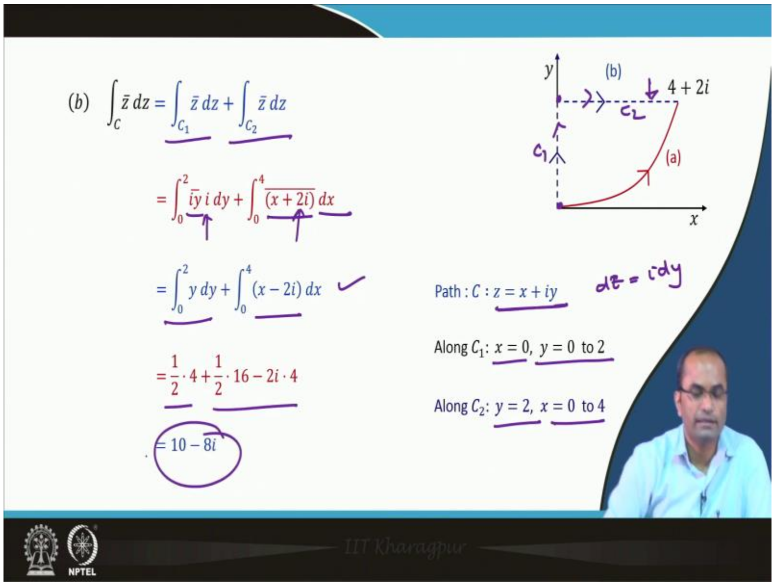
Okay, so this z bar already observed, it is not analytic so we have to I mean therefore we expect different values along the different path, it will it may not depend only at the beginning and the end point of the interval. So, let just compute this, so first z is equal to this corresponding to these two points if we look at the parametric equation this z is equal to t square and plus this i t so the t equal to 0 we are at 0 and t equal to 2, t equal to 2 we have 4 plus 2i so the end points, so the in terms of t we are varying from 0 to 2. So that means this line integrals 0 to 2 t then we have substituted here the parameterized form that means t square plus i t and its derivative, so the d z over dt given that we have 2t n plus i. So this is here and then integral over dt

(Refer Slide Time: 29:47)



So this simple integral we have to evaluate now. So this product we can make and then over t we can integrate. So after this integration we observe that the value is coming this complex number 10 minus 8 by 3 i.

(Refer Slide Time: 30:02)

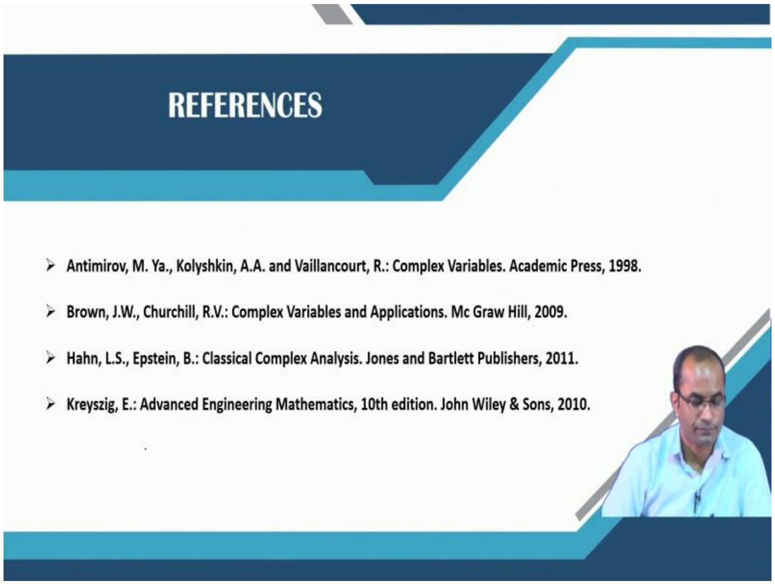


Now, the second part we will take choose the other path here, so first we will go from 0 to this 2i and again in the we will vary in the direction of x. So we have 2 curves basically the C1 and then we have the C2 curve there, so the C1 is x equal to 0 and y is varying from 0 to 2 if we take this side is equal to x plus i y along this C2 the y is constant, the y is kept constant x is varying from 0 to 4.

So, in the first one we will for z we will put because x is 0 so we have iy so I y and the absolute the conjugate and then the dz will be just i, so dz in this case will be idy. In the second case this will be just dx and we have substituted for this z because y was too there so y is fixed here, x plus 2i and along this dx we can vary.

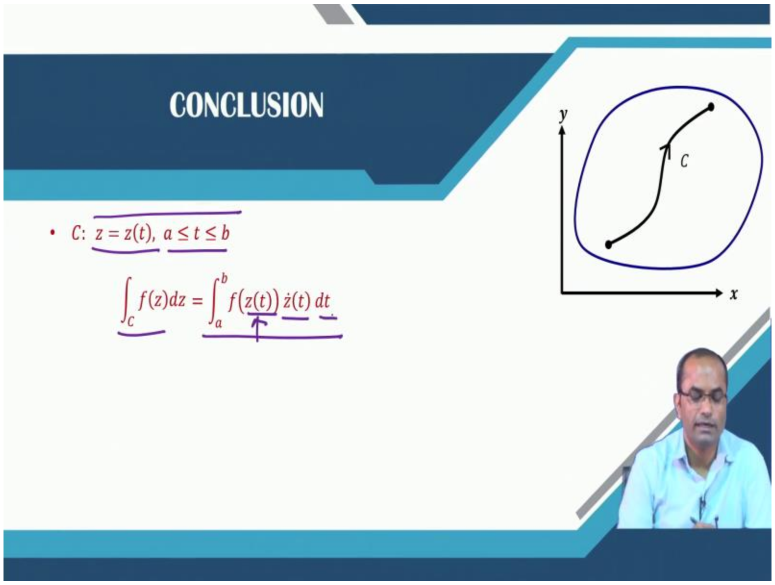
So, here we have y dy because this i-i will be minus i there and i, so this will become 1 then you have this x minus 2i so this tx, so just to get this integration it can easily be performed. So we are getting a 10 minus 8i as the value of this integral along this path p. So what we have observed that along these 2 path we have 2 different values which is because of this z bar was not analytic. If the function is analytic then it will not depend on path, it will depend on only on the beginning and the final point, the end point.

(Refer Slide Time: 31:50)



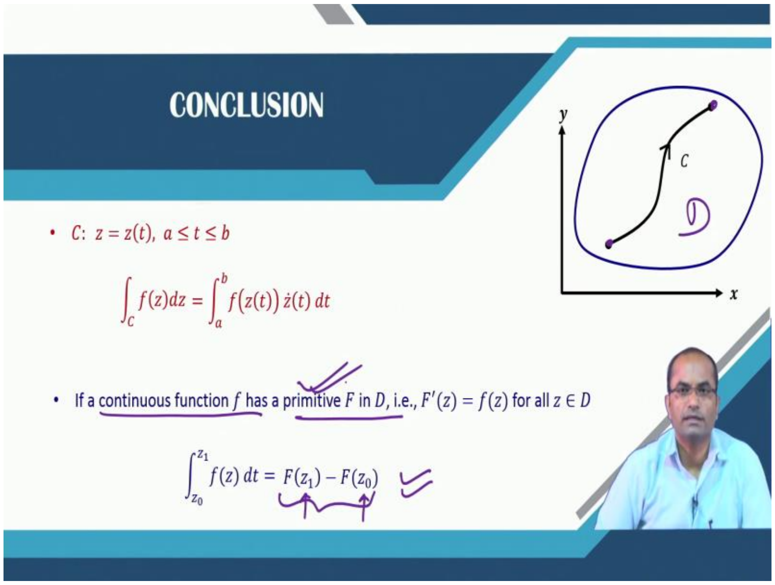
So these are the references used for preparing this lecture.

(Refer Slide Time: 31:57)



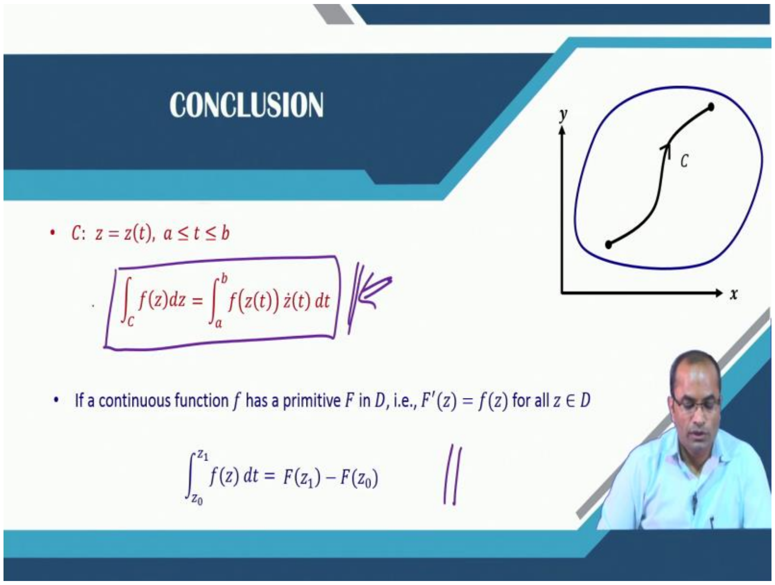
And just to conclude, so what we have learned? We have learned these line integrals so the basic definition for the line integral or how to evaluate this, so we need to parameterized c z equal to z t and then this curve integral can easily be evaluated by substituting this parameterized form in this f and then its derivative and integral over dt.

(Refer Slide Time: 32:23)



The second case we have seen that if function is continuous and has a primitive or we can replace this we say that f is analytic in a domain D and then we can take actually any path because the (integrant) the integral value will not depend on the path, it will depend only on the final point and the beginning point. So here this is also very useful in many cases, once we know that the function has primitive or in other words the function is analytic in a given domain we can also use this second part very useful.

(Refer Slide Time: 33:03)



But this can be used for any cases whether function is analytic or it is not analytic, we can use this evaluation here based on this parameterized form of the curve. So that is all for this lecture and I thank you for your attention.