Engineering Mathematics II
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Lecture 13
Analytic Functions

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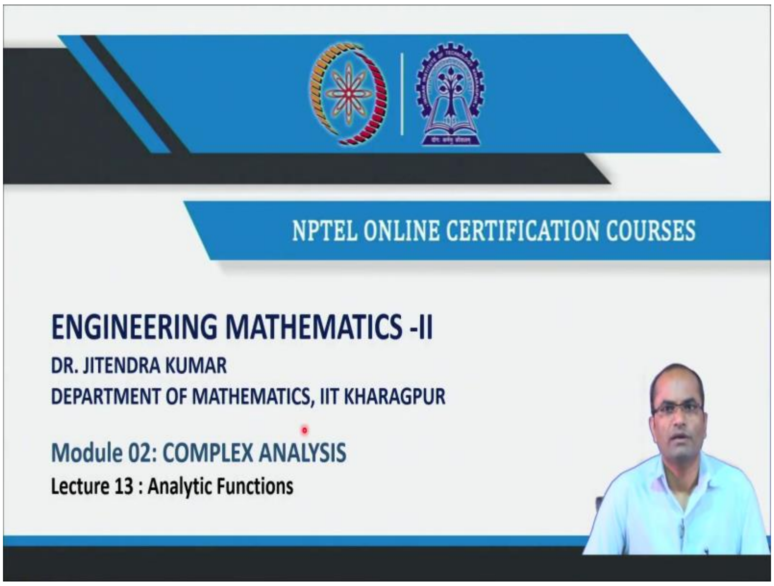

Welcome back to lectures on engineering mathematics 2. And today we will be talking about a analytic functions.

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So, in this lecture, we will discuss analytic functions, which was already partly discussed in previous lecture and then we will come to what are the harmonic functions, there is a relation between these analytic functions and harmonic functions. And then we will go for the construction of analytic functions.

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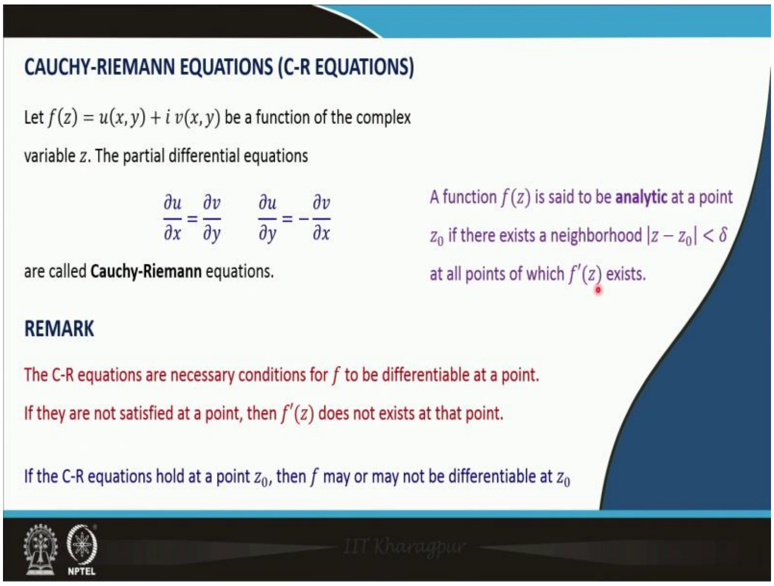

So, just to recall what we have learned in previous lecture, that was the Cauchy Riemann equations, that is CR equations, the necessary conditions for analyticity or differentiability that was already discussed. So, just to recall that your function f z, u x y plus i vxy be a function of the complex variable z, then these partial differential equations that is ux is equal to vy and here we have uy is equal to minus v x. So, these equations are called the Cauchy Riemann equations.

And then what we have learned in the previous lecture the CR equation, so these Cauchy Riemann equations are necessary conditions for f to be differentiable at a point and if they are not satisfied at a point, then the derivative f prime that does not exist because these are necessary conditions.

But if they exist, then we cannot say whether the f prime exist or not that we have to check because these are necessary conditions not the sufficient conditions. We have also learned that these equations become sufficient conditions if these derivatives here ux, vy, ux and here this vx if they are continuous in addition to these relations, which has to be satisfied as a necessary condition.

So, if they are continuous these partial derivatives appearing in CR questions, then these conditions become sufficient condition for a differentiability or analyticity. So, this is what we have already discussed in the previous lecture. So, it is a written here also that if the CR questions hold at a point z0, then f may or may not be differentiable at z0 then we have to check because these are just necessary conditions. But if they are not satisfied, then we can definitely conclude that the function is not differentiable at that point.

So they will be helpful to prove non-differentiability of a function. So, just to recall again that what was the analytic the definition for analytic function. A function f z is said to be analytic at a point z naught if there exists a neighborhood z minus z 0. So, around 0 if there exists a neighborhood delta neighborhood.

And at all points of which f prime z exists. So, for analyticity the function need not to be just differentiable at the point z0, but it has to be also differentiable in the neighborhood of z0. So, this is the difference between analyticity and differentiability. So, the function may be differentiable at a particular point, but it may not be analytic because for analytic, the function has to be differentiable or so in the neighborhood of that point z0 where we are talking about the differentiability.

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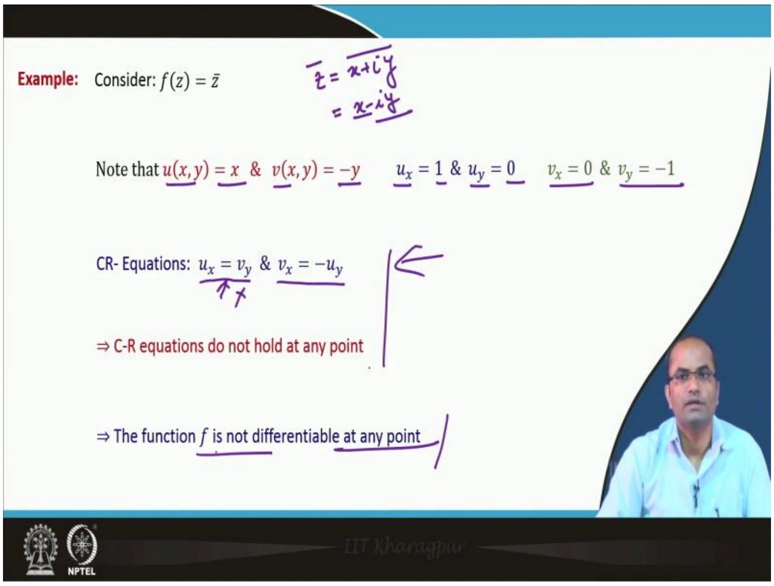

So with this consideration we can move further now. So, here if we consider that f z equal to z the conjugate. So, this is the function here and we will see with the help of CR equations, that how what we can conclude about it differentiability or analyticity? So, here we can break this function as in the component.

So, here the z bar, so z is x plus iy and then we are talking about this conjugate that means, x minus iy. So, we have u is equal to this x and then v, these the imaginary component we have the y, so with minus sign so minus y. And now with this u and v we can check the Cauchy Riemann equations.

So, we will get here the ux partial derivative of u with respect to x. And that is 1 here, partial derivative of v with respect to first with respect to, so partial derivative of u with respect to y because it was x, so this will be 0. And then for v with respect to x will be 0 and with respect to y this is going to v minus 1. So, having all these partial derivatives what we see? Because we know that Cauchy Riemann equations at u x is equal to u, v y and v x is equal to minus uy. These are the Cauchy Riemann equation.

So here just to take a look that ux was 1 and vy was minus 1, so they are not satisfied this equation for instance, is not satisfied. So the CR equations do not hold even at a single point they do not hold anywhere in the domain because this ux is one and vy is minus one. So they can never be equal at any point.

So, here the conclusion is that this function is nowhere differentiable, the function is not differentiable at any point. So, this we had discussed in the previous lecture, but where we have seen the direct differentiation of this function and we concluded that the function is not differentiable at any point.

But what we have observed that with the help of CR equations, we can conclude the same that the function is not differentiable, because the CR equations are not satisfied at any point. Again to repeat if CR equations are satisfied, then we have to check the differentiability from the fundamental definition. But here the CR equations are not satisfied at any point. So, we do not have to check even differentiability at any point we can simply conclude that the function is not differentiable at any point.

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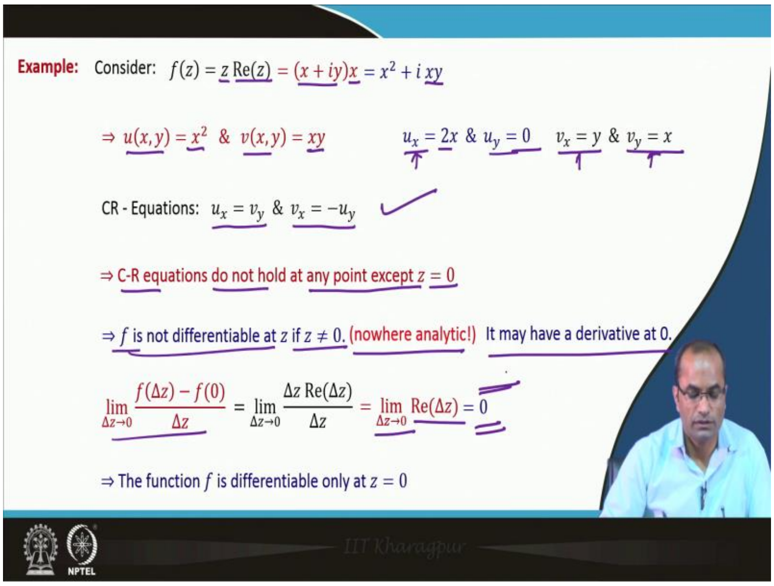

So, coming to the next example, where we will again observed that the function is z and the real value of this z. So, which we can write, so z is x plus iy and then we have x for the real z. So, again we can break into u and v. So, the u xy is x square and v the function is this xy there. So, again we can compute these first order partial derivatives that means, the ux which is 2x here and uy will be 0 because this is function of x square alone. And when we have vx that is y and vy will be x. So, these are the partial derivatives.

So, again if you take a close look for a first equation of the CR equations, that means ux is 2x and here u, vy is x so they are not equal for any point but when we have we are talking about for instance origin that means x0 y0. So, they are equal that means the CR equations ux, vy and vx and then we have uy. So, at origin they are satisfied because everything is 0 net origin all these partial derivatives are 0 at origin. So, the CR equations are satisfied at origin.

So, CR equations do not hold at any other point except z 0. So, that is the point to be discussed here that other than this 0 because of this 2 x and there we have x and here also 0 and there are y. So, if we take any other point then the origin then 0 0 then the CR equations are not satisfied, then what we can conclude here? That at z equal to 0 the function may be differentiable may not be differentiable. We do not know because the necessary conditions are satisfied at 0.

But at any other point then origin definitely we know that the function is not differentiable, because here equations are not satisfied. So, here just to conclude again for differentiability at z equal to 0 we need to check because we cannot conclude from the CR equations. Second about the analyticity at z equal to 0 we can definitely say that the function is not analytical at z equal to 0. Though it may be differentiable or may not be differentiable that we have to check.

But analyticity at this stage itself, we can conclude that the function cannot be analytic at z equal 0. Because if the function is analytic at z equals 0 then it has to be differentiable in a neighborhood around this z equal to 0. But CR question says that function is cannot be differentiable at any other point tan 0 and at 0 we have to check the differentiability. So, for analyticity, the conclusion is clear the function is nowhere analytic, but for differentiability we need to proceed now, whether the function is differentiable at z equals 0 or not.

So, f is not differentiable at z if z is not equal to 0, so other than origin function is not differentiable, this is what written here. And as I discussed, we can conclude that the function is nowhere analytic because here we learned that the function is not differentiable at any z if z is not equal to 0. So, it may have a derivative at 0, this is what we still hope that it may have derivative may not have derivative because the necessary conditions are satisfied.

So, we will check with the fundamental definition that f delta z minus f 0 over delta z that is delta z real z over delta z, f 0 is 0. So here we have the real z and the delta z goes to 0. So the real delta z means the delta x increment in delta x only. So that when delta z goes to 0 naturally this will go to 0, so we have the derivative here.

So, the function is differentiable at 0, but it is nowhere and analytic and CR questions were satisfied. And here as a conclusion, we also observed that the function is actually differentiable at z equal to 0 but note that function may not be differentiable. CR equations just not tell about differentiability of this z equal to 0 but if they are not satisfied if the equations are not satisfied, we can definitely conclude that the function is not differentiable.

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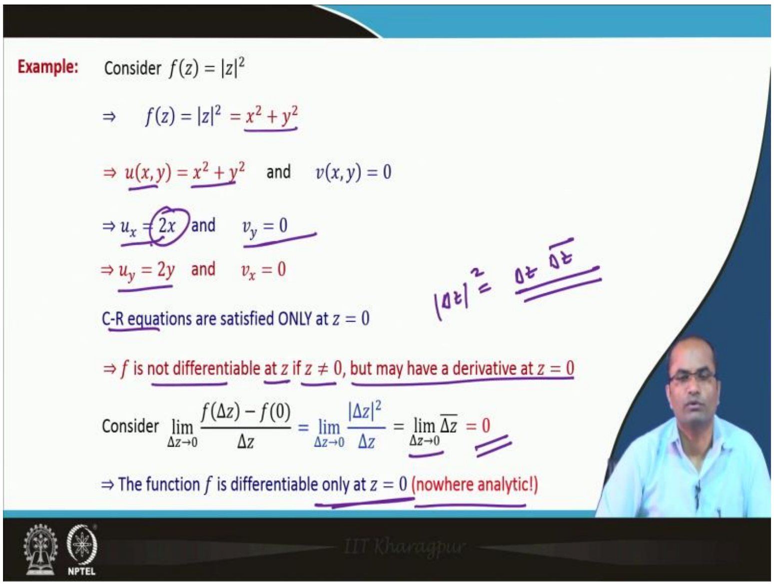

So well, in this example f z is equal to absolute value of z whole square we will observe now. So, again f z is absolute value z square we can break into this real and imaginary part. So, here we have, because absolute value of z is the square root x square plus y square and we are talking about whole square, so this is just x squared plus y squared.

So, uxy is vx square plus y square and vxy is naturally 0 here there is no imaginary part. So ux is 2 x and u vy is 0. So uy is 2 y and vx will be 0. So again, the CR equations we can observe that ux is equal to vy. So ux is equal to vy this is possible only when z equal to 0.

So adding z equal to 0 and that is the only point. So CR questions are satisfied only at z equal to 0. So the same situation what we had in the previous example that CR equations are satisfied only at origin. So, here also we have the same situation that means the function is nowhere analytic, but for differentiability we need to check using fundamental definition of derivative. So, f is not differentiable at z, if z is not equal to 0, but may have derivative at z equal to 0. So, we have to again apply this fundamental definition, where we observe that we have delta z square over delta z.

So, this delta z square absolute value we can write down we have learned this already that delta z and delta z conjugate. So, then delta z will get cancelled and we have delta z conjugate and the limit as delta z goes to 0, this conjugate will also go to 0. So, we have again in this problem, the derivative at z equal to 0. But since the function is differentiable only at one point, but for analyticity we need a neighborhood where the function has to be differentiable.

So, here we can again say that the function is nowhere analytic, but it is differentiable at z equal to 0.

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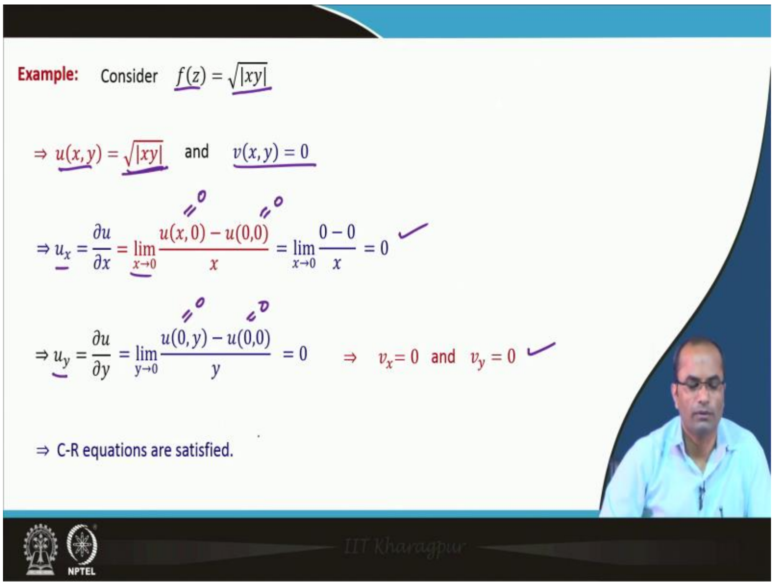

This example where we have this f z equal to the square root and absolute value of this x y we have taken. So, in this case, the uxy is exactly given the square root of this absolute value of xy and the vxy is 0. So, the ux we can get, so partial derivative with respect to x of this given function, we have to apply the we can apply this fundamental definition because this will be easier.

So, limit x goes to 0, ux 0 minus u 0 0 divide by x. So here u 0 0 will be 0 and this since y is 0 and the product is there in u, so this is going to be also 0. So this partial derivative at this 0 0 is going to be 0. The partial derivative of u with respect to y, again at 0 we want to compute, so this is going to be 0 and this is again going to be 0. So, we will have again 0 and v was just a constant 0. So vx is 0 and vy is also 0.

So at 0 0 we know the whole situation about the derivatives and what we observe here again, that the CR equations are satisfied at this origin, we are talking about the origin. So here the CR equations are satisfied at origin.

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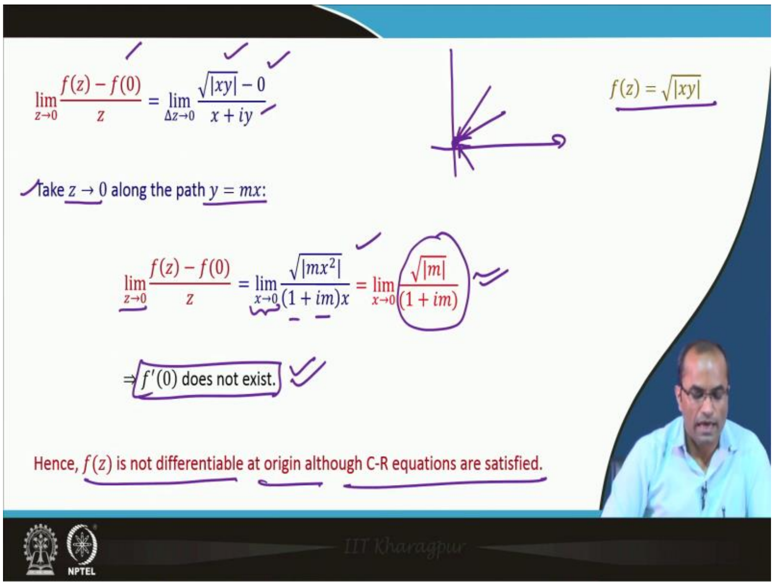

And then let us move further. So for the differentiability we want to check its differentiability at z equal to 0 we need to again apply this fundamental definition f z minus f 0 divided by z. So, this f z is a square root the absolute value xy minus we have 0 and then there we have x plus iy.

So, we take a path here now z approaches to 0 along this y equal to mx, so we have fixed the path that we are approaching to this 0 along these paths here which are straight lines. And now we have this limit which z approaches to 0 will reduce to like letting this x to 0 because why is mx, so y is also going to 0 when x is going to 0.

And then square root so m and y is x square, so one plus y is m, x and the next get common, so from there also. So, we get this which is independent of this x here. So, that is the limited itself, which depends on m. So, you choose a different m you are getting a different limit that means, the limit does not exist. So, this f prime 0 does not exist.

So, what is interesting in this example as compared to the earlier one, there the derivative exists, the CR questions were satisfied at origin. In this problem CR equations are satisfied at origin, but that derivative does not exist and again to remind that the CR equations are just necessary conditions, they are not sufficient conditions to declare to claim the differentiability or analyticity.

So, here this f prime 0 does not exist, though the CR equations are satisfied. So, which is not a surprising result which is find the f prime makes this f prime may not exist. So, f z is not differentiable at origin, all those CR equations are satisfied at origin.

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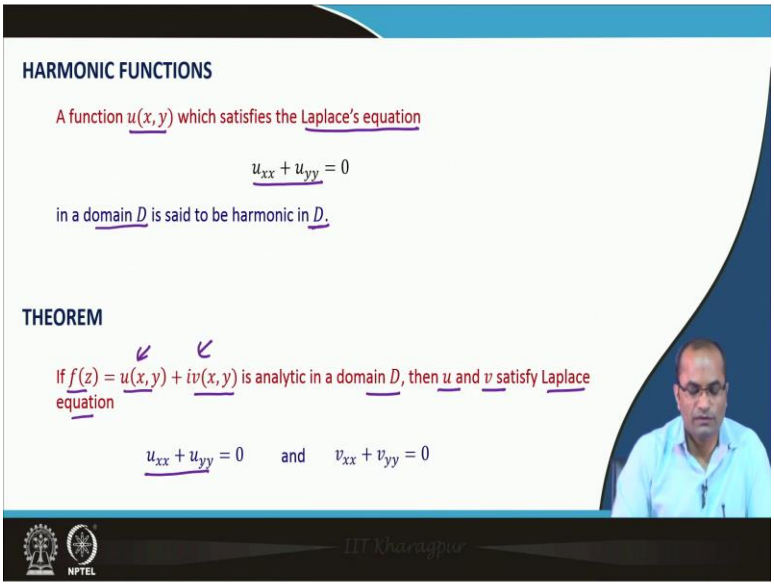

Coming back to these harmonic functions, what are the harmonic functions? So, a function uxy which satisfy Laplace equation. So, if a function satisfy this Laplace equation, then we call such a function harmonic function in D. So, if the equation is satisfied in this domain D, we will say that u is harmonic in D. So, what is the connection between this harmonic function means a function which satisfy this Laplace equation to the analytic functions that we will state now in this theorem.

So if f z is equal to uxy plus ivx is an analytic in a domain D, then u and v., so its components here u and these v functions they satisfy Laplace equations meaning. If this f z is analytic then this u and v are harmonic that is the connection between the analyticity and this harmonic functions. So, that means, if f z is analytic then this equation uxx plus uyy equals to 0 is satisfied, and vxx plus vyy equal to 0 is also satisfied.

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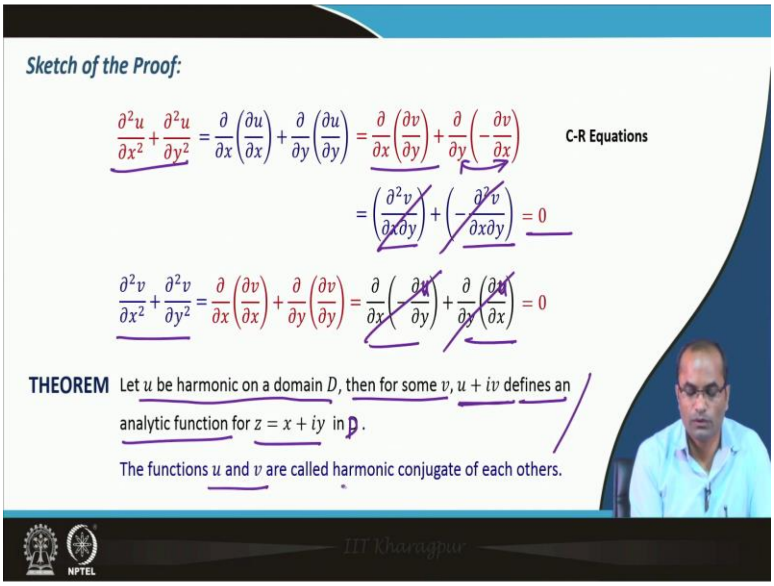

Just to give you the sketch of the proof, not the detailed formal proof, but so we will get some idea out of it. So, we have taken these Laplace equation the left hand side of the Laplace equation, del so second derivative with respect to x second derivative with respect to y. So, here with the partial derivative with respect to x then we have del u over del x, del over del y and then del u over del y just read it in this, in this form.

And then we know the CR equations that means del u over del x is equal to del v over del y. So, that relation we can use here del over del x, del u over del x is replace with del v over del y using the CR equations and then we have here del over del y and this term is replaced again with the CR equations. So, del u over del y is minus del v over del x.

So, now if we assume the continuity of these second order derivative, so we can change the order of differentiation. And changing the order of differentiation means that here we have del 2 v over del x over del y and here we have changed the order. So, del x del 2 v over del x del y. So, they become same and that means this is 0. So, we have that, that you satisfy this Laplace equation. Similarly, we can discuss for v. So, concerning we also we will apply the CR equations and then we will conclude that these equations are satisfied, these v also satisfy the Laplace equation.

So, here we have del v over del x, so that this equation we have satisfy, so del u over del y and then here we have del v over del y. So, we have del u over del x, that is the CR questions and again using the or changing this order of differentiation we will see that this become 0.

So, again we have so seen that the v also satisfies the Laplace equation. So, both u and v these real and the imaginary part of analytic function, they are harmonic.

There is another result, so if u is harmonic, so if we have a function which is harmonic in a domain D then there exists a v. So, for some v, so there will be a corresponding v corresponding to this u there will be a v such that u plus iv defines an analytic function for z is equal to x plus iy in D in that domain D.

So, what do we learn here that if a function is given which is harmonic which is harmonic in a given domain D, then actually we can find another function v such that u plus iv will become analytic in the function in the domain D. So, here this we will not go for the formal proof of this theorem, but with the help of examples we will learn that how to find v if u is given or if v is given then how to find u. So, here again the finding u or v the Cauchy Riemann equations play important role. And these functions u and v are called harmonic conjugate of each others.

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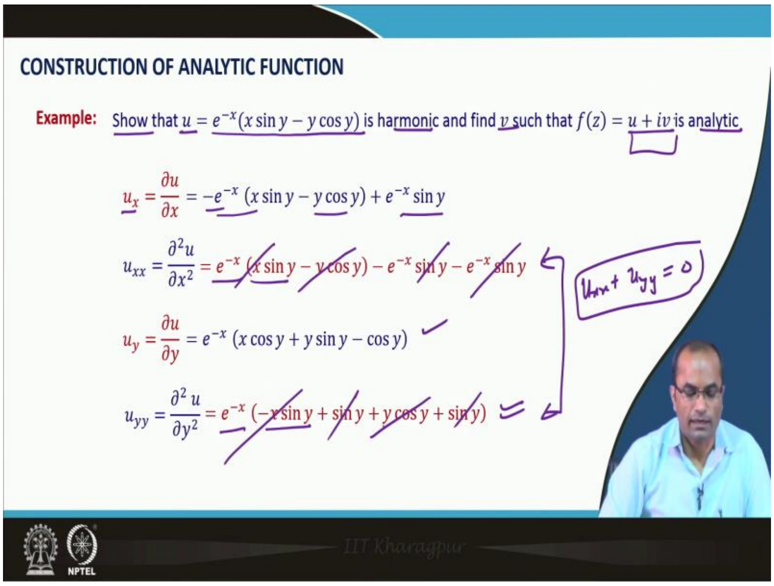

So, in this section we will be talking about the construction of analytic functions for given u we will find its v and if given v we will find u. So, here we will show that this u therefore the function u is given here is harmonic and then we will find v such that this u plus y is analytic. So, to check that this is harmonic we have to find the second order derivatives.

So, I will not go into the details. So, we have already discussed how to find the partial derivatives. So, here the first order partial derivative can be given in this form. And then again once more the partial derivatives of this u we can differentiate this again with respect to 2 x.

So, there you have the product tools here and at one place actually, so other two cases, we have only one function as a function of x. So, we can get these partial derivative of second order.

And similarly, we can get uy, so with respect to y first order partial derivative and then by differentiating once again treating x constant, we can get the second order partial derivatives. So, having these first and second order partial derivatives, what we observe from these two from the from the second order derivative that u e power minus x e power minus x and we have minus x sin y here the plus x sine y.

So, this will when we add this will cancel out and this y cos y term here there is m with minus, so that will also cancel and then sign y term with e power minus x here with the minus sign that will cancel out and sin y sin y and again here we have the e power minus x there, so here also e power minus x, so that will also cancel out.

So, here what we have observed that the ux, x plus this u, yy equal to 0 that means, this u is harmonic. And we have learned just in the previous theorem that if we have a function u which is harmonic then we can find the another function v which will be also harmonic but we can see that this f z u plus iv is analytic. So, we will find another function v now, so that this u plus iv becomes analytic.

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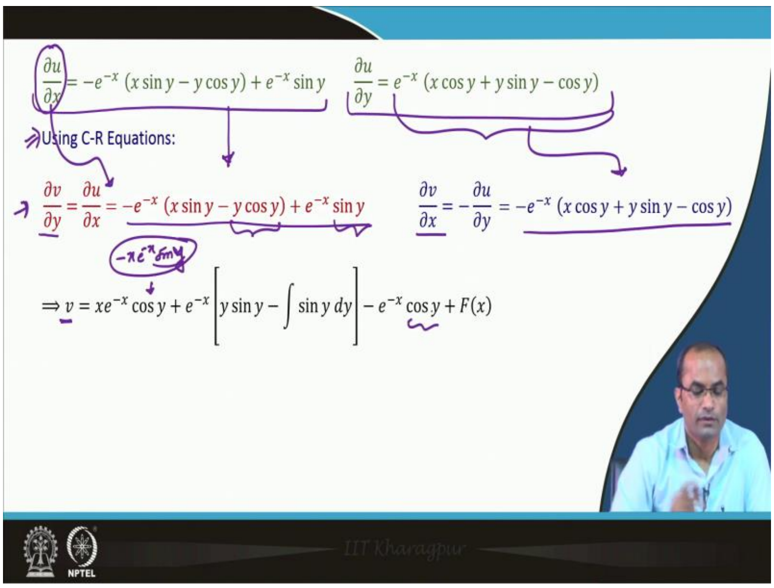

So, how to find the other function that process we will explain here now. So, we from the previous slide we have these two first order derivatives ux and uy. So, using the CR equations, the relations we know that the CR equation says that del v over del y is equal to del u over del x. So, this del u over del x is given already there. So, we can set this equal that e power minus xx sin y minus y cos y in e power x sin y. So, this is exactly coming from this given relation of del u over del x.

And the from the CR equation we observed that del v over del y is equal to del u over del x. So, hence now we have out of the CR equation that del v over del y is equal to this here this expression. So, which we can integrate now to find v a possible form of v. So, if before we go here we will use the another CR equation as well, which says del v over del x is equal to minus del u over del y. So, here del u over del yv we have already here, so which can be just use here. So, we have here del v over del x is given as this expression. So, now coming back to the first one again this del v over del y is given with this equation.

So, here if we integrate, so we will get v. So, we are integrating partially with respect to y. So, here the x will be treated as constant again. So, for instance, in this first term we have minus x e power minus x and e power minus x sin y. This was the first term here. So, that integration of this the sin x with respect to y, so we have sin, sorry this is sin y. So, with respect to y we will have minus cos y. So, this is what cos y is coming and minus sin is adjusted now.

So, x e power minus x cos y. Similarly, we can do for all other terms. So, at this place, we have again the product rule, so we have to apply for integration and in the third term we have just the sin so, that will become another cos there.

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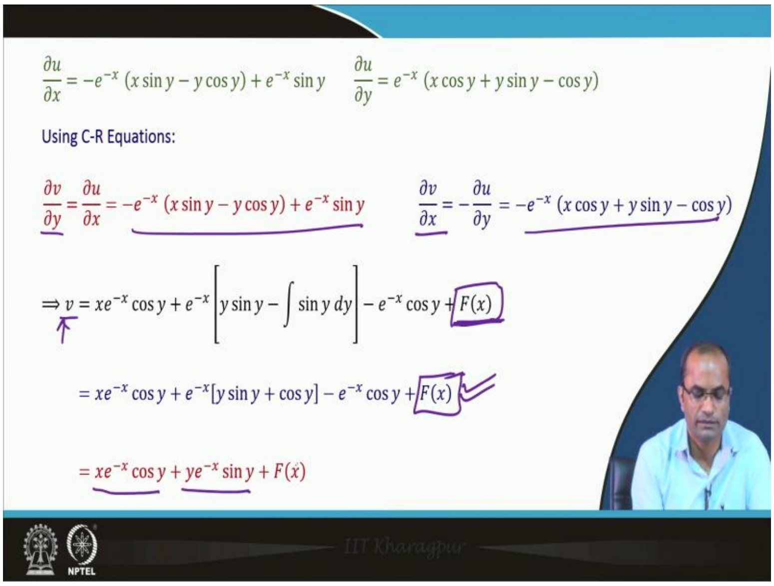

So, we can integrate this and so from this equation we got the v here which is the possible form now, so again just by simplifying this, we got that this is the the v here coming from this del v over del y equation. So, now having v, we can now differentiate with respect to x, so we will get del v over del x and del v over del x, we also know from this equation by putting them together we will observe that we can get this extra function because what is this? This is a so called arbitrary function of x. This is a constant of integration which we call but here we are integrating with respect to y. So, this constant of integration can be a function of x.

So, this can be evaluated and then we will substitute back to this v and we are ready to find this v in this way. So, this just the some more simplification tells that v is x e power minus x y power minus x sin y and this affects.

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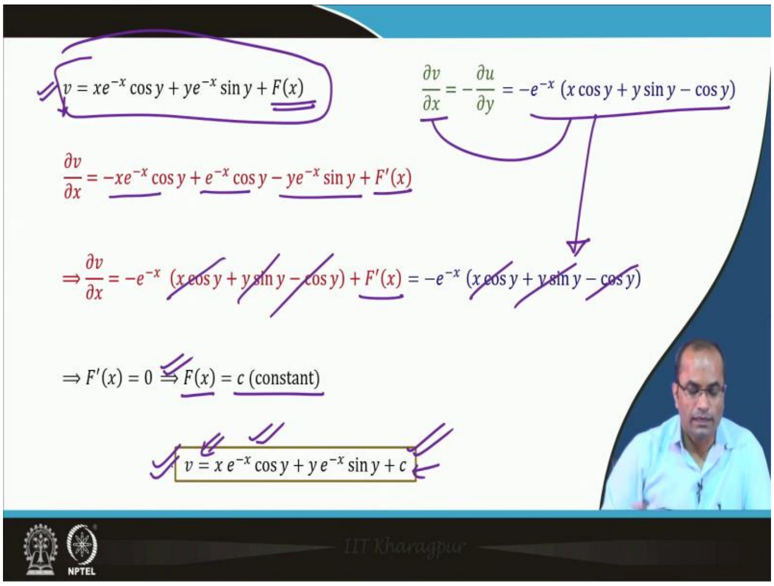

So, just to conclude again the v we have observed now, but in terms of this f the arbitrary function and del v over del x also we have from the CR equations. So, from this v we can compute this del v over del x which is which is given here and this f prime x will appear and this we can just simplify. So, we have del v over del x this expression. And we can set equal to this because this is also del v over del x. So, these two should be equal.

And having this we will observe immediately that x cos y, y sin y and also this cos y they are the same both the sides. So, this f prime x is actually 0. So, that means this is a constant because the df over dx is 0 that means the f is constant, f does not depend on x, this is what this equation is telling. So, Fx is just a constant C there and we know that v is given here. So, if we put this Fc, Fx as a constant, then we got the expression for v. So, this constant can be anything either one can choose 0 by to 1, 2, 3 anything or any complex number can be chosen for this v.

So, for given u first we have checked at this given u is a harmonic function and then we have using the the CR equations, we are able to find this v such that this u plus iv will become will become a analytic function.

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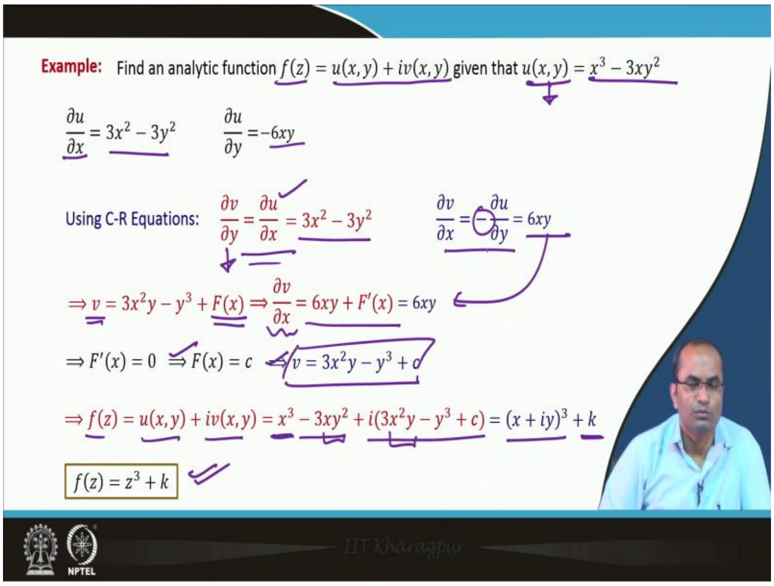

So, another example where the u is given as x cube minus 3 xy square and we want to find this analytic function f z such that this ux plus iv is becoming will become the analytic function. So, the same process we will follow we will have del u over del x and then del u over del y from given u we can find these two partial derivatives and then we will apply CR questions which says del v over del y is del u del x.

And this has given already 3 x square minus 3 y and from the second CR equation so we can just write down that this is 6 xy with minus sine, but there is minus sine here. So del v over del x is 6y. The same process this we will integrate now the integration gives this constant of integration which is a function here of x and by getting the derivative with respect to x, we got 6 xy a prime x and here this is equal to 6 xy which is the given equation already. And we observed that this F prime x is 0 meaning this Fx is c the constant term.

So, we got v here that 3 x square y minus y cube. So, our f z is uxy plus ivy. So, we have here x cube minus 3 x square exactly substituted here the u and v and then we will observe because the cube is coming here also we have the homogeneous from the 3, here we have 3 x square y. So, it is a cubic formula for this x plus iy power 3 and this ic we have just a constant there. We have renamed it to k and x plus iy can be replaced by z.

So, this f z is given as z cube plus the constant k, this is an analytic function satisfy the CR equations which we already set here equals, so they satisfy the CR equations and naturally these partial derivatives are continuous. So, this f z has to be a analytic function because the sufficient condition for in terms of CR equations are that these u and v must satisfy CR equations and the partial derivatives are continuous then they are sufficient for claiming analyticity of this u plus iv. So, this function is an analytic is analytic function.

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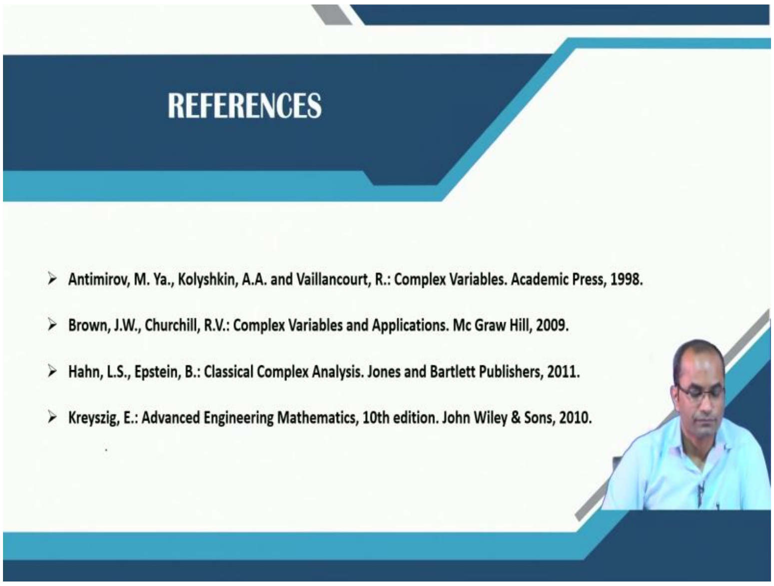

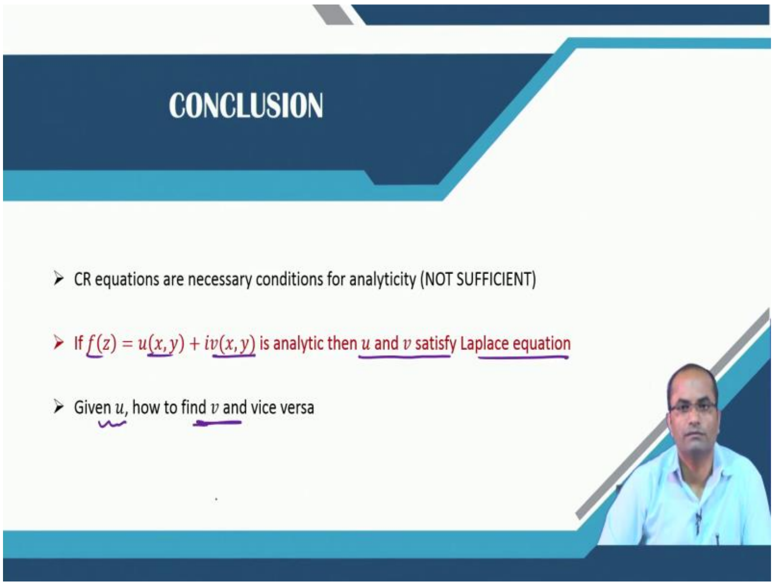

So, these are the references we have used for preparing this lecture and to conclude the CR equations are necessary conditions for analyticity not sufficient this we have also emphasized in the previous lecture a very important concept. And then this f z, u plus ivx is analytic then u and v satisfy Laplace equation. So, u and v are harmonic functions, this is what we have seen today and more importantly we have seen that given u which is which satisfies the Laplace equation. So, given u has harmonic function, how to find this v and vice versa given we have to find u that we can get using with the help of CR equations. And so, that is all for this lecture and I thank you very much for your attention.