Engineering Mathematics II
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Lecture 12
Differentiability of Complex Functions

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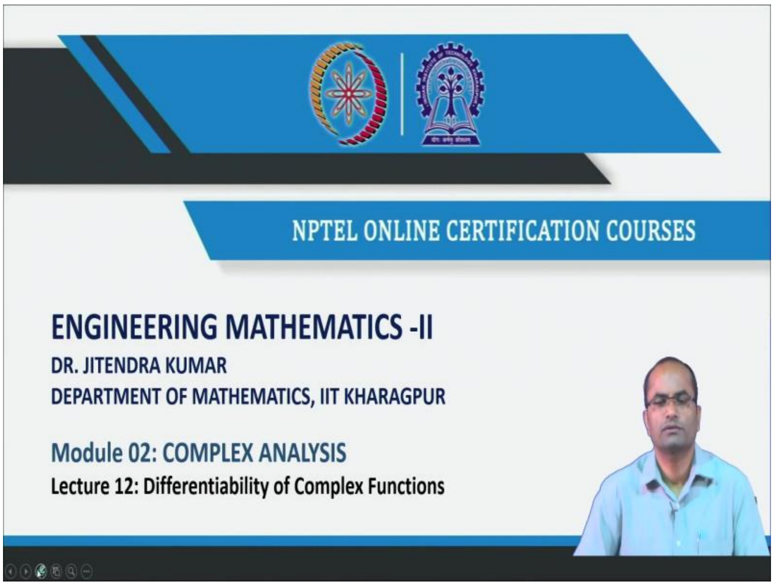

So welcome back to lectures on engineering mathematics 2 and this is module number 2 on complex analysis lecture number 12 on differentiability of complex functions.

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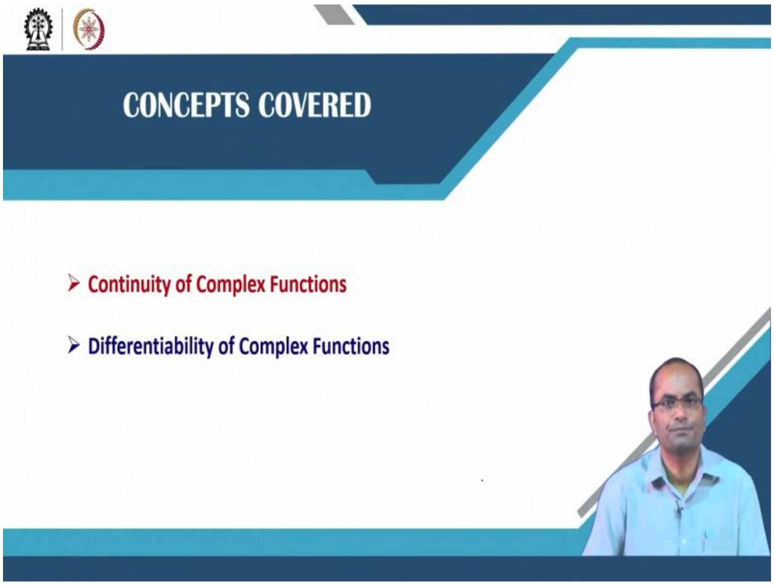

So, in this lecture we will be talking about the continuity of complex functions and then we will come to the main topic of differentiability, analyticity, etc.

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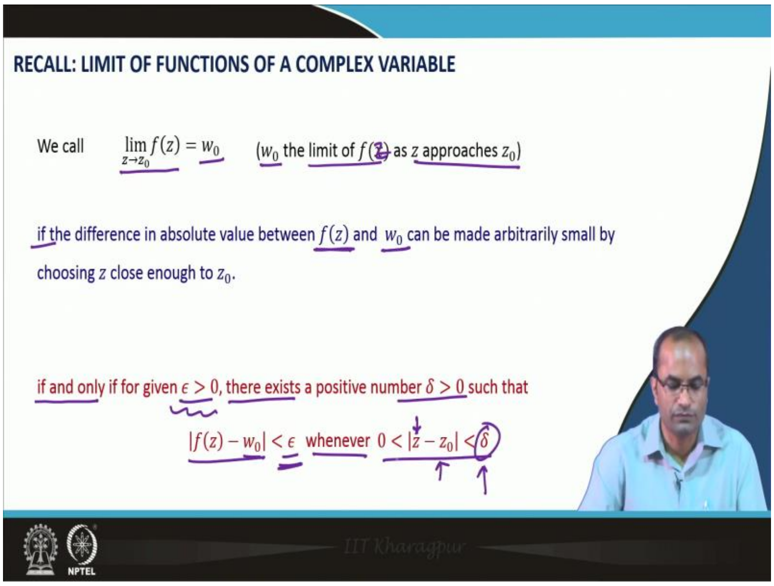

So, coming back to just to recall that what is the limit of functions of a complex variable we call that this limit here as z approaches to z 0, of f z is equal to w z when w naught is the limit of f z as z approaches to z 0, if so this is the first version. So, there were 2 versions discussed in previous lecture.

So, if the difference in the absolute value of this f z and w naught can be made arbitrarily small by choosing z close to close enough to z naught. So, that was the one definition we understood about the limit that if the difference of the absolute value this f z and w naught can be made arbitrarily small then we call that the limit of this f z as z approaches to z naught approaches to w naught.

And the second part was that if an only if for given epsilon. So, this was more mathematical way of expressing it. So, if and only if for a given epsilon, there exists a positive number delta such that this difference f z and minus w naught whenever it is less than epsilon, this is less than epsilon whenever we have z from this delta neighborhood delta minus was z minus z naught less than delta.

So, we have one neighborhood here. So, if we choose a point z from this delta neighborhood, then the f z should lie in the epsilon neighbourhood, this is the idea of this mathematical definition that we can choose any epsilon. So, for any epsilon there should be a delta there should exist a delta such that if we choose z from this delta neighborhood, the f z should lie in this w naught neighborhood. So, this was the idea of the limit. And now, we will relate this limit to the continuity for f z at a point z naught.

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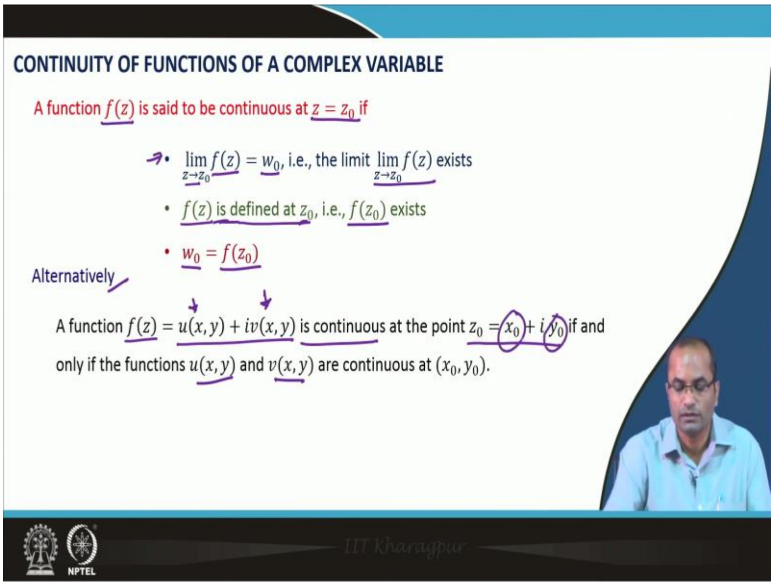

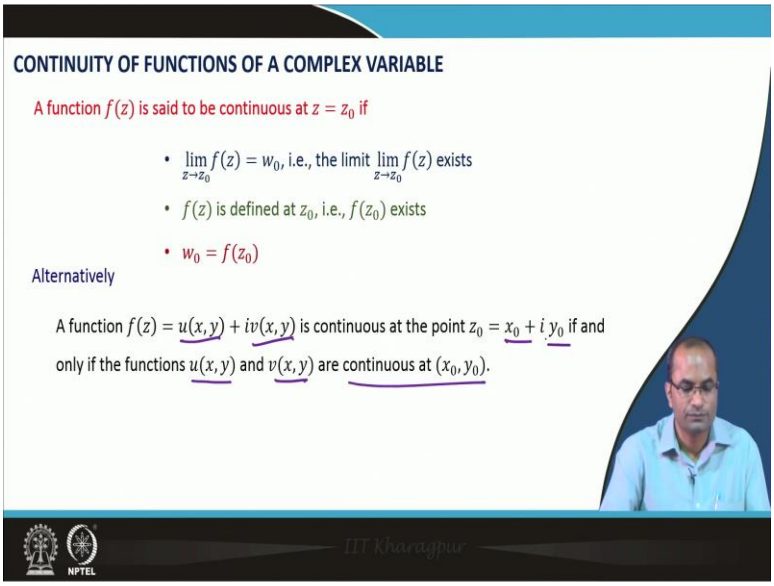

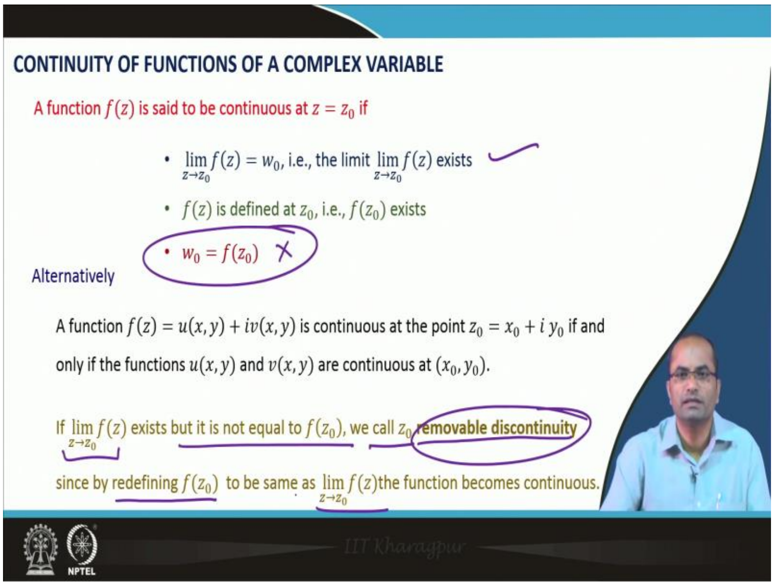

So, coming to the continuity of functions of a complex variable, what we say that a function f z is said to be continuous at z equal to z naught if the first condition is that the limit f z as z approaches to 0 let us say this is w naught meaning that the limit should exist. So, the point here is that the limit f z as z approaches to z naught should exist, if it does not exist, we cannot talk about the continuity of this function at z equal to z naught. So, this limit should exist and the second is that f z should be defined at z 0 indeed around z 0 also otherwise you cannot talk about the limit.

So f z must be defined at z naught that means this f z naught exists just not that, that for getting this limit f z may not be defined at z naught but for continuity the f z must be defined at z naught and this f z naught must be equal to the limit w naught, which we have just discuss in this point 1. So, the w naught must be equal to f z naught so, if we have these 3 conditions fulfilled meaning that the limit of f z should exist at z approaches to z naught the second is the f z must be defined at f z naught and the value of this f z naught must be equal to w naught.

Alternatively, we also can define this or it is equivalent to say that the function f z which is u x y plus i v x y is continuous at the point z naught equal x naught plus i y naught if and only if the function so this u and v the component here of this f z u and v they are continuous at this x naught y naught point.

So, this is another way or we can easily define this continuity in terms of this functions of 2 variables u x y and v x y if they are continuous at a point x naught y naught, then we call that this function is also continuous at this point x naught i y naught. So, this is what we call continuity and there is a so called terminology on removable discontinuities.

So, what is this so, if this limit exists, so, this first point which we have defined there so, the limit exists, but it is not equal to f z naught that means, the third one is not fulfilled here that this f z naught is not equal to the limit which we got with this f z as z approaches to z naught. So, we call than this such a point z naught removal discontinuity a point of removable discontinuity, since by redefining f z naught so, if we just redefine the value of the function at that particular point z naught to be same as the limit then the function becomes continuous.

So, this is what we call removable discontinuity as such the function is discontinuous, but if we redefine the function at a point z naught then this will become continuous because the limit already exists. And we have defined this function in such a way that the function value at this point becomes exactly equal to that limiting value that limit. So, then the function becomes continuous and we can then we say that this is a kind of removable discontinuity means, this is a discontinuity, but can we easily removed?

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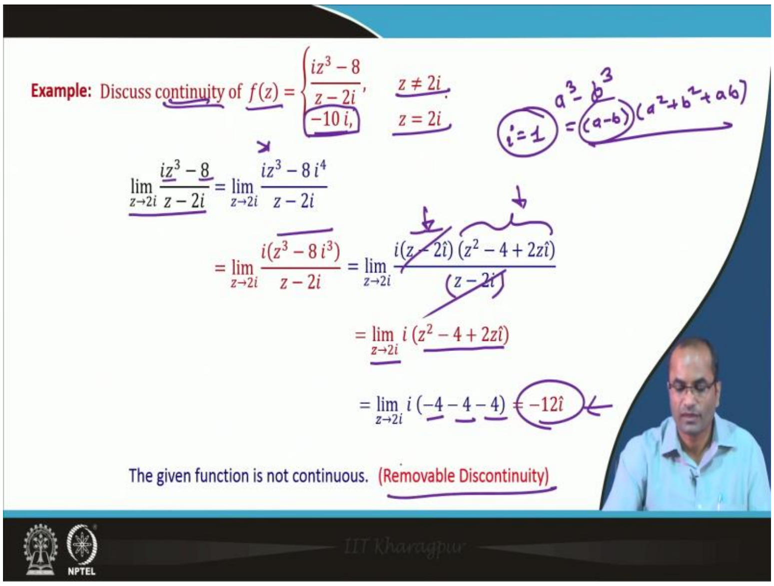

Well, so just a simple example to discuss this continuity so, we have this function, which is defined as this iz cube minus 8 divided by z minus 2i and whenever z is not equal to 2i when z is equal to 2i because z is equal to 2i the z minus 2i will become 0 and then the function is not defined.

So, z equal to 2i we have defined this here minus 10i. So, we want to see whether this function is continuous or not. So, we have to basically get the limit of this function f z as z approaches to 2i. So, to get this limit just look at this expression here iz cube minus 8 and z minus 2i the simple way of getting this because we can easily identify here that we have z cube we have here 2 cube.

So, if we can use this formula a minus b cube, which is a minus b and then we have a square b square plus a b. So, if we can use this perhaps this factor will be cancelled and then we can pass it straight away the limit without any difficulty. So, to do this we have written just this 8 and i4 so the i4 is nothing but it is a 1. So, we are not disturbing the expression. So, we have iz cube minus 8i 4 and z minus 2i and if we take this i common from this numerator, then we will have this i and then z cube minus this 8i cube which we have the form this a cube minus this 2i whole cube.

So, that we can expand now to get this z minus 2i that is the factor we got here and the second factor is remain with the z square with 2i square, so 2i square we become minus 4 and then the product so, 2zi so this has to be divided by this z minus 2i so this (())(09:27) this factor gets cancelled, which was problematic to pass the limit.

So, we have the i and z square minus 4 plus 2zi and z approaches to 2i, so when z approaches to 2i now, we can just substitute this and we will get the limit so we have i and z square so, 4i square that is minus 4, here we have minus 4 already and z is a place by 2i so 4i square so again minus 4.

So, we have minus 12i the value of the limit but what do we see there that at z equal to 2i, the value of the function was minus 10i so, this is not equal to minus 12i and as a result this function is discontinuous, but this is a kind of discontinuity which can be removed easily. So, if we defined if we redefine our function at z equal to 2i as minus 12i then the function will be differentiable will be sorry the continuous. So, this is a removal example of removable discontinuity, which we have discussed here.

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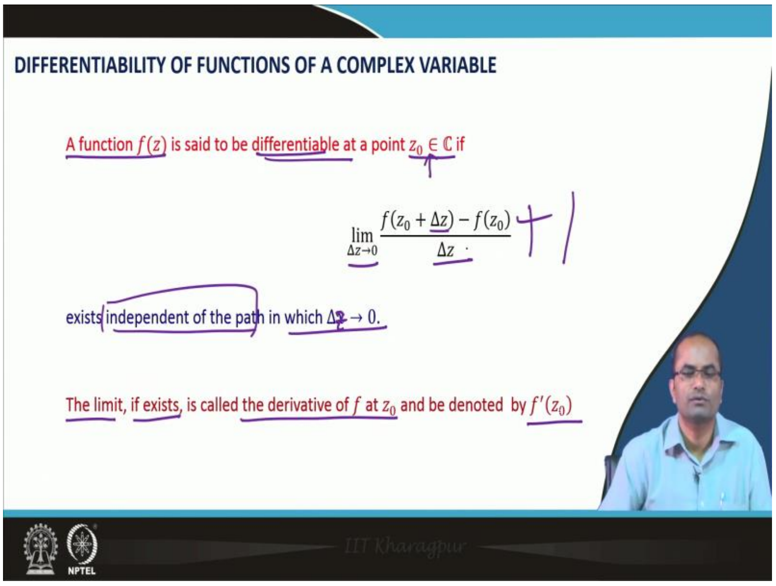

Now, we will move to the most important topic of this lecture that is actually the differentiability of functions of a complex variable. So, to define the differentiability we say a function f z is differentiable at a point this z0 from the set of complex number. If this quotient has a limit, so what is this quotient f, z0 plus. So, we are talking about the differentiability at z0.

So, we have increment here with this delta z so z naught plus delta z minus this f z naught and divided by the delta z the similar the same definition what we have for real functions as well. So, we have increment there and we have already discussed for vector valued function and these complex numbers, they have 2 component x and y.

So, we can also think like the vector functions there, and we have already discussed, so f z naught plus delta z minus this f z naught and then we have to divide by this delta z and taking this limit. So, if this limit exists, then independent of the path that is natural otherwise you will not call that this limit exists if we have fixed a path and got the limit, so, that is not basically the limit.

So, the limit should exist independently of path that is obvious and when this delta z approaches to 0, so this f z is set to be differentiable at a point z naught if this limit exists independent of the path in which this delta z approaches to 0. So, the limit, if exists is called the derivative if this limit exists that is called the derivative of f at this z naught and usually denoted by the f prime z naught, so the standard notations what we are using earlier as well.

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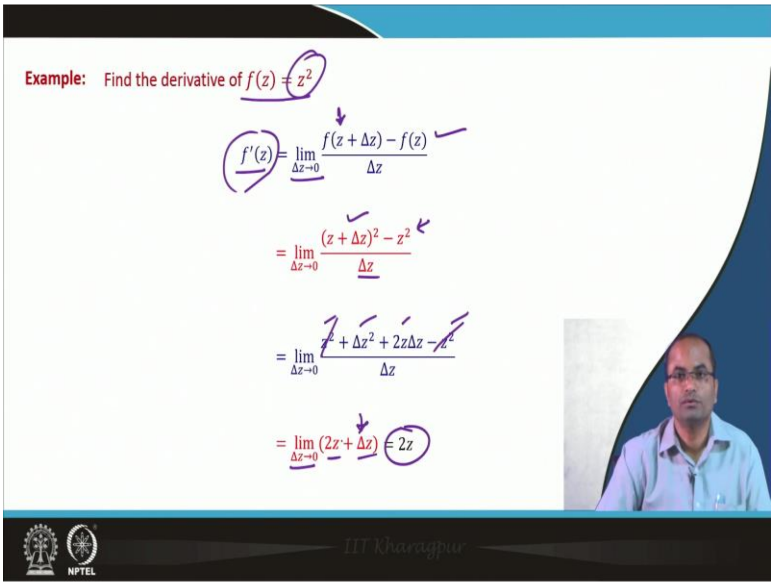

So, just to look at a simple example that how to find this differentiability or how to find the derivative So, find the derivative of this f z equal to z square so, we will use this fundamental definition that f prime z if this limit exists then we can say that this is a z prime z. So we assume that this limit exists because we know in this example the limit is going to exist. So limit delta z approaches to 0 z plus delta.

So we have taken a general point z here, we did not fix the point so z plus delta z minus this f z divided by delta z and the function is that square so this we can replace now the z plus this delta z square minus this z square and this delta z, so we can open this square there. So we have z square then delta z square and we have 2 times z delta z and then minus z square is already there. So this is z square and this z square gets cancel. And we have here 2z plus this delta z and this value and delta z approaches to 0, so this will go to 0 and we have this value equal to 2 z.

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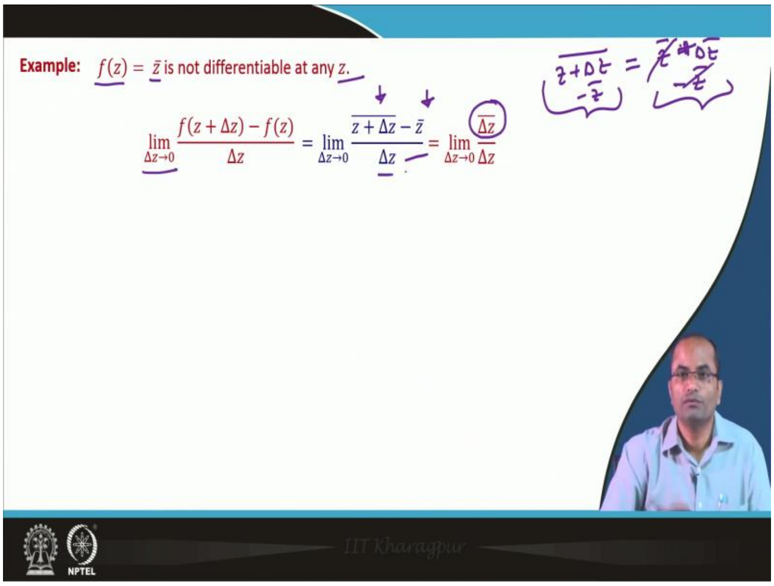

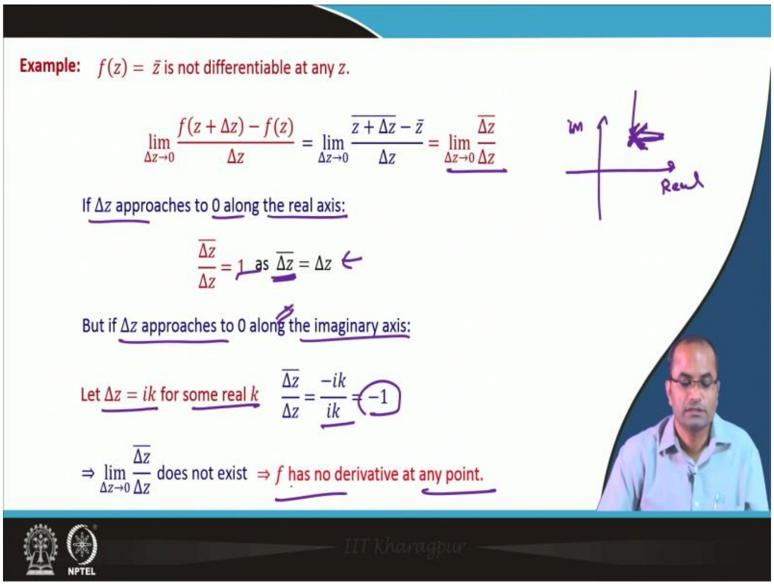

So, another interesting example in this theory of the complex variables we have f z equal to z bar. So, the conjugate of this z f z is defined as the conjugate of its argument, its input and this function is not differentiable at any point, this function is not differentiable at any point z in the domain. So here if we take this limit delta z to 0 f z plus delta z minus f z divided by delta z. If we talk about this function, then this f z plus delta z, it is given by the conjugate of this z plus delta z minus we have f z so, which is defined exactly with the conjugate z and then we have to divide by the delta z and look for its limit.

So, that means, so this z plus delta z if you take the absolute value the conjugate, then the conjugate not the absolute value. So, we have the z absolute value minus the delta z absolute value. So, which we have used here and then we have the minus this z also there, so this expression, if we look at here this gets cancelled, so we have.

So, this was plus there, so not minus then we have this delta z the conjugate and then divided by this delta z, which is already here. So, now we will look whether this limit exists or not. We will look for this limit now, so we choose that this delta z approaches to 0 along the real axis. So, we have the real axis here and we have the imaginary axis.

So we will in the first we will approach this delta z to 0 wherever z is there. So, we will approach to this point along this or parallel to this x axis along this real axis and see that what is the limit. So, along this real axis if we approach to this 0 that means, the imaginary component is fixed.

So, we can talk about that along this real axis, this conjugate of delta z when we are approaching to this point. This delta z the conjugate will be dealt z because we are talking about now, there is a variation in delta x itself there is nothing in delta y happening. So, we are approaching to this point just by the along the real axis.

So, this delta z the absolute the conjugate of delta z is just delta z and so we have this value as 1 because this will be equal to delta z so this is 1 and therefore, the limit along this x axis we are getting as 1 and now if we approach this along the imaginary axis, so now we are approaching along this imaginary axis.

So, this is like the delta z is like i into k so there is no x anymore. So, here delta z is equal to ik therefore some real number k and then in this situation, the limit is going to be minus 1. So, along 2 different paths, we have the value 1 and we have the value minus 1 and therefore, this limit does not exist and hence this f has no derivative f is not differentiable at any point, at any point in the domain of these complex numbers. So, this is a very special example, which where we have seen that this z is the conjugate of z is not differentiable at any point in the given domain.

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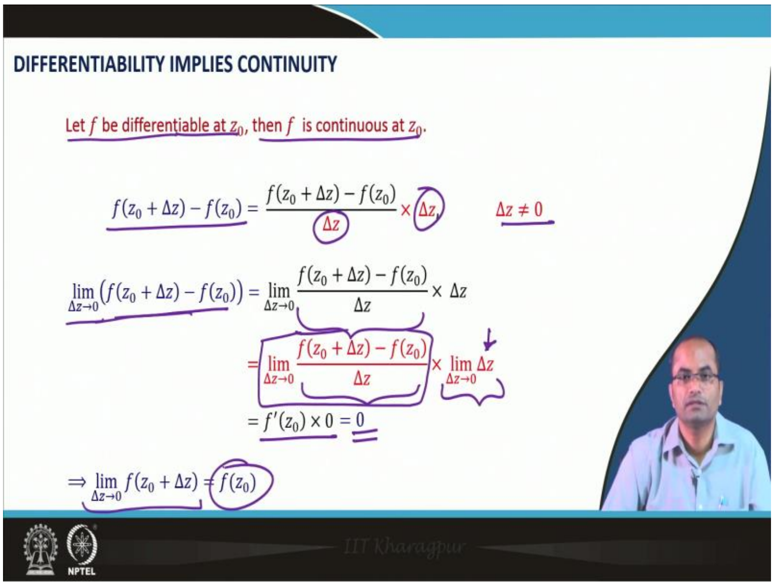

Now, here we can just take a look quickly that the differentiability implies continuity that means, if a function is differentiable this is continuous. So, we assume that f be sorry let f be a differentiable at z naught and we will show that f is continuous at z naught so, we will consider this expression hear that f z naught plus delta z and minus this f z naught. So in this case, we have f z naught plus delta z minus f z naught, we have divided by this delta z and also multiplied by this delta z and assuming the delta z is not equal to 0.

So if we take now the limit both the sides that means limit delta z approaches to 0, f z naught plus delta z minus f z naught this equal to this limit into this delta z. And remember that this limit, we can break into 2 parts here. So, the first the limit of this then the limit of this and here the first expression here tells if the function is differentiable, this limit will exist if the function is differentiable, this limit will exist.

And then here we have the delta z approaches to 0 and delta z that is 0 so something exist there into 0. So, then we have a 0 right hand side. So, that means this is f prime z naught into 0. So, that whole thing is 0. So, this limit here is 0 which implies that the limit delta z approaches to 0 f z naught plus delta z is equal to f z naught and this is the definition of the continuity. So, this proves that the function is continuous if it is a differentiable function.

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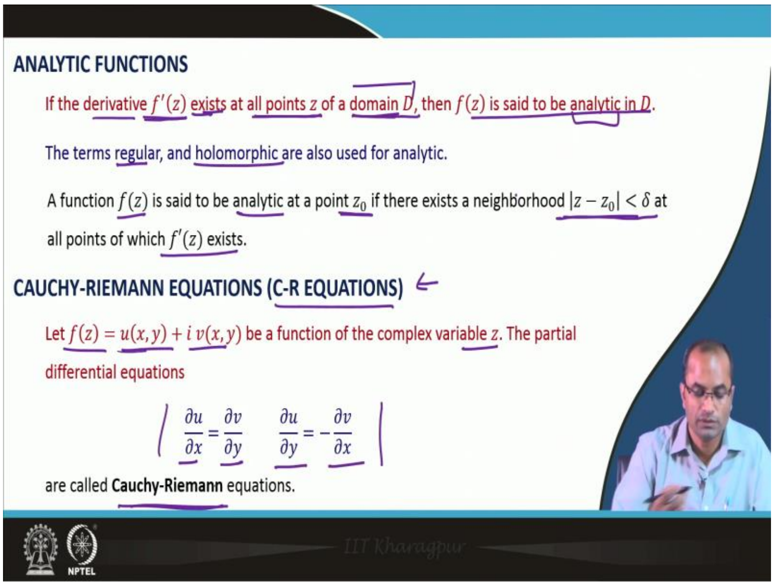

So, there is extension of this differentiability in the context of this complex variables or complex functions and we call it like analyticity. So, analytic function, so we will just understand now what is analytic functions what are the differentiable functions we have already seen.

So, if the derivative f prime z exist at all points z in a domain in a domain D, then f z is set to be analytic in the domain D. So, looking at this now, we do not see any difference on differentiability and analyticity. So, if the derivative f prime z exist at all points z of a domain D, then we also call that f z as differentiable in D, but now we are talking about here that f z is analytic in D. So what is the difference?

So, let me just clear it out when we are talking about this domain, which is open domain usually we consider whenever we are considering a domain. So here, if this derivative f primes z exist at all points z of the domain, then the f z is also differentiable in D f z is also analytic in D.

So there is no difference as such. And the term regular and holomorphic are also used for analytic functions, the main difference will come when we talk about a point that f z is differentiable at a points z naught and f z is analytic at a point z naught then the real difference of this analyticity and differentiability will come into the picture.

So, here the f z is set to be analytic at a point z naught if there exists a neighborhood, so that is important that and this shows exactly that analyticity is little more than differentiability, differentiability does not imply analyticity. Because, the function is said to be analytic at a point if there exists a neighborhood at all point of which f z exists meaning for analyticity f z, the functions should not only be differentiable at z naught but it should also be differentiable in the neighborhood of z naught so, that is the difference where we can understand what is analyticity and what is differentiability.

For differentiability at a point we have just defined that if that limit exists in the previous slide, we have seen that caution that limit exists, so the function is differentiable at z naught, but now for analyticity, the differentiability at z naught is not sufficient, but if the function is not only differentiable at z naught but also in its neighborhood there exists a neighborhood, neighborhood can be very small, but there should exist a neighborhood at all of these points in the neighborhood the function should have a derivative it should be differentiable, then we call the function is analytic.

So, there is a well famous well known this Cauchy Riemann equations these are called the C-R equation, which are used for verifying that the given function is analytic or not without going through all the fundamental definitions of differentiability and then we have to for analyticity at a point we have to also check whether a neighborhood exists where the function is again differentiable then only we can claim that this is analytic function.

So this C-R questions help to certain extent to identify that this function is analytic or not. So here is the statement so if f z u x y plus i ui be a function of complex variables z, then the partial differential equations this ux the partial derivative of u with respect to x is equal to partial derivative of v with respect to y and partial derivative of u with respect to y is equal to minus times the partial derivative of v with respect to x.

So, these equations are called the Cauchy Riemann equations. So, first this is just the definition of the Cauchy Riemann equation. So, for a function f z, u x plus i v y if we have these 2 functions 2 equations are satisfied or these 2 relations are satisfied and these relations are called the Cauchy Riemann equations.

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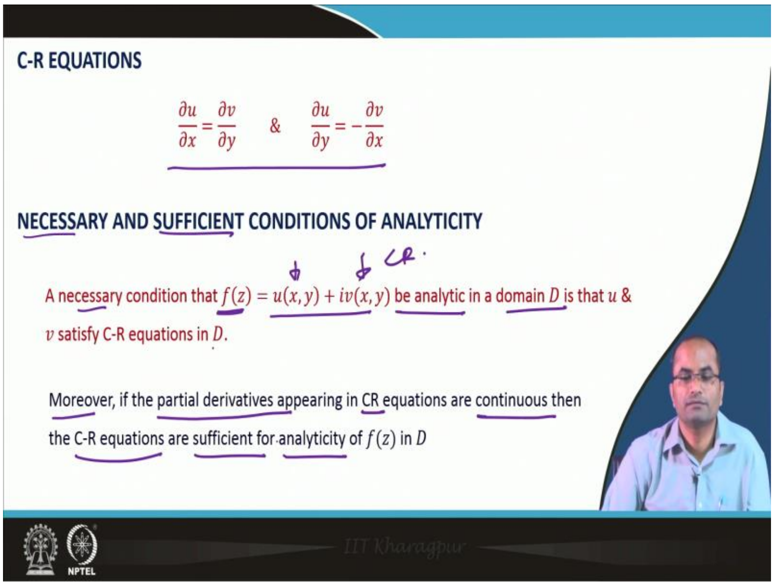

So, now next so we have the C-R equations. Now, the important result of this discussion is the necessary and sufficient conditions of analyticity. So, a necessary condition for that this f z be analytic in a domain D is that the u and v satisfy C-R equations in D. So, this is the necessary condition that this function will be analytic in a domain if this u and v will satisfy C-R equation, So, what do we mean by this necessary condition? It is clear that if these conditions are not satisfy, if C-R equation are not satisfied, then this function is not going to be analytic.

Indeed, we will see in the proof itself, it is not about the analyticity, it is also about the differentiability. So, if the C-R equations are not satisfied at a point the function will not be differentiable or here we are talking about the whole domain. So, if the C-R equations are not satisfied in this domain, then the function will not be differentiable even forget about the analyticity because analyticity is always more, I mean either they have the same notion if you are talking about the domain, otherwise the if we are talking about a point, then analyticity is more than differentiability.

So here are the necessary condition for differentiability or for this analyticity in a domain D is that, that u and v must satisfy the C-R equations, so they must satisfy the C-R equations otherwise this function is not going to be analytic, this is what the necessary condition means. So, this condition will be very useful to prove that a given function is not differentiable or it is not an analytic.

The second is where we have the sufficient condition that if we find that the partial derivatives appearing in the C-R equations means this u x v y u i and v x if they are continuous, then C-R equations are sufficient for analyticity of f z so, we have the condition for under which we are sure that the function will be analytic or function will be differentiable.

So, these are the conditions that the C-R equations all the derivatives appearing there these first order partial derivatives if they are continuous then the C-R equations are sufficient to prove to claim that the function is analytic on definitely differentiable.

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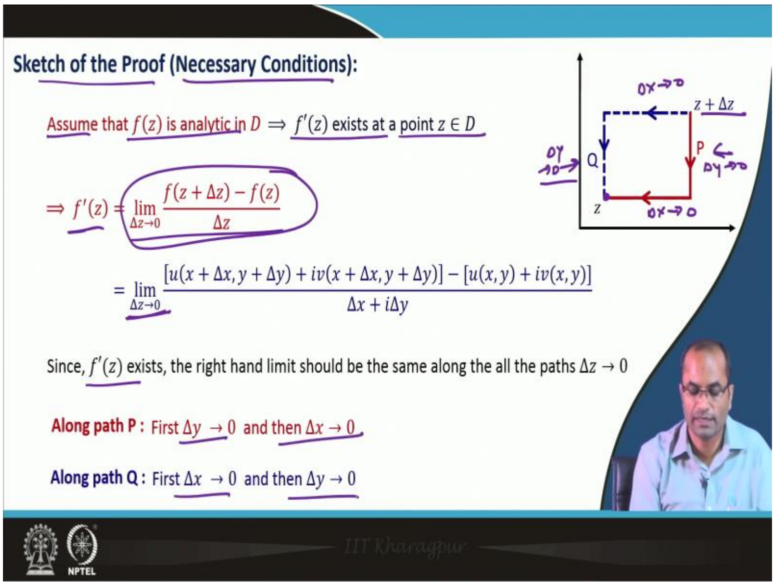

Just we will go through a quick proof, where we will just see the necessary conditions, because the proof for the sufficient condition is a little more involved. So, we will skip here. So, assume that this f z is analytic indeed that is the assumption that means, the f prime z exist at all points z in this domain D that means, this quotient has a limit which we are calling a derivative f prime z exist then this f we will right in this component y, so u and i v and then we have v and u x y v x y and divide by delta z which is delta x plus this i delta y and the limit this delta z approaches to 0.

So, since we know that the f prime exists that is our assumption that f z is analytic which implies of course, that the f prime exists the right hand limit should also be same along all the path we take we approach this delta z to 0, because if this limit exists if this limit exists, then irrespective of the path, the limit should be the same that is f prime z. So, here we approach through 2 paths this deltas z to 0 and then we will conclude those C-R equations.

So, these 2 paths we have taken one path is p here then other path is Q, so we are in the P we are first letting delta y to 0 and then we are so here we have delta y2 0 first and then delta x to 0. And in this part we are letting delta x to 0 first and then delta y to 0. So, these are the 2 paths along these 2 paths we will approach from this z plus delta z to this z and we will see that what comes out. So along this path p, as I said there delta y to 0 then delta x to 0 along path Q, we have delta x to 0 and delta y to 0.

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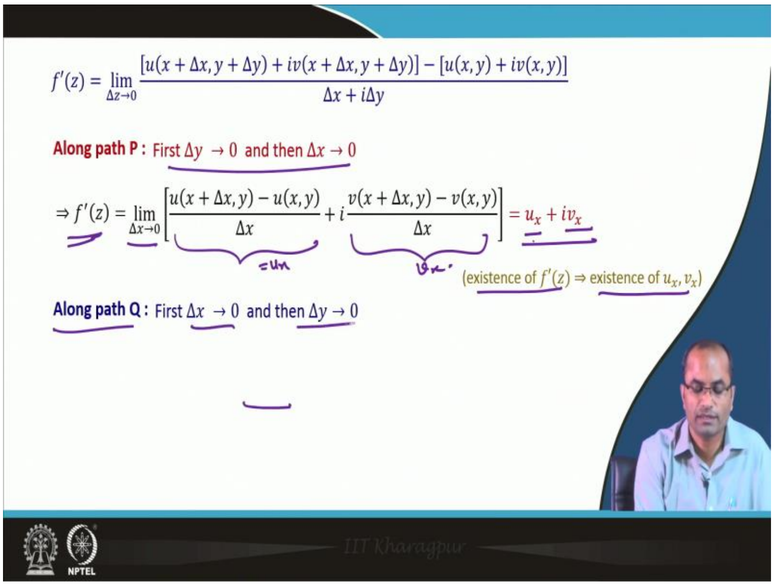

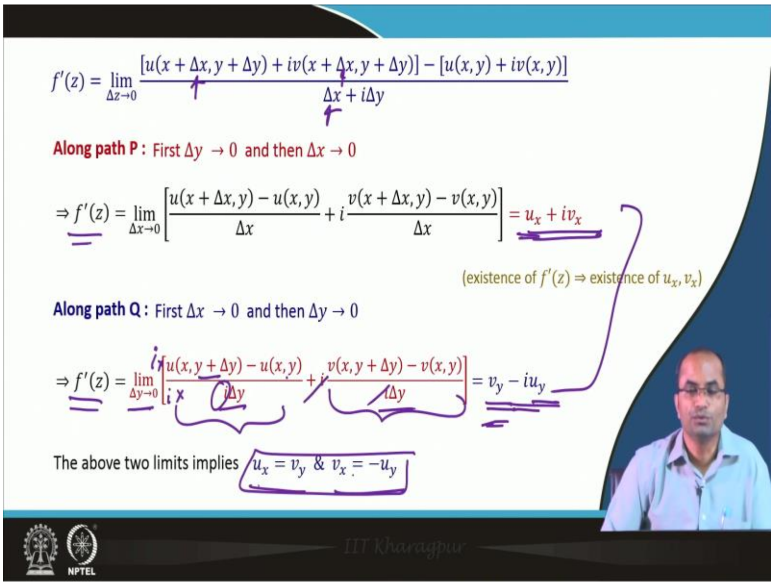

So let us take this expression for the limit and then along path P, so first we will add delta y to 0. So putting their delta y to 0 first and then later on this delta x to 0, so what we will have we have this expression here, when delta x to 0 this we know and this we know these are the partial derivatives of u and this v that means this is u x and this is v x, this is ux and this is v

x. So if this f prime z exists, so naturally, this implies here also that the u x and v x has to exist because this limit is equal to u x plus i v x along a particular path and along any path if we go the limit should exist because we know that the limit exists it should be same f prime z. So this u x plus i v x is equal to this f prime z.

That means this u x and v x both exists. So the existence of f prime z also shows here the existence of these partial derivative. Anyway, now we go along with the path Q where dx we will let first to 0 and then delta y to 0. So in this case, again the f prime z equal to this limit where we have delta y to 0, and delta x is already set here in this case, this delta x0.

So we have this i delta y, i delta y there and we are taking the limit delta y approaches to 0. So in this case, again we will see that here the variation is in y so we will get the partial derivative with respect to y when we take delta y to 0 and that i we can adjust with this minus i by multiplying i in numerator and denominator.

So, here the delta y, so in this case i i goes to 0 and delta y goes to 0. So, this is v y and then this will go, be u y and we will get minus i there because of this i when we multiply by i and divide by i there to this first expression. So, we will get this equation. So, what do we get that f prime z has a value u x plus i v x and also it has a value v y and minus u y. So, these two are same that and when two complex numbers are same, so we can compare their real and imaginary part.

That means u x must be equal to v y, the u x must be equal to v y and the other one the v x must be equal to u y. So, from here we got this conditions that, if the function is differentiable at a point, then these 2 conditions must be satisfied. If this one of this conditions is not satisfied, then the function will not be differentiable at that point and hence it will not be analytic at that point. So this are the necessary condition for differentiability or analyticity and very useful for proving that the given function is not differentiable or it is not an analytic.

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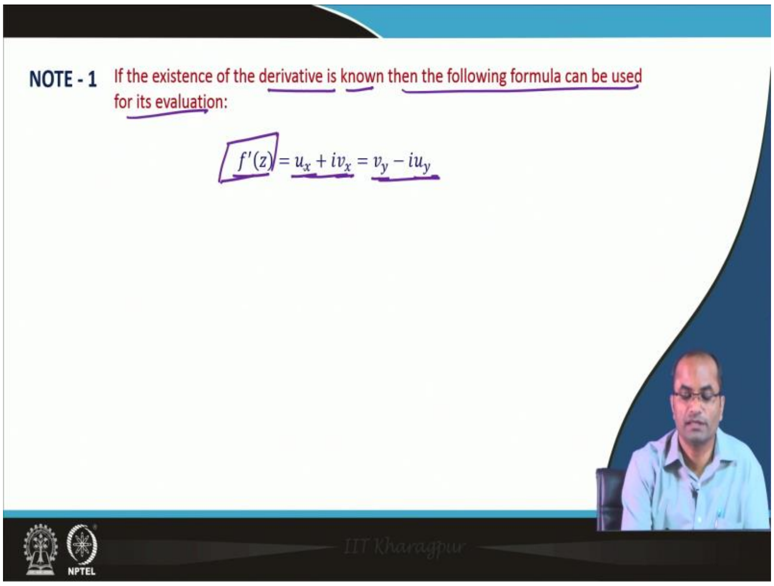

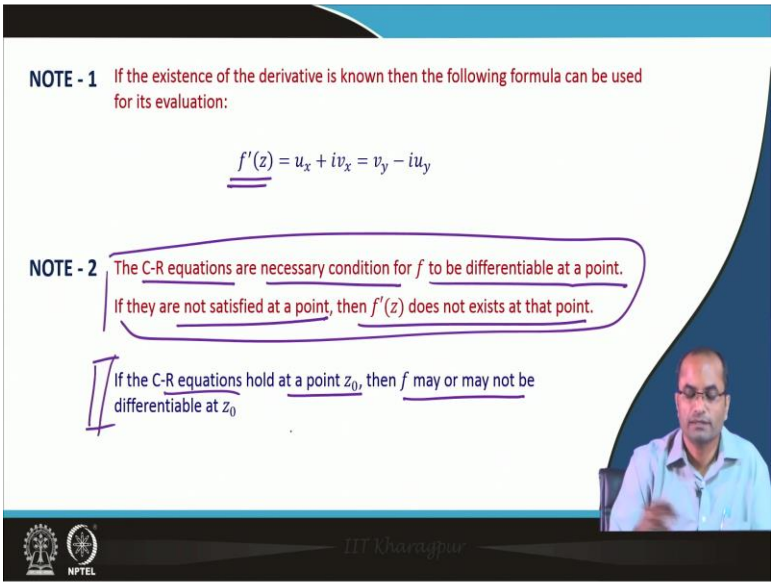

So, here just 2 quick notes, so here we have if the existence of the partial derivative is known, then the following formula can be used for its evaluation for evaluation of the derivative. So, if we know u x v x or we know v y u y just in the previous slide we have seen if we know that the derivative exists. If we know that the derivative exists, then this derivative we can compute just directly by having this partial derivative with respect to x or with respect to y.

So, this is a nice formula for getting the derivative without the fundamental definition for instance, the note 2 the C-R equations are necessary condition for differentiability at a point and they are not satisfied that f prime z does not exist at that point. So, the function is not differentiable at that point and C-R equations hold at a point z naught then f may or may not be differentiable, because these are necessary conditions not sufficient condition for sufficient condition we need additional restriction of continuity of these partial derivatives.

So, if the C-R questions hold then we cannot say anything whether the function will be differentiable or analytic at that point or it is not analytic at that point for that we have to go for the further test that is the continuity of this u x u y v x v y and so this C-R equations are very useful for proving that the function, the derivative does not exist, but for proving that it is differentiable or analytic we have the C-R equations are not sufficient, they are just necessary.

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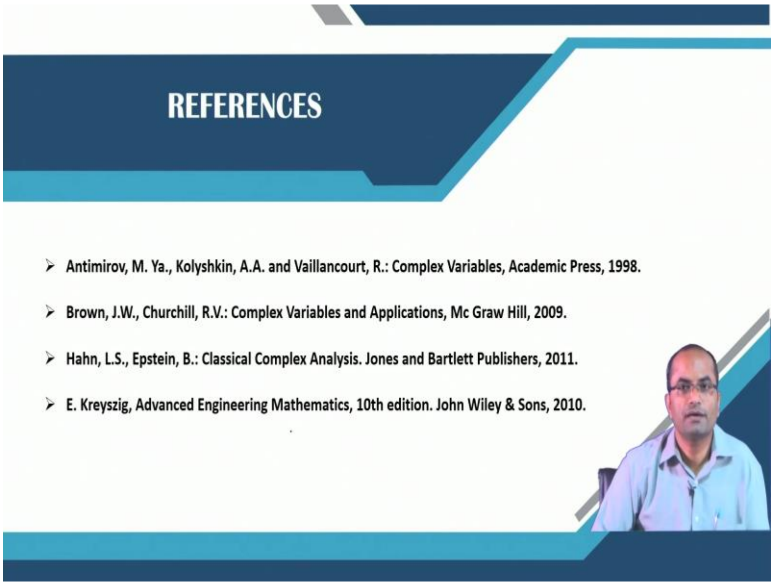

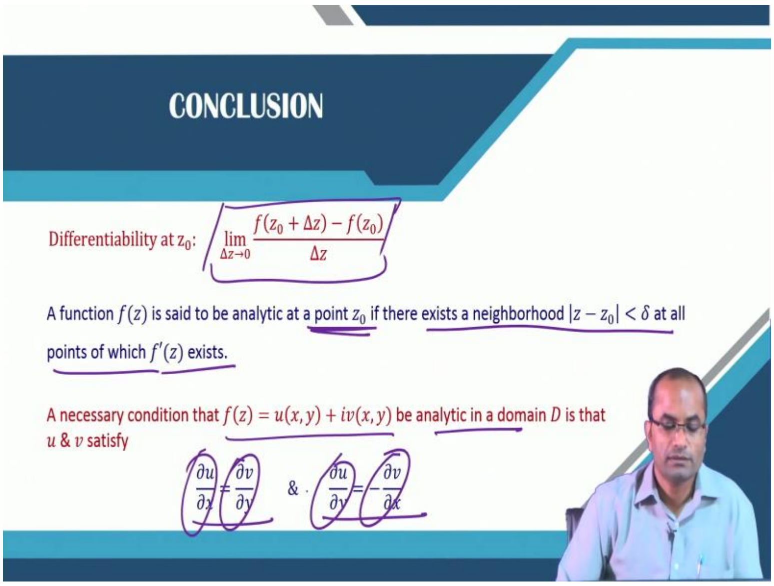

So, these are the references we have used for preparing this lecture and just to conclude here, so we have talked about mainly about the differentiability at z naught where we can we observe that if this quotient exists this is then the function is differentiable and this value is called the derivative of the function and the function is set to be analytic at a point z naught because here were the 2 definitions differ, so for the function is set analytic if there exists a neighborhood at all points of which f prime z exists So, the differentiability is little more than just sorry analyticity is little more than the differentiability.

So, which can be observed this when we are talking about a point and the necessary condition we have learned for analyticity or differentiability at a point is that these C-R questions must (())(36:20) satisfied. And if these partial derivatives appearing here are also continuous, then these conditions become sufficient for claiming that this function is differentiable or even analytic. So, that is all for this lecture and thank you for your attention.