Engineering Mathematics II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology Kharagpur
Lecture 11
Complex Number and Functions

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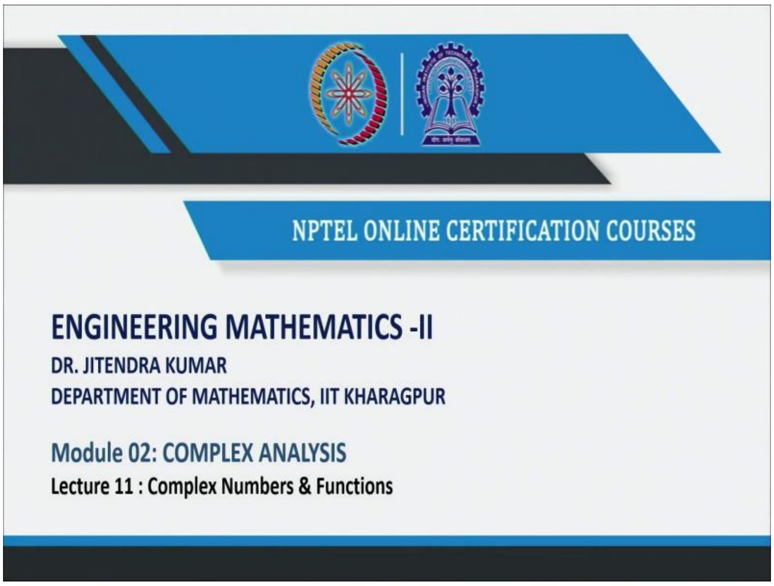

So welcome back to lectures on Engineering Mathematics 2 and today we will begin with the module number 2 that is on Complex Analysis. So this is lecture number 1 on Complex Numbers and Functions.

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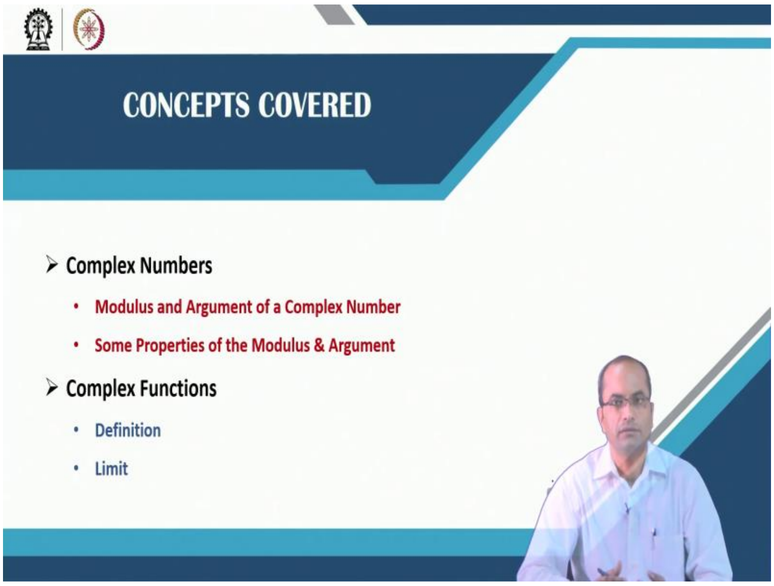

So today we will cover, we will introduce the complex numbers. So this is going to be a review because you are the, you are familiar with the complex number by now. So this modulus and the argument of a complex number will go through and some properties of the modulus we will cover and then we will come to the complex functions and in particular we will be talking about what are the complex functions and their limit.

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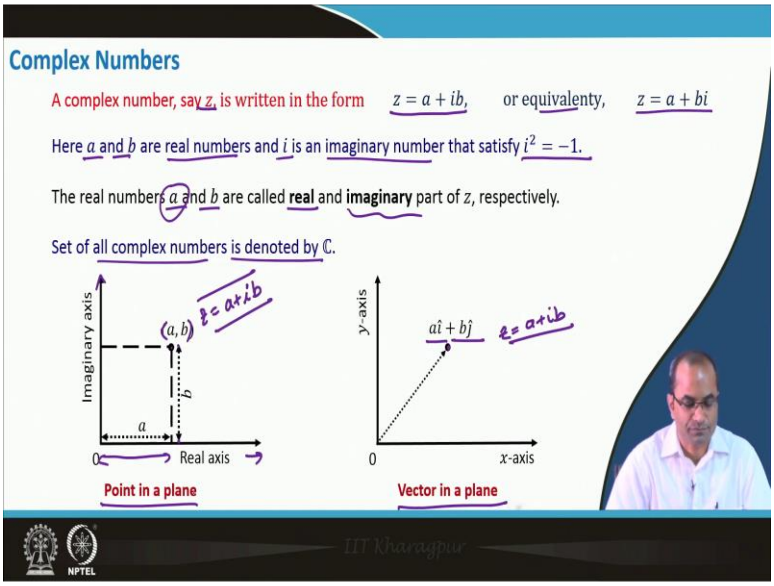

So coming back to the complex numbers, so a complex number we usually denote it by z. This is written in the form z equal to a plus ib or equivalently we also write z is equal to a plus bi. So ib or bi, where this a and b are the real numbers and i is called the imaginary number which satisfies i square is equal to minus 1.

So these real numbers a and b are called real, so the a is called the real and the b is called the imaginary part of z respectively. So all complex numbers will be denoted by this C that is the set of all complex number and just to show this geometrically, so there are two ways, one we can think is as a point in this 2 dimensional plane.

So this is, we call here in this connection, or in this context the real axis and this y axis we will be calling as the imaginary axis, so here we have a, the component in the real axis and the b which is the imaginary part of z we have this distance from the real axis to this point b, so this a point in this 2 dimensional plane we can treat it as a b which is the complex number we have denoted by this a plus ib.

The other way we can actually geometrically represent this would be in a vector setting, which vector we have already covered. So this a b this point or this z is equal to a plus ib, we can think as a vector pointing to this point a b that means ai plus bj. So i and j are the unit vectors in the direction of this x axis and y axis. So we can also define this in as a vector in a plane, or we can define this complex number as a point in a plane.

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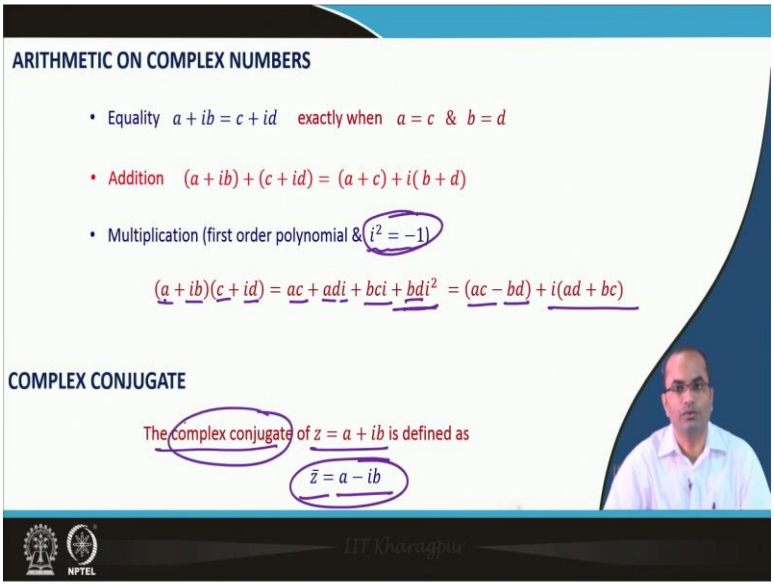

Well, so some arithmetic on complex numbers to get familiar again or just to recall some of the properties. So equality means if a plus b is equal to c plus this i, this imaginary number id then this is only possible when we have both the real parts that is main, that means a equal to c and also this imaginary part that is b equal to d. So these two numbers are equal if we have a equal to c and b is equal to d. The addition we can define again, we have to add the complex, say sorry the real part as well as the imaginary part separately.

So if we have two numbers here a plus ib and then c plus id so their sum will be the addition of the real number that will form a real part of the addition and this ib will be added to this id so that will form the imaginary part here. So a plus c and plus i b plus d that will be the addition.

So coming back to the multiplication we have to just consider that this i square is equal to minus 1 and we will go with the product as we do for polynomials. That means a plus ib, and c plus id if you want to multiply then we will do the usual multiplication we follow in case of the polynomials.

So a will be multiplied by c, so we have ac, a will be multiplied by id, so we have ad and with the i then we have i with this, ib with c so we have this product here, bc product with i and also we have ib and then we have id there so this is i square multiplied by this bd and i square is minus 1 so this is like minus bd. So we can write down ac minus bd, ac minus bd, and with this i we have ad and bc.

So it is a very simple, usual multiplication, one we just considered that this i square will be replaced by minus 1. So similarly we can define the multiplication, so we have defined already. We can also define the product which we will come little later to define that.

So first let me just introduce this complex conjugate because that is another important part here to be discussed. So the complex conjugate of this z is equal to a plus ib, the number a plus ib is defined as the conjugate means this i will be replace this by minus i, so that means we have the complex conjugate a minus ib. This is the complex conjugate of the number z, a plus ib, this is what we call this complex conjugate.

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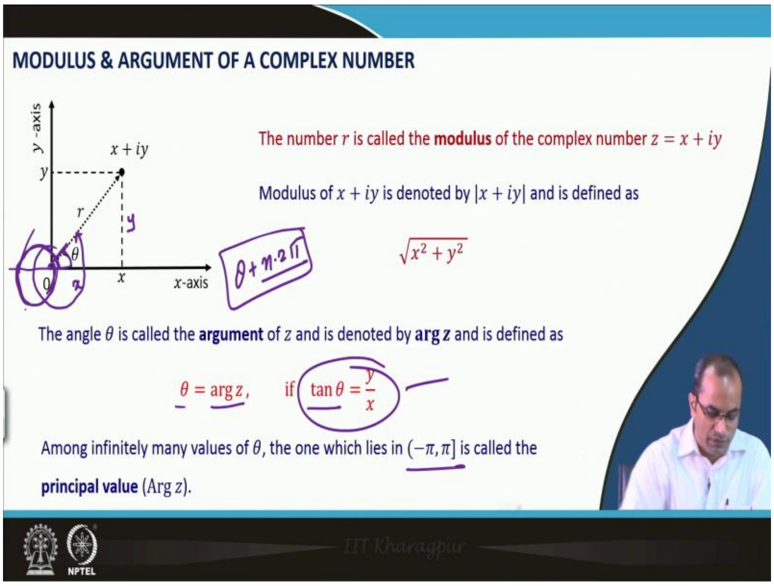

Now what are the modulus and the argument of a complex number? So let us suppose this is the point here x, y or the number this x plus iy in the plane, x axis, this is the y-axis here. So this is the distance here x and this is the distance here y. Therefore this point is x plus iy, or x comma y whatever ways we define this here.

So the distance from the origin to this point will be denoted by this number r which is going to be, since this is x and this is y here or this point is x comma y so from this 0 0 to x comma y the distance will be x square plus y square and the square root.

So here this r, the distance from the origin to this one is important and then this theta, so just recall how do we characterize in the, in the polar coordinate, so similarly we have here, this is theta, the angle from this x axis and then we have this distance r. These are two, two numbers which, with the help of these two numbers we can again define the given complex number.

So now this number r is called the modulus of the complex number, and the modulus of this x plus iy is denoted by this sign, absolute value x plus iy and this is defined as the square root x square plus y square. As I discussed that this r is the distance from origin to this point which is nothing but the square root x square plus y square. So r is x square plus y square, square root, and this is what written here, this is the modulus of this complex number x plus iy.

The angle theta which is given here, this line makes from with the x axis, this is called the argument of z and this argument is denoted by arg z and it is defined as that theta is the argument of z. Theta we know now, what is theta, that is the angle here, this axis, this line is making with the x axis.

So theta is the argument and since this is the distance x here, this is the y so we have tan theta is equal to y over x. This relation must hold for this theta and since if this is the angle theta, if we go again with 2 pi, if we add 2 pi we will be back to this angle theta. So if theta is the argument then theta plus 2 pi is also the argument or theta into, so theta plus this n into 2 pi, n is an integer you take. So if this theta is the argument, theta plus 2 pi into n will be also the argument, that will also satisfy this tan theta is equal to y by x and representing the same angle with this rotation.

So there are infinitely many such values of theta but we will just chose between this minus pi to pi. So if theta is from this x-axis to up to this pi here, the angle is coming here from 0 to pi we will take and then otherwise in this direction we will go with this.

So from here to here we will cover with the pi, 0 to pi and then in this direction we will cover with minus 0 to pi, so any theta we can have between this minus pi to pi and this is what we called the principal value. So this theta which is between minus pi to pi is called the argument.

So the principal value of argument which is usually denoted by this big Arg z so this is what we call the principal among infinitely many values of theta because, as we discussed that we can add 2 pi, we can add 4 pi, again we are back to the same angle. So among all these if we choose the one which lies between this minus pi to pi we will call this as principal value of argument of z. So that is going to be just the unique value.

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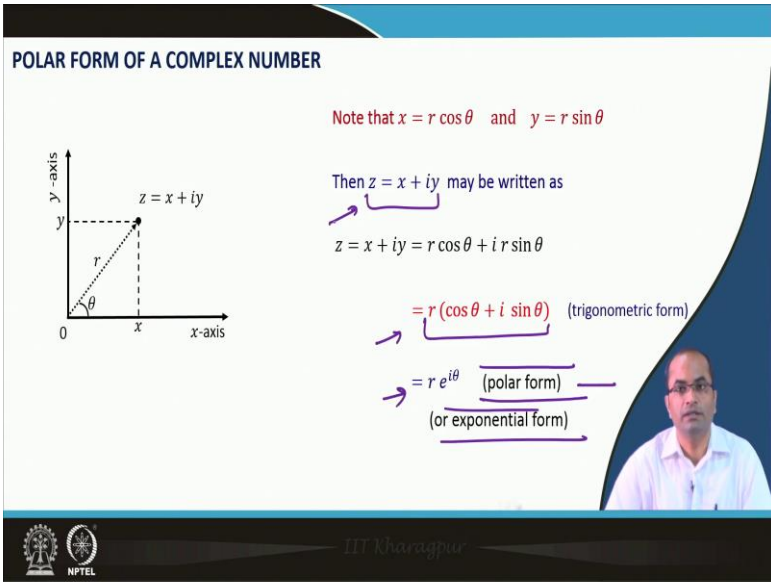

Well, so the polar form of complex number we will define with the help of this argument and this theta z so not that this x, because this is the x distance here and this is the y distance. So we can have this relation that x is equal to r cos theta and y is this r sin theta from this triangle we can identify this relation that x is r cos theta and y is r sin theta.

Then this z equal to x plus iy can be written as, so with this relation we can write down z is equal to x plus iy, x is r cos theta, y is r sin theta and if we take this r common we have cos theta plus this i sin theta, this is the trigonometric form we call of the complex number z which is represented here by this x plus iy.

So the trigonometric form will be r into cos theta plus i sin theta. Or we can replace this cos theta plus i sin theta with this r into e power i theta. So e power i theta, this Euler identity we have the cos theta plus i sin theta. So we can write this as r e power i theta and then this is also called the polar form or the exponential form of the complex number.

So we have the three forms, the one the standard one z is equal to x plus iy, then we have the trigonometric form and then we have the polar or the exponential form which is written here in r e power i theta, the trigonometric form r cos theta plus i sin theta and then we have the usual, in the Cartesian coordinate, so x plus iy.

So now we are familiar with the polar form of a complex number, the modulus, the argument, so we can go through some properties of complex numbers which will be used later.

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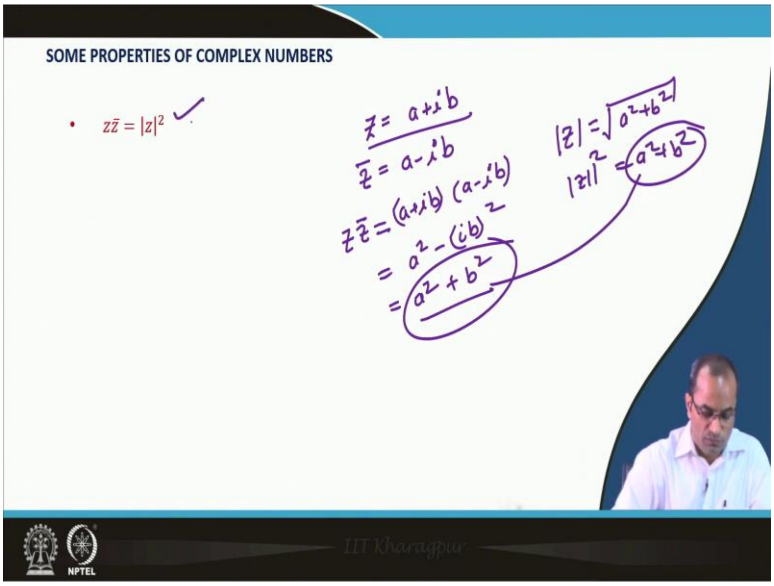

So again this is to recall because many these properties are already discussed in the complex numbers so and we begin to learn. So here this very nice, important property we have that z and the, if we take its conjugate, so this is equal to the absolute value of z square. So this can be easily seen that if we take z is equal to a plus ib for instance, this number and we know z is equal to a minus ib and if we look at this product here, zz bar that is going to be a plus ib and the product with a minus ib and if we do this product, so it is a plus ib, a minus ib that means we a square minus ib whole square.

So a square and then minus i square, so i square will be 1 so we have the plus here b square. So that means this is a square plus b square, and if we look at the other side that is modulus z is given which is, once the number is a plus ib the modulus z is a square plus b square. That means this modulus z square is a square plus b square. So we do see that both are equal. That means z bar is equal to a square plus b square and that is equal to mod z whole square, so these two are equal. Well, so we will erase this here now.

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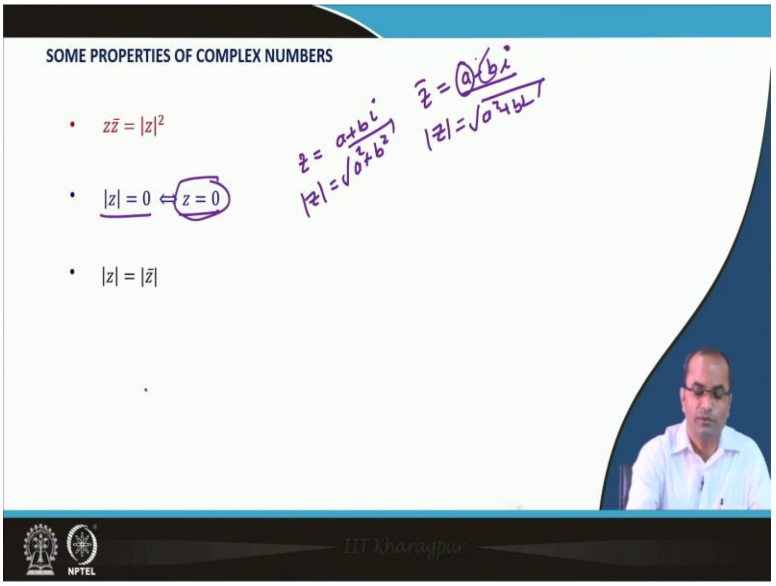

So the next we have this property that the absolute value of z equal to 0 which is only possible when this z equal to 0 or if z equal to 0 we have the absolute value must be 0. So this is clear we have already seen what is the argument. So argument was distance there from the origin to that point and that will be 0 if exactly you are at the origin.

So if you are just away from the origin so there will be some distance and that cannot be 0. So the 0, z is equal to 0 implies that the absolute value is 0 and the absolute value of complex number 0 implies that this number is 0.

So here we have the absolute value z equal to the z bar. So this is also trivial because we have z a plus ib for instance and this if we take the absolute value this is a square plus b square and the square root and similarly if we take the conjugate that is a minus this bi, and if we take now the absolute value so this is going to be the same, a square plus b square. This square and plus, minus b square you can say but it is b square only so these two values are same.

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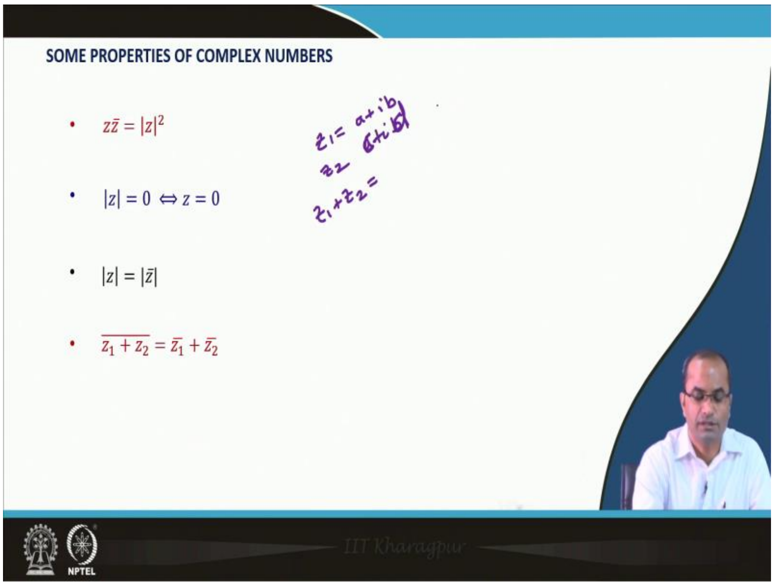

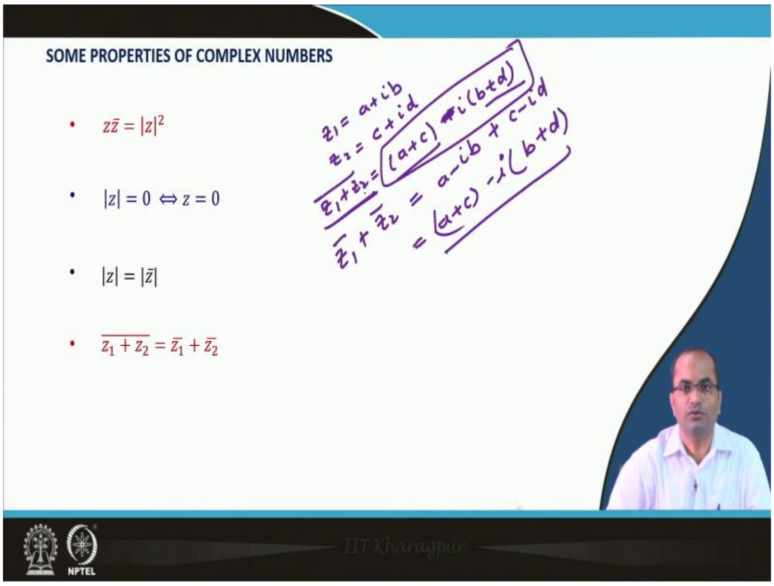

So this z, the absolute value of z or the modulus of z is equal to the modulus of its conjugate. Regarding this property so the again we can prove similarly because if we take the two number again a plus ib and a minus ib as your z2 and if we add the two, z1 plus z2 so here let us say, c and d. So z1 is a plus ib and then we have c plus id. So again we will take. So a plus ib and z2 we are talking about a new number, so c plus id. And if we add the two, so z1 plus z2, we know how to add so a plus c and then we have i and b plus d.

Now we are talking about its argument. So we can take the argument here, that is, this square plus this square and the square root, we are talking about and similarly we can get the individual one here, that is, just the absolute value so we can take the absolute value of this, so this plus will becoming just the minus there and when we talk about the individual conjugates so again we have a minus ib and then plus the c minus id.

And result would be the same because we have a plus c and then minus i if I take common here, b and plus d. So this and this is the same. So therefore we have the conjugate of z1 plus z2 equal to conjugate of z1 plus conjugate of z2.

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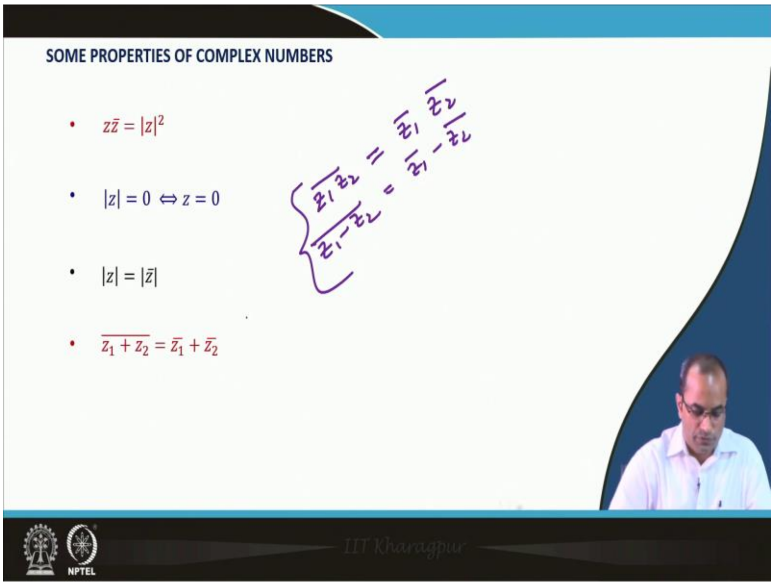

Similarly we can also prove that the z1 z2 if the, we have the product, so again z1 and z2, the conjugate one can prove this also for the minus, one can prove that z1 minus z2, the conjugate is going to be z1 conjugate minus z2 conjugate so these two also we can easily prove as we have done for the plus.

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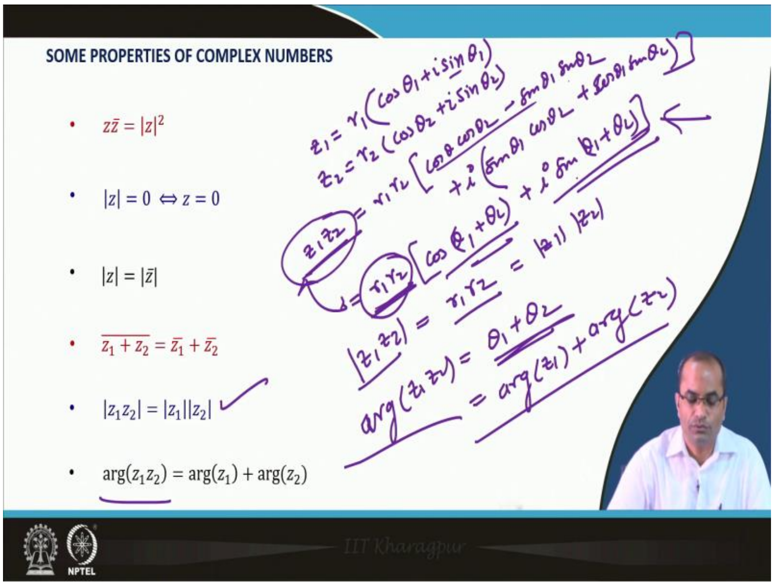

Coming to the next property so we have the product here, z1 z2 and equal to this modulus z1 and this modulus z2 so concerning this property so we will take a z1 for instance in the polar form. So r1, or in the trigonometric form, so cos theta 1 plus i sin theta 1 and the z2 we can take r2 and then cos theta 2 plus i sin theta 2. So these are the two different numbers, z1 and z2 and we are talking about the product so z1 z2, this product we can talk here, so this is r1 r2.

This product will come and the product of the two here so that will be cos theta 1 then we have cos theta 2 and then only I will consider first, so this i i will be minus so minus sin theta 1 and sin theta 2 will be there, then we have with the i here sin theta 1 and cos, so sin theta 1 cos theta 2, then we will have plus, so here cos theta 1, cos theta 1 and sin theta 2.

So this we will get out of the product. That means we have r1, r2 and if we combine this trigonometric identity so we have cos theta 1 and this plus, plus this theta 2. And then we have plus i and this is sin theta 1 plus theta 2. So what do we get out of it? That, so we have this relation that z1 z2 is equal to r1 r2 and cos theta 1 plus theta 2 i times sin theta 1 plus theta 2. Now looking at what is the, so this z1 z2 is another complex number which is again in this form r cos theta plus i sin theta.

So we can easily look into this now that the z1 z2 the absolute value, the modulus of this is going to be just r1 r2, because this is the modulus part. You have r cos theta plus i sin theta. So then we have this r1 r2 and r1 r2 is nothing but the absolute, the absolute value of z1 and this z2 or the modulus of z1 z2. So this is proved now that modulus z1 z2 is equal to modulus z1 plus modulus z2 with this relation.

The another property which again we can directly look at, that is the argument z1 z2. So here again from this relation we can, we can find out, we can realize that the z1 z2 is equal to this argument. So the argument of, argument of z1 z2 again from this relation is nothing but the theta 1 plus theta 2. And then from here theta 1 plus theta 2 we know already that the theta 1 was the argument of z1 and theta 2 was the argument of z2.

So we have this relation as well. So these are two important identities which may be used later and this relation here plays a crucial role that the z1 z2 is equal to r1 r2 and theta 1, this cos theta 1 plus theta 2, i sin theta 1 theta 2. Indeed if z1 z2 are same so we will get here like z for instance, so z square will be r square and cos 2 theta plus i sin 2 theta. Or we can generalize these results when we have z power n then here r power n cos n theta plus i sin n theta also we will get instead of this. So this equality we can generalize and we can refer to some other properties based on this relation.

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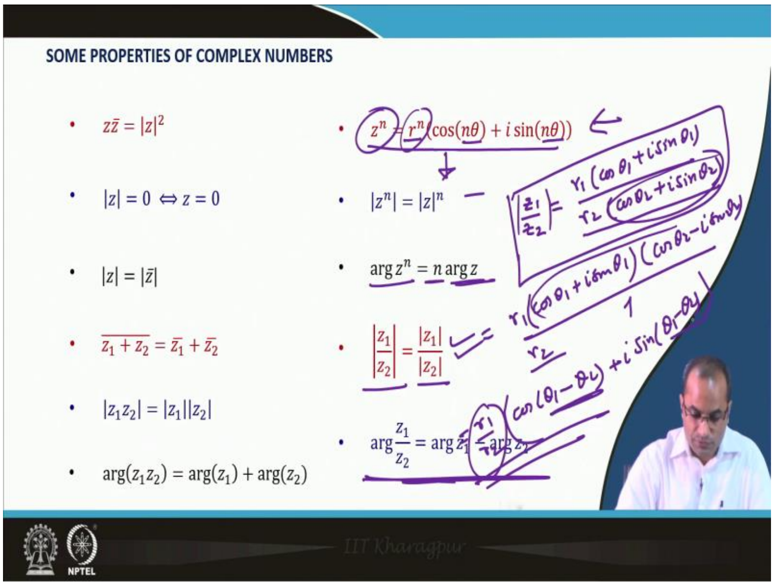

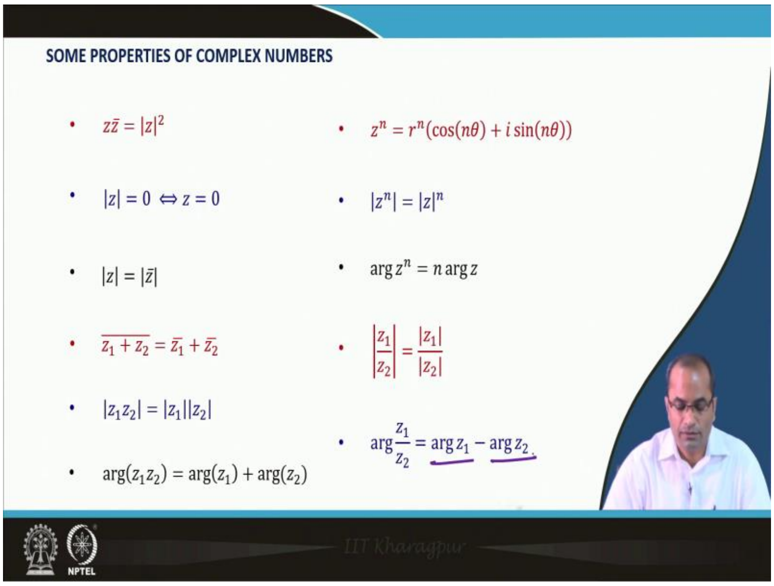

So let us moving further now, yeah this is exactly what I was talking about from this last identity the product of the two number which were written in trigonometric form. We can directly conclude from there that z power n, so we have instead of z1, z2 now the same number z, so here the product of r was coming so now we have r power n because we are talking about n numbers and cos.

So there it was theta 1 plus theta 2 for two numbers. We have the same number which is multiplied n times. So here n theta and then we have again n theta there. So this is the identity which is just coming from the previous one. And this is of course the implication of the above result because if we take the modulus of this, so modulus of this is nothing but the r power n, and r is the modulus of z. So this is just the coming from the above identity. Or similarly the argument of z power n if we want to get, so we have this n theta, n and theta is the argument of z.

So this is also directly coming from this identity. If we talk about the z1 divided by z2, the argument z1 divided by argument z2 will be appearing here. The idea is again same so we can take that trigonometric form. So z1 divided by z2 we have r1, we had there cos theta 1 plus i sin theta 1, and then we have here r2 and then cos theta 2 plus i sin theta 2.

So with this relation we can again rewrite this. So we can multiply here, divide by the conjugate and also we can multiply by its conjugate. So when we divide here by this conjugate here cos theta minus i sin theta, so what will happen. So if we do so, so just one more step. So cos theta 1 plus this i sin theta 1 and if we multiply by this conjugate that means cos theta 2 minus i sin theta 2 and divided by, so here it will become with minus, so cos theta whole square and then minus i square sin theta whole square.

So this minus i square will be again this 1, so i square is minus 1. So that will be just 1 there and then we have r2. And we can multiply there the numerator part, what we will get finally that r1 r2 is nothing but, we have cos theta 1 minus theta 2 plus this i sin and theta 1 minus theta 2.

So this relation we will get and this we can directly now get that the argument or the modulus of this z1 divided by z2 is nothing but r1 by r2, r1 was the modulus of z1, r2 the modulus of z2. So we have proved this identity and similarly we can also look at this one where argument z1 over z2 we are looking now. So argument is theta 1 minus theta 2, so theta 1 is the argument of z1 and then the argument of z2. So many such properties we can just prove with the knowledge of this multiplication we have just learnt there.

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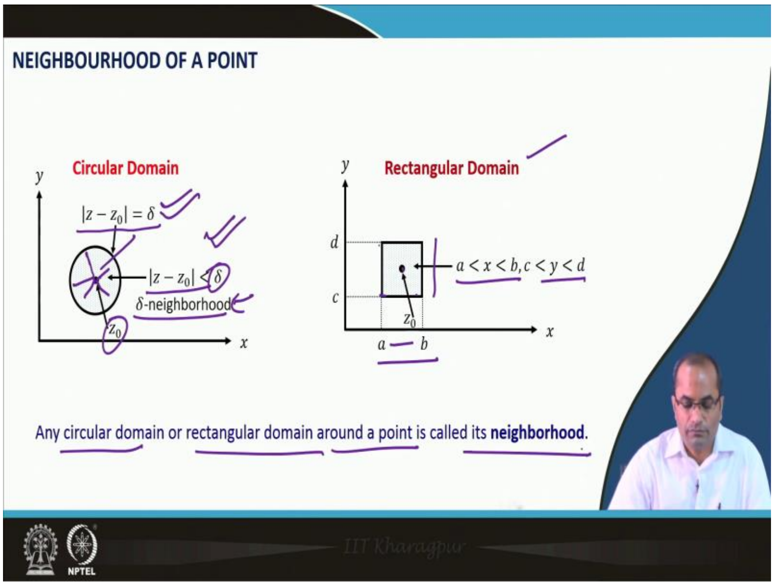

So we directly now go to the function, the complex function and some of this its properties will be discussed. So before that we have to talk about the neighborhood of a point. So the neighborhood in a circular domain, is a circular domain about this z0 and whose radius we have taken this delta, so this is called the delta neighborhood.

So it is a circular region around this z0 with radius delta, it is a disc here and this is what we called the neighborhood of the point, the delta neighborhood of a point. The delta is the radius. Usually we talk about very small number delta which define the neighborhood of this point.

So this boundary here, the z minus z0 is equal to lambda, the inner part is z minus z0 less than delta because the distance to any point there inside is less than delta. Any point we can take and when we are on the boundary, this distance the, is equal to delta therefore this is, this defines the boundary, the circle z minus z0 is equal to delta.

So another could be the rectangular domain. Some for some convenience sometimes we can also think about the rectangular domain. So this is the neighborhood of this point z0, rectangular one, a b is this distance and d minus c, so b minus a and d minus c. So around this we have a rectangle with these distances here, from a to b and then there c to d.

So that could also be a rectangular domain which is defined here but the circular domain is more common and will be used frequently whenever we are talking about the neighborhood, so any circular domain or rectangular domain around a point is called its neighborhood that is, in simple words we can think of a neighborhood.

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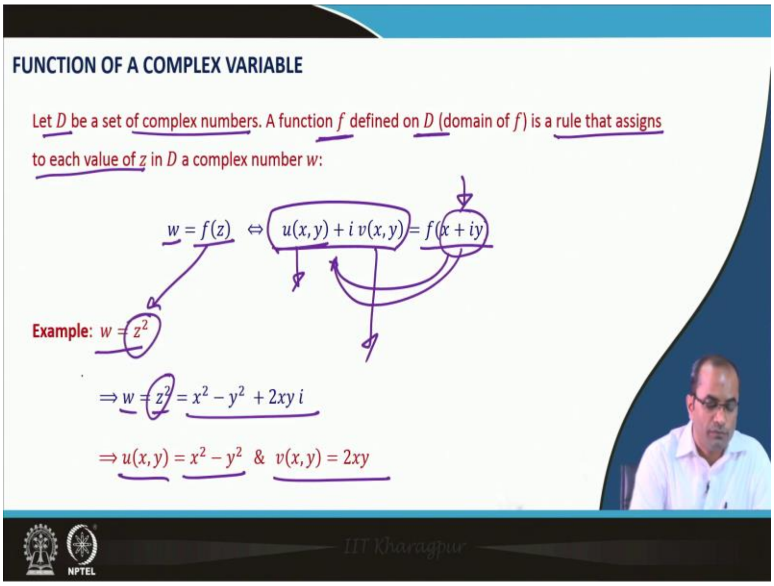

Now defining the function of a complex variable, so let us suppose this D is a set of complex number. So it is a part of the full complex number C and a function f is defined on D which is, usually we call the domain, when the function is defined on this D, is a rule that assigns to each value of z in this set D, a complex, another complex number a w.

So this f is taking as input from this complex number and again producing a complex number. So that is a function, a mapping from complex number to a complex number. Usually we denote this w is equal to f z or we write it as f. So x iy has a input to this function, a complex number and then we are getting again a complex number u which is a function of x y plus i v x y.

So from complex number to complex number, that is the mapping this function is doing now. So for example w is z square, if we talk about w is z square so its f z is here z square and if we write in the expanded form so z square, z is x plus iy and if we perform this product we are getting this x square minus y square plus 2 xyi. So here we have this u, u is x square minus y square, this function and v is the function 2 x y. So we can always write in u plus iy form, given function or in this compact the z square form.

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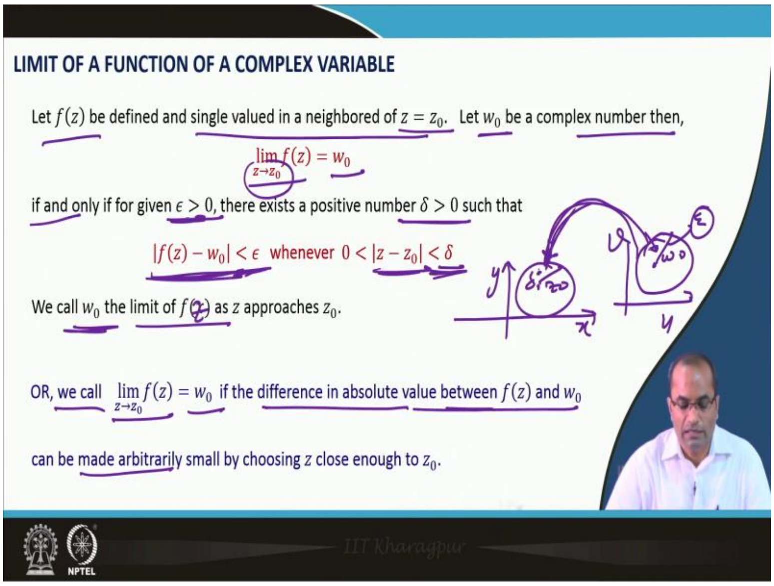

Coming to the limit of function of a complex variable, function of a complex variable. So f z is defined and is a single-valued in the neighborhood of this, z is equal to z0. So the function is defined in the neighborhood of z0 point and we can talk about the limit now. So let this w is 0 be a complex number such that the limit of this f z as z approaches to the point z0 we are getting the complex number w naught.

What does that mean? If and only if for a given epsilon so this is more mathematical definition we give for the limits, so if for given epsilon there exist a delta such that this f z minus w, the difference between this f z and minus w naught, the distance between the two is less than epsilon whenever we take the z from the neighborhood of this z naught. So the distance between z and minus z0 is less than a delta naught, the distance between this is less than epsilon.

So for given epsilon, for any given epsilon, for any given epsilon we can find such a neighborhood then we call that this is the limit. So this is exactly what we consider this limit in the real analysis. So there is no difference. And we call this w naught the limit of this f x, f z indeed when z approaches to z naught.

Or we call in more a simple words that this limit f z, z approaches to z naught is w naught. If the difference in absolute value of this f z and w naught can be made arbitrarily small by choosing this z close to a z naught. So we have the two neighborhoods here, the one is around this z naught point with delta radius and there is another in uv so here we have x y, let us say here we have the function f z we are talking about, the w naught and this is the neighborhood, this epsilon neighborhood.

So for any given epsilon however small if we can find neighborhood here around this z naught so that this f z, any point in this neighborhood assigns the values in the neighborhood of this w naught then we call that this is the limit of this function.

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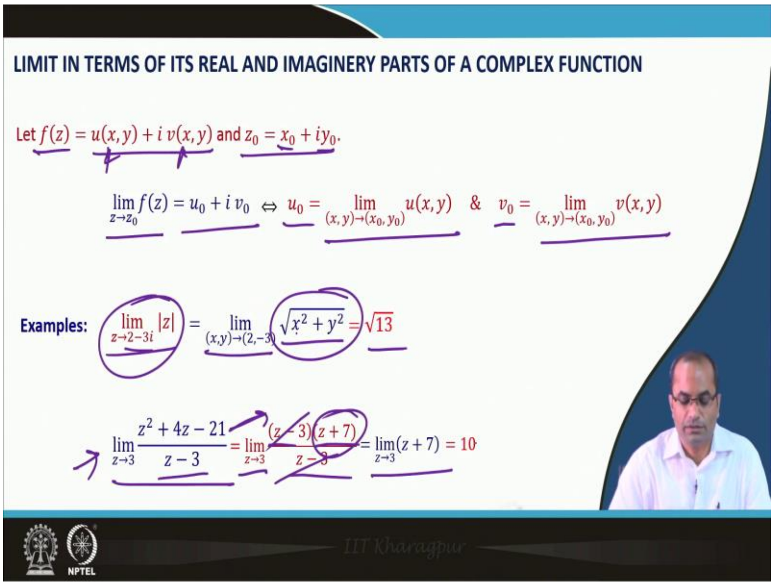

Well, so limit in terms of real and imaginary parts of a complex number we can define. So if f z is u x y plus i v x y and this z naught is x naught plus y naught then we can say that f z as z approaches to z naught is u naught plus i v naught, and what is this u naught and v naught? These are the limits of this u and v as x y approaches to x naught y naught. So this u naught is this limit, function of two variables and this is the limit for this v function as x y approaches to x naught y naught.

So for example if we consider, we want to get this limit z approaches to 2 minus 3 i of this absolute value of z, so we know absolute value of z is x square plus i y square so this is only the real part, and we will just compute that what will happen when x y approaches to 2 minus 3 of this function which is a straight forward so 4 and plus 9, that is square root 13, that is the limit of this absolute value of z as z approaches to 2 minus 3.

So when z approaches to 3 we have this z square 4 z minus 21 divided by the z minus 3, we want to get. So this is another approach we can follow. So we observe that the numerator can be written as z minus 3 and z plus 7 divided by this z minus 3. So this z minus 3 gets canceled and when z approaches to 3 this z plus 7 is getting just the value 10. So we have this limit as 10 for this given function.

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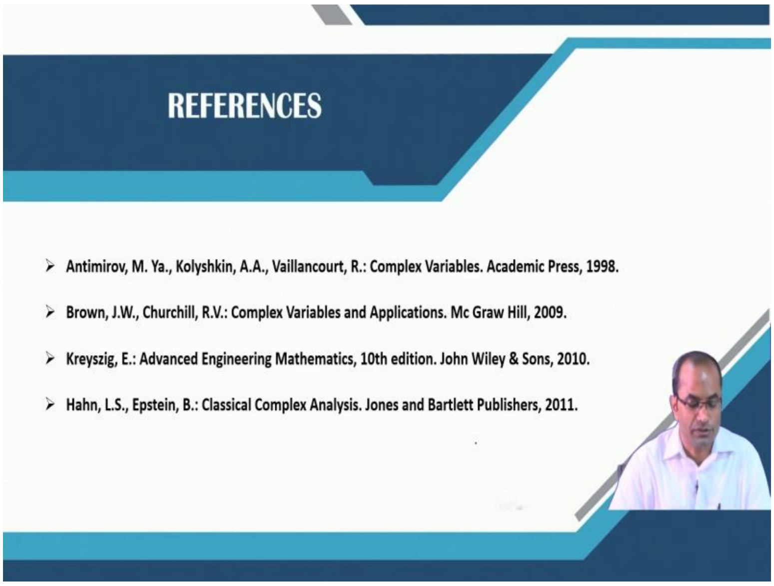

So these are the references we have used for preparing this lecture.

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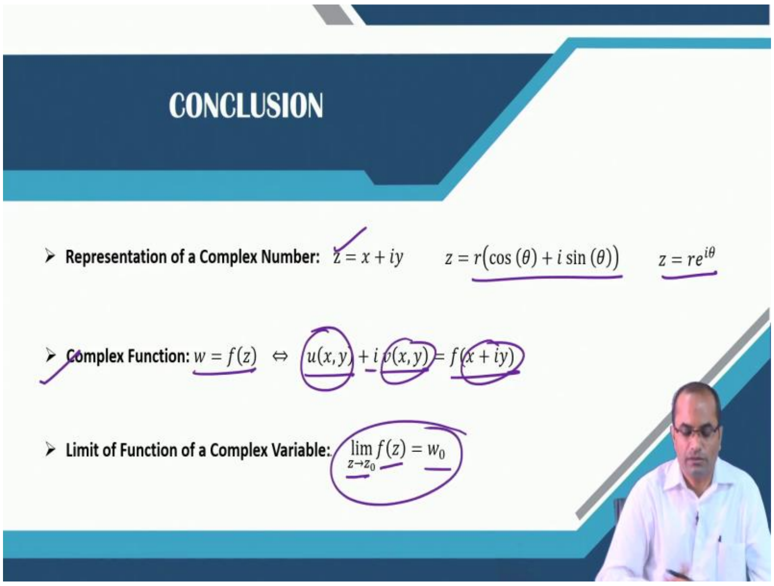

And just to conclude we have learnt that how to represent a complex number and there were two ways we have discussed that we can think as a vector in a plane or we can think as a point in a 2 dimensional plane. We have also seen this trigonometric representation of the given function and as well as this exponential representation of a complex number.

And then we discussed about the complex function, w function of the z f, so this may be also expanded as a u x y and this i v x y and f, the z is written as x plus iy. So in this form and we have also discussed that what you mean by the limit of a complex function, so f z when z approaches to z0 we have w naught, so we have also seen the epsilon delta definition and its physical interpretation. So that is all for this lecture and thank you for your attention.