Engineering Mathematics II
Professor Jitendra kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 10
Divergence Theorem

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So welcome back to lectures on Engineering Mathematics-2. This is lecture number 10 on divergence theorem.

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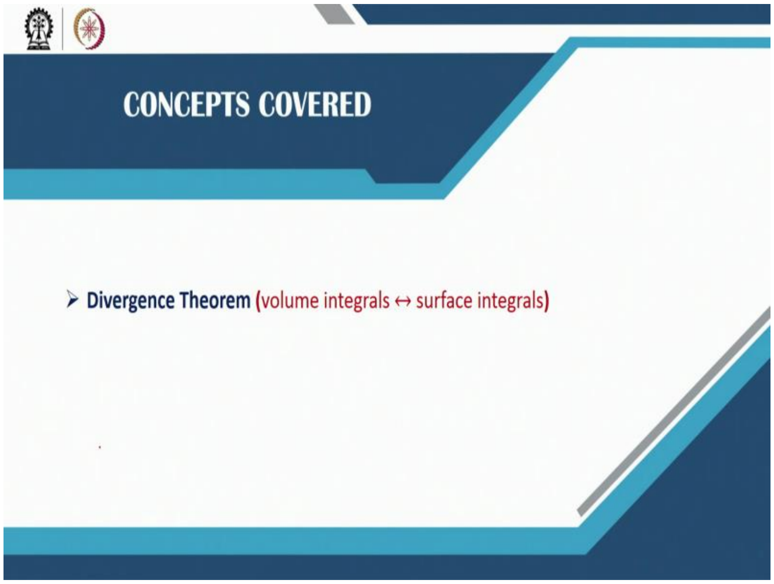

So, in this lecture we will be talking about the divergence theorem and that is basically the volume, how to get this from volume integrals to the surface integral. So, that connection you will see in this lecture.

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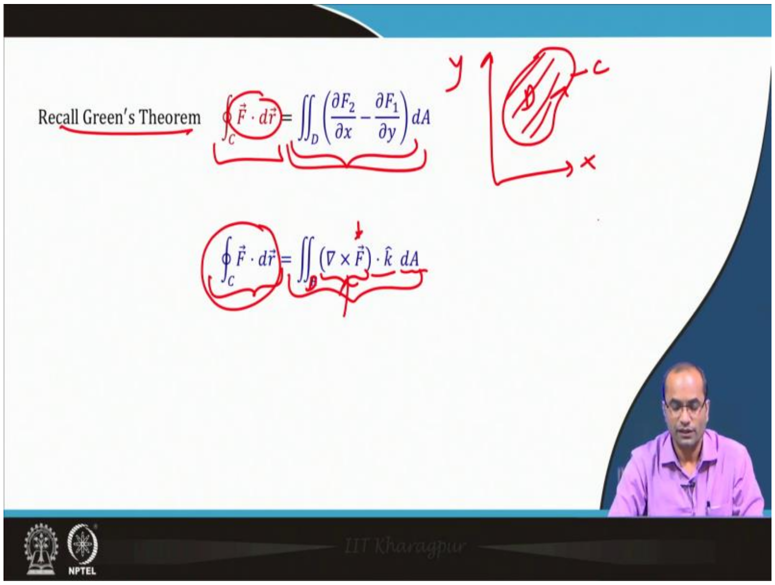

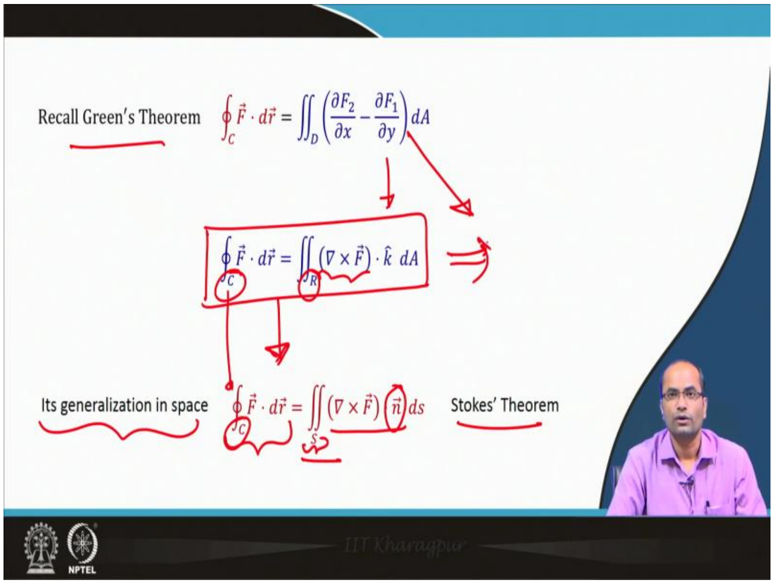

So just to recall that we have already discussed Greens theorem. So the idea was that if we have a closed curve here in 2 dimensions then we can interchange here this closed, the integral over this closed curve to the area integral which is done over this region, which is covered by the curve C. So, this was the Greens theorem and we sometimes call this Greens theorem in a plane because we are talking about these 2 dimensions.

And just to recall that this Greens theorem can be also written in this form, so we can put this D or we can put there R, so this R or D is the region covered by this curve C, so this can be written in terms of this curve of this vector field F. So again this curve integer or the lining integer is equal to this curve and dot product with K and this integral has to be done over the region bounded by or enclosed by this curve C.

So, with this form of the Greens conclusion we were able to extend it for 3 dimensions where the curve can be in a 3 dimensional space and then this R which was the region in the plane will become a surface. So that is what we have seen, its generalization in space, this curve integral. So, now the C is in this space and this the area bounded by this curve, which is a surface now, we can do the surface integral and this N was the normal to the surface. So, this was the Stokes theorem, which we have already discussed in the last lecture.

Now, what we will do that this is one form of writing this Greens theorem, where this extension was possible, this generalization was trivial to observe. What we will do now there is another form which we can again rewrite this Greens theorem, instead of this curve we will use now the divergence and from that form which will be written in the divergence, we can again extend to another result which is called as the divergence theorem.

So, there the extension will be again to 3 dimensions, but the extension, the type would be different because here the extension was the 3 dimension but the curve was a curve again 3D and that area which was in 2D in this case, was a surface. Now, the extension which we are talking about in this lecture will be that this curve integral will become a surface integral in 3 dimensions and then this region which was the surface in this Stokes theorem will become a volume integral.

So that is another generalization or the extension of the Greens theorem which we will study today, which is the so called the divergence theorem.

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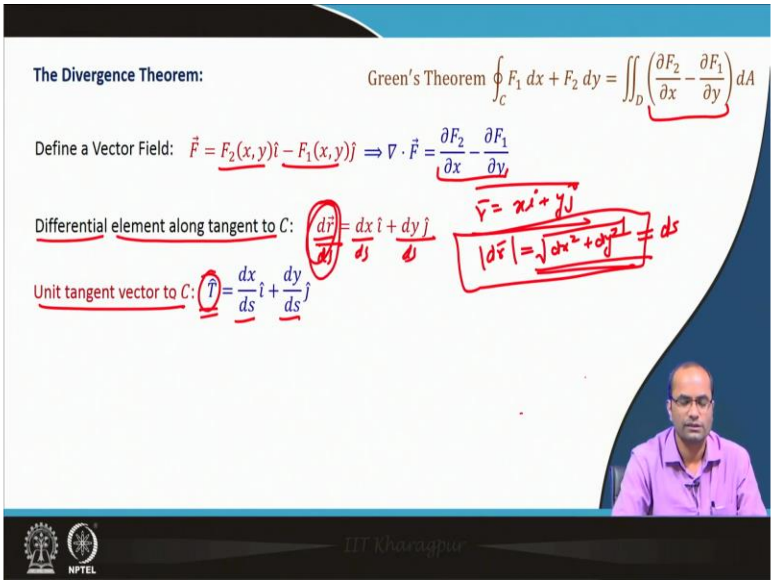

So, let us discuss the divergence theorem. So, we have this again I have written this Greens theorem once again. So, we define here the vector field F in such a way that we have these 2 components F2 and the minus F1. So, the minus sign is taken just to get actually the desired form of the Greens theorem, but we can actually work with like F1i and F2j as a standard notation.

But here just to show exactly that we are getting this form which we have written in the slide, so we will take this vector field and the 2 dimensional again. So, F2 xy, the first component and the second component we have taken this minus F1 xy . Having this if we compute this divergence of this F, so what will happen? The partial derivative of F2 and the minus the partial derivative of F1 with respect to y will come as a result of this divergence.

So which is precisely the integrant of this area integral and that is the reason we have chosen this kind of form for this vector field F. So, if we know already, all this knowledge we have gained in previous lecture that the differential element along the tangent that means, basically the dr we are talking about the derivative of R, so that is dx and dyj, we can write down because R is usually xi plus yj.

So, from here we can dr, the differential element on the tangent is S dxi and the second component is dy.So, having this if we divide by its length, so the length of this dr or the magnitude, so we have like dx square plus this dy square. So, we can write down this dr as dx square plus this dy square and if we divide this by this factor, which is also we have defined earlier we can call it as ds, so ds is the incremental element on or the differential element on the arc.

So, if you divide by this factor ds to dr and dx and dy. So what we will get? This is going to be the unit tangent vector because this is along the tangent and we have divided by its magnitude. So, now the magnitude of this dr over ds will be 1. So, that is the tangent, the unit tangent vector we are calling it. And we have dx over ds and then we have dy over ds because we have divided the whole equation by ds here and then here and then here. So this dr over ds is the unit tangent vector here we are denoting just by this d hat.

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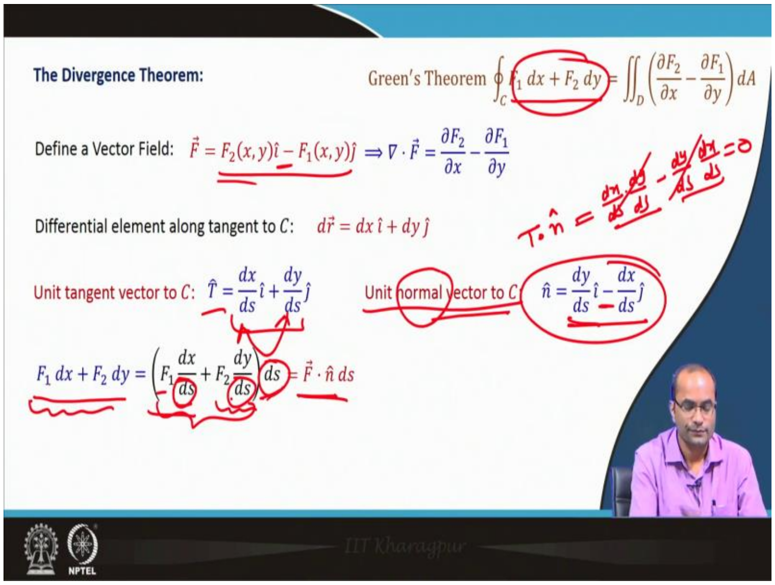

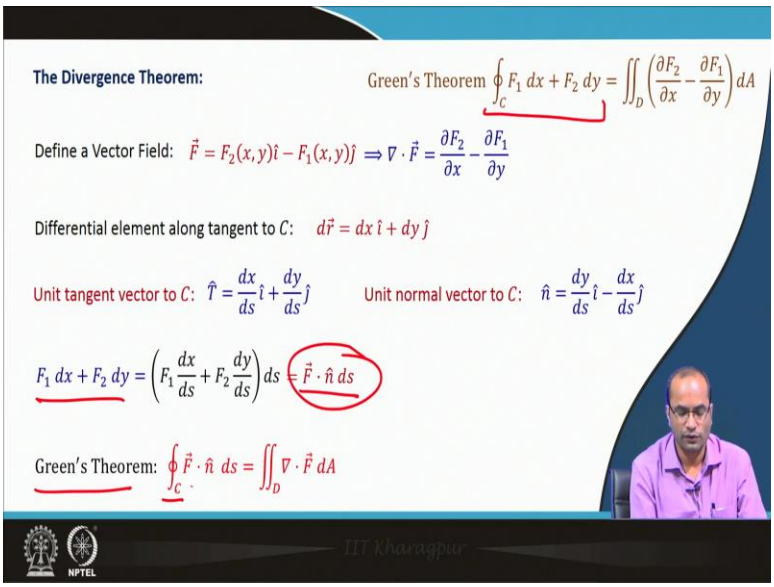

And now, so this is the unit normal vector to C, we are going to get out of this tangent vector. So the unit, this normal vector will be the perpendicular to this tangent vector which is written here, dx over ds and dy over ds. So we can just swap this that is a trick and put a minus sign there. So this is normal because we can just check here the T dot n, the dot product of the two will be exactly 0 because we have a dx over ds from there and then dy over ds from this end, and then we have minus sign, so dy over ds and dx over ds.

So this gets cancelled because they both are the same. So we get the 0, so they two are perpendicular so this is tangent and this is normal. And since its magnitude is also 1, so we have precisely this normal vector to this C at a point xy, at a general point and then we can write this F1 dx plus F2 dy, the left hand side, the integrant of this left hand side in this Greens theorem, as F1 dx ds, F2 dy ds. So ds, ds and then we have here ds to compensate this differential element.

And now we can observe that this F1 dx over ds plus this F2 dy over ds can be written as the F, the dot product with the normal vector. So again, so F was this F2 minus F1 and if we do this dot product with this n here, so we will get F2. With F2 we will have a dy over ds which is given here, and then with minus-minus will be again plus because they are two minus here, so F1 will be with dx over ds. So, this is precisely F dot product with this normal vector to C and then we have this ds factor.

So this F1 dx plus F2 dy is F dot n and this ds. So the Greens theorem now we can write down or rewrite it again. So, this closed interval over this curve C, the closed curve C, the F1 dx and F2 dy is now F dot n ds. And so again going back here, so equal to the right hand side which was this area integral, partial derivative F2 with respect to x minus partial derivative F1 with respect to y was the divergence of F. So, here this integrant we have replaced with the divergence of F.

So, what we have? We have now this result which is a Greens theorem again but just a way of writing like earlier we have done in terms of the curl, now we have done this in terms of the divergence and then this normal vector has come into the picture. So, the curve integral of this F dot n ds is equal to the area integral of the divergence of F and over the enclosed region by the C.

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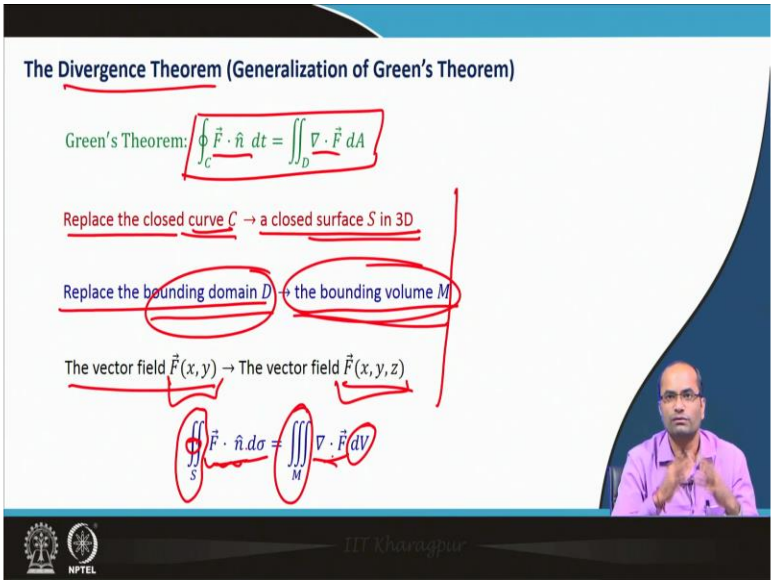

So, now moving further from this result we will come to the point of this generalization of the Greens theorem which is the so called the divergence theorem. So, we have just observed that we have the Greens theorem which can be written in this format and if we replace, so now we are going for the generalization without proof. So this is just one way of understanding this, the formal proof is this, pretty lengthy.

So, what we can do? We can replace this closed curve here. For the generalization, the curve will be replaced by a surface. So, we are exactly going into the 3 dimensions from the (1 dimension) from the 2 dimension. So, in the 2 dimension we have the curve now and now we are talking about a surface. So, a closed surface S will be replaced in 33 dimension. Now, again the closed surface. So the surface is closed now.

And we will replace this bounding domain. So there is a closed surface that means there is a bounded area or the, sorry bounded region in 3 dimension. So, that is the bounding volume we are mentioning here. So, this bounding domain D will be replaced by this bounding volume which is M and the vector field which was in 2 dimensions for the Greens theorem, we will now extend to the 3 dimensions.

So, having just these three consideration that this curve will be replaced by the surface, the bounding domain will be replaced by the volume and the vector field which was in 2 dimensions will be replaced for 3 dimensions, in 3 dimensions. So what we have as a result? We have F dot n d sigma. So, again this curve integral is replaced by the surface integral which is again over the closed surface and then we have F dot n d sigma.

So, the integrant is F dot n, here also we have F dot n. So, this n is again the unit normal to the surface which is here outward unit normal. And then the right hand side again we have divergence which is the same integrant. So divergence of F, but now we are integrating over the volume, over the whole region enclosed by the surface S.

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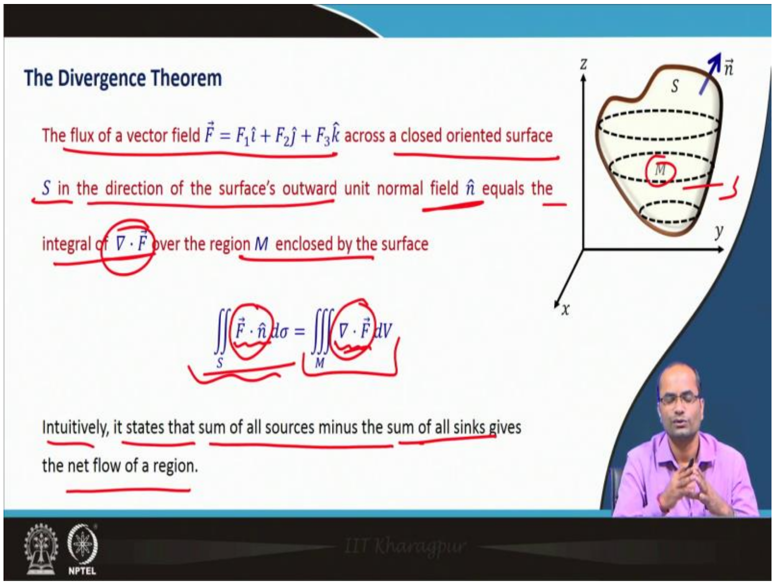

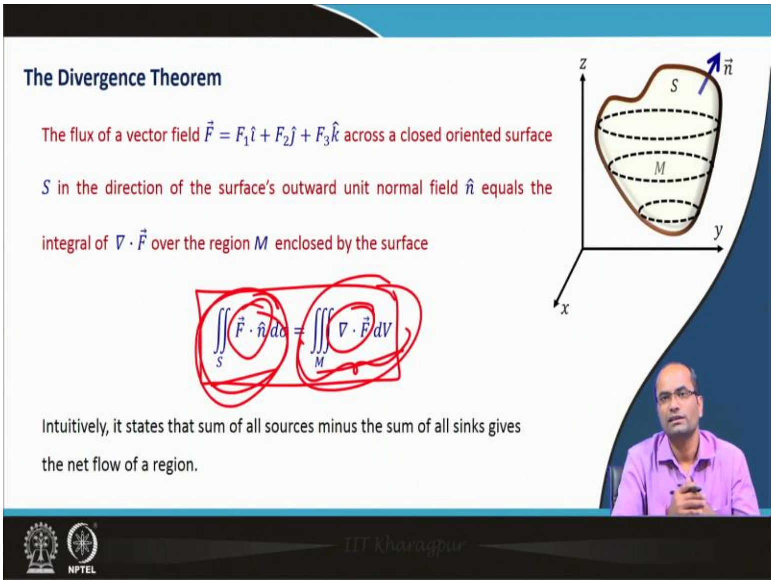

So, just to conclude again what is the divergence theorem, the flux of a vector field F1, F2, F3, the 3 components across a closed oriented surface, so these terms we have already discussed before. So we have a closed surface and it is oriented. So, we have the normal which is defined in 2 dimensions normally. That means we have the 2 surfaces which are distinguishable and in the direction of the surface outward unit normal n.

So we have the outward unit normal on this surface. This is equal to the integral of, so the flux means the F dot n. If we integrate the F dot n over the surface, that is equal to the divergence of S over the region M, which is enclosed by the surface. So I think the idea is clear. So, we have a region which is bounded by the surface here S and we have the outward normal, the unit normal vector N to the surface, M we are denoting the enclosed region which is a mass or the volume now.

And then we have this result which we have just discussed previously, as an extension of the Greens theorem that the surface integral of this F dot n, so this is precisely the component of this F in the direction of the unit normal and we are integrating over the surface which is equal to integrating the divergence of F on this volume which is covered by the surface the closed surface S. So intuitively it states that sum of all sources minus the sum of all sinks, this is what the divergence we have already seen this physical interpretation of the divergence, its tendency of the expansion or the compression.

So all these we are now integrating over the whole region and that is just equal to the net flow out of the region. So, we are just integrating over the surface at one end and the normal component of this flux. So, these 2 integrals are equal.

So that is a interesting result which we will use in now or apply for solving several problems, because we will observe that in most of the cases which I have demonstrated here in this lecture or I will demonstrate now, so this integral, the region integral, this volume integral is pretty easy as compared to the surface integral.

So, in some problems we will verify the divergence theorem that means the both of the integrals we will evaluate and some problems you will observe that this is quite complicated, having this flux over the surface but through this area, through this volume integral, we can easily compute this flux.

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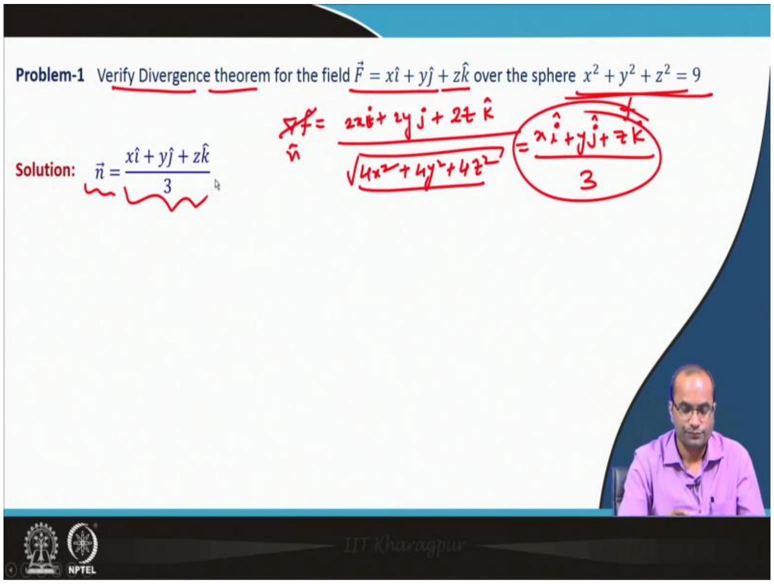

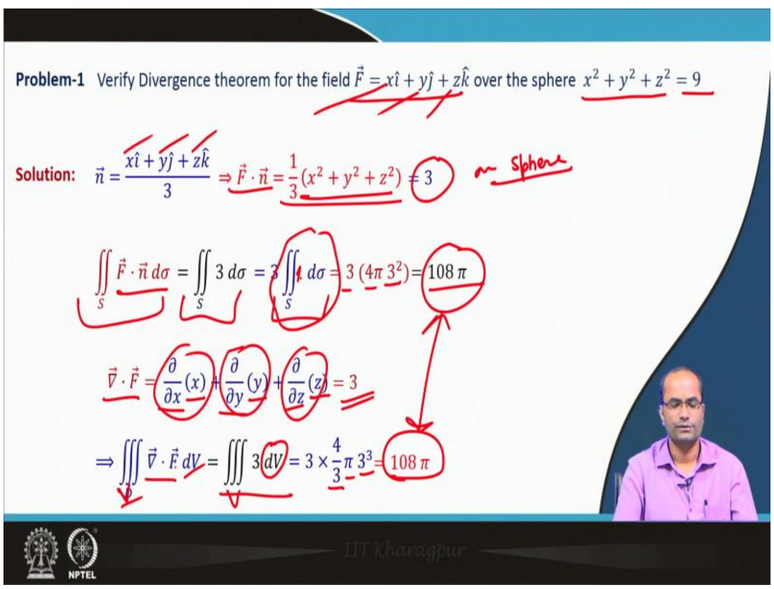

Okay, so coming to the example problem here, we will verify in this the divergence theorem for this field which is xi, yj and zk, so the 3 components over this sphere, whose radius is 9. So, we have the surface, we have the flux and then we will verify the divergence theorem. So, for that we need to get the unit normal vector because we need to compute F dot n.

So, unit normal vector on this sphere can easily be computed if we just get the gradient of this F. So, this is gradient of first this F, F is the given surface. So, if we compute this gradient we have 2x the ith component then we have 2y the jth and then 2z multiplied by this unit vector K. And to have the normal, so the unit normal we have to also divide by the magnitude of this one. So for y square and then for z square. So, these two can go out and that will cancel out from the numerator.

So, we have then the xi, then we have yj, then we have zk, this unit vectors and then we have this 2S is already settled, so we have x square plus y square plus z square that is 9, so which is 3 here. So, this is what we have written the unit normal vector is xi, jy and plus that zk divided by 3. So now, the F dot n we can compute, because we need in the surface integral. So F dot n, so F is given. xi, yj, zk and the dot product with this n will give 1 by 3, then we have here x square, then we have the y square, then we have here the z square.

So this is x square, y square and plus z square and which is 3 on the surface, on the sphere. Because on the surface x square plus y square z is 9, because in the surface integral, we will be integrating this F dot n over the surface, so this F dot n is 3. So because this x square plus y square plus z square is 9 there and 9 divided by 3 is 3. So this F dot n d sigma, the surface integral, we will evaluate now. We know this F dot n is 3. So its a very simple now. So 3 and this surface integral.

So 3 and just the surface integral with just 1 as its integrant. So that is just the surface area of the given surface and we know the surface area of the sphere. So, we have 3 times and 4 Pi R square, R is 3, the radius of the sphere is 3. So, we got this 108 Pi, the surface integral. The second we will evaluate that volume integral, so for that we need to get the divergence of F. So for the divergence of F, we have partial derivative with respect to x of the first component which is x, partial derivative of y, the second component of F, partial derivative of z and the third component of F which is Z.

So this is 1 here and again here also 1 and at this place also we have 1. So this is simply 3, the divergence of F. And now we can compute the other side integral of this divergence theorem, which is the divergence F integrated over the volume V, or we have written D here. So, this is exactly the 3 times this integral over the volume V. So, 3 is again the constant and this integral the triple integral will give the volume of the sphere which is 4 by 3 Pi and this R cube and again if we see this, this is coming to be 108 Pi.

So these two values are same. So this is the verification we have done for the divergence theorem. We will consider some more problems to practice now.

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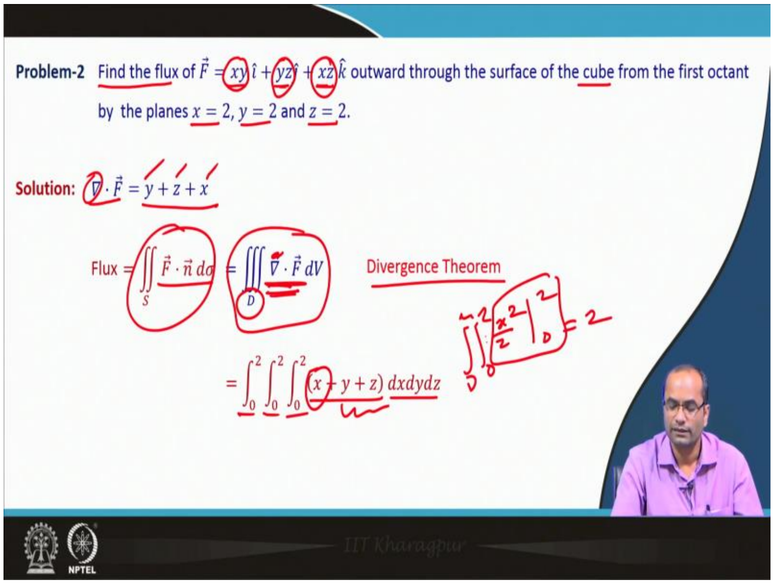

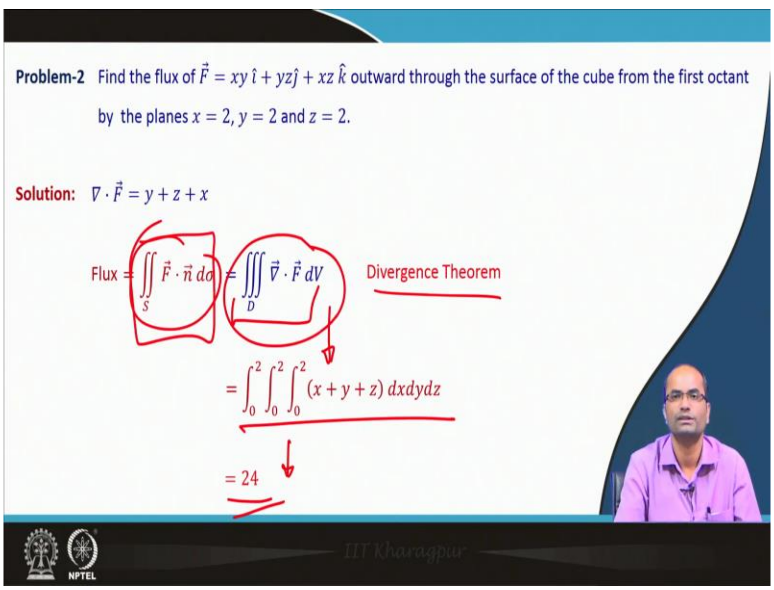

So here the question is find the flux, where we have component xy, yz and xz outward through the surface, that means F dot n we are talking about and the surface is given the cube in the first octant which is bounded by these planes, x is equal to 2, y is equal to 2 and z is equal to 2. So, coming back to the divergence of F, the divergence of F will be because you have xy and yz and then we have xz, the 3 component.

So, here the partial derivative of this xy with respect to x will give y here, the partial derivative with respect to (z) y will get z here and here partial derivative with respect to z, so we will get x here. So, the flux we can get with this F dot n integrated over the surface and that is given by, so we are not, I am not usually using this arrow sign on this gradient operator. So, here are the integral over this volume which is bounded by this cube.

So, the divergence F, divergence F we have computed here that is x plus y plus z. So using this divergence theorem, this flux we are computing using this volume integral. So that is in our case it is a cube which is lying in the first octant. So, that means the x is varying from 0 to 2, y is also varying from 0 to 2 and z is also varying from 0 to 2. So, we have these limits. So this is the integrant and then dx, dy, dz.

So, we have eventually 3 integrals here. One is over x, then y and z and all 3 will give the same value. So, the first one if we compute we will get like x square by 2 then the limit 0 to 2 and there are 2 more integrals sitting there 0 to 2 and 0 to 2. So, this is for, just for x, similarly for y and z we will get. So this is the value coming as 2 there and if we integrate then the, this one with respect to y we will get another 2 and then we will get 2. So, we have the 8 and then there are 3. So, we will get then 24 as a result of this whole integral.

So, here we have not verified, but we have just used the convenience of this divergence theorem to evaluate this flux using the volume integral. If you would have computed this flux using this surface integral then we have the 6 phases and for each we have to evaluate this F dot n and then sum it up, naturally we will get 24 but it seems that this is much easier now with this volume integral. So, that is one of the applications of this divergence theorem with this conversion from the surface integral to the volume integral actually works.

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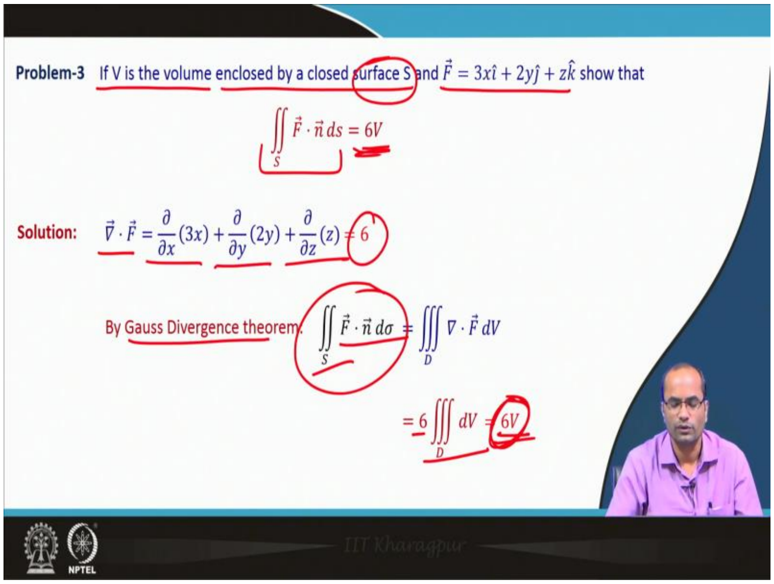

So, here we will see that if V is the volume enclosed by a closed surface S and this F is given by 3 times x, 2 times y and z times this K, then we will show that this flux integral F dot n ds is nothing but 6 times the volume of this surface. So here we will compute naturally we will apply the divergence theorem to see that how the 6 times V is coming and the idea is exactly from here.

So the divergence of this 3x, 2y and z is nothing but 3 plus 2 plus 1. So that is 6 here. And if we apply the divergence theorem now, so the cost divergence theorem says that F dot n over the sigma which is just 6 here, so 6 and this volume integral which is the volume of the surface enclosed by this this surface S. So, we will get 6 times the volume. So, this flux integral is nothing but the 6 times the volume. So very a simple example here to see the importance of this divergence theorem again.

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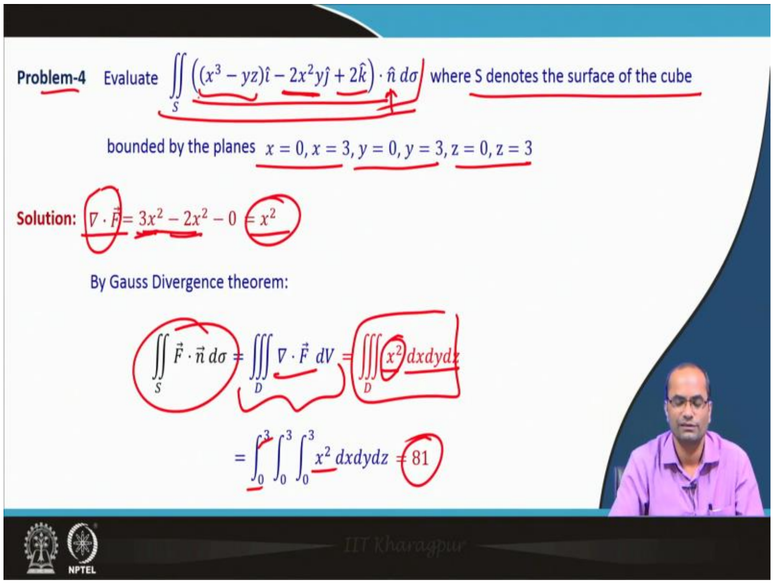

So, the next problem we will evaluate this integral, so this is a surface integral if you look at. So, we have some kind of F here then we have F dot n d sigma. So, this is exactly the flux we are talking about in divergence theorem. And this S is the surface of the cube again, its bounded by this up to x is equal to 3, y from 0 to 3 and z is 0 to 3. So it is similar to one of the problems we have just done so, the F dot n, this d sigma, so, again if we integrate this directly on S this is going to be complicated.

So, what we will do? We will use the divergence theorem. For that we need to get the divergence of this given vector field. So, what is the divergence? If we partial derivative of this component with respect to x we will get 3x square and then partial derivative with respect to we will get minus 2x square, partial derivative with respect to z, that is 0 there. So we have 2x square or 3x square minus 2x square and the value is just x square.

So, this x square the divergence, this x square is to be integrated over this cube. So the divergence theorem says that this flux integral is nothing but this volume integral where we have the divergence here and this is x square. So, dx, dy and dz, we have to integrate this which is a simple integration now, because the limits x, y z, both all 3 are going from 0 to 3 and we are just integrating X square.

So one can do this integration and we will get the value 83. So, again one more application where we have seen that this volume integral is very simple as compared to the surface integral which we would have done with in all these phases.

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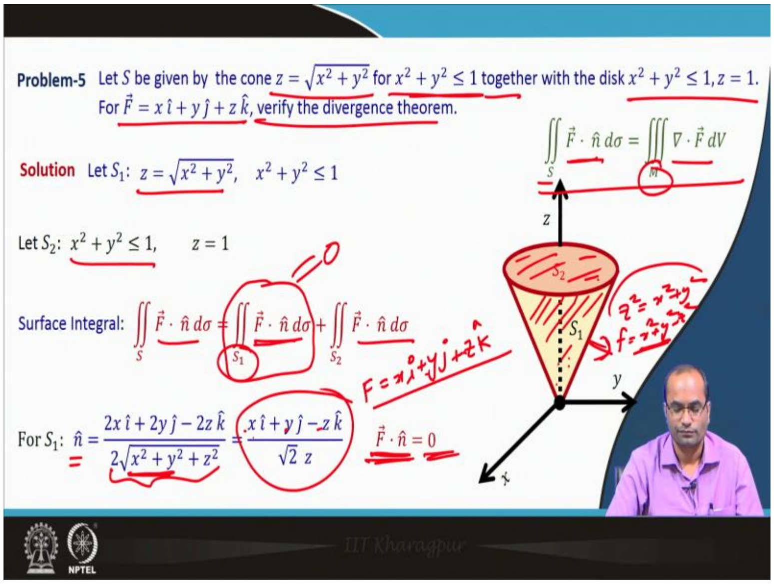

So, this is just the last example where we will be talking about that, that how to verify again this divergence theorem on this problem where we have a cone, the surface is given as the cone, z is equal to square root x square plus y square and the cone goes up to where we have x square plus y square less than or equal to 1. So, this is a open surface, this cone So, if we close it with this disc, so on the top we are closing it with this x square plus y square minus less than equal to 1 with this disc there at Z is equal to one plane.

The F is given by xi, yj and zk. So we have to verify this. That means, this F dot n over the surface and then the divergence F over this covered volume which we are denoting here by m. So, we have z is equal to square root x square plus y square, one surface which is the cone there and then the another surface we have the disc which is the cap on this cone. So, this is the situation. This cone is denoted by the surface N and its cover there we have denoted by S2.

So surface integral if we compute first that means we have 2 surfaces, S1 and S2. So, this integral we have written in 2 integrals, S1 and over S2,. This F dot n, so for S1 surface, we need to compute the normal. So, for normal we have the surface here z square is equal to this x square plus y square and for the outer normal, so let us take our F here, like x square plus y square and minus z square.

So, from here x square plus y square minus z square, we will compute the normal. So the gradient and it will be divided by its magnitude. So on this surface, however we have this x square plus y square is equal to z square. So we can replace this by z square. So we will get this expression for the unit normal and then F dot n if we compute, so what was the F? The F was xi plus this yj and then we have this zk. So if we compute this F dot n so what will happen?

We have x square there, we have y square there and we have minus z square and then divided by this factor square root 2z. But if you look at the numerator, we have x square plus y square and minus z square. And from the on the surface, this is going to be 0 because our surface is z square is equal to x square plus y square. So, this dot product with this F is going to be 0 that means this first integral, the first surface integral over this S1 is going to be 0. So F dot n d sigma over this S1 we will be getting as 0.

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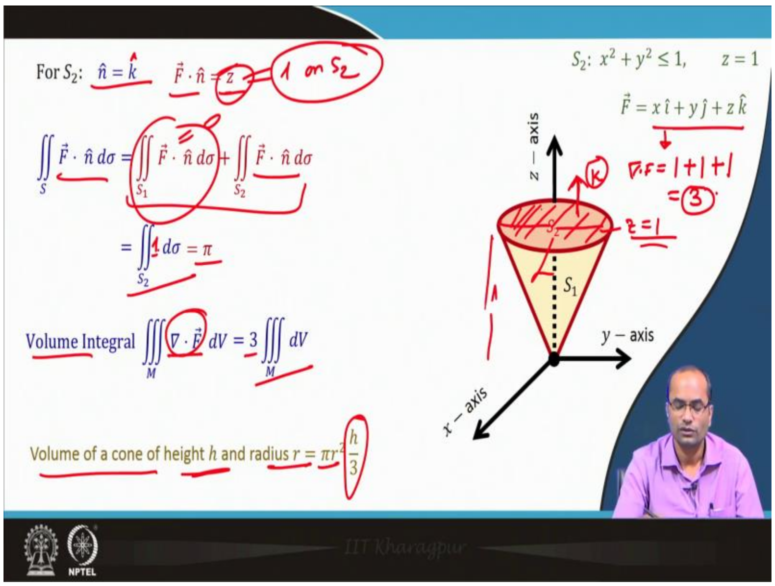

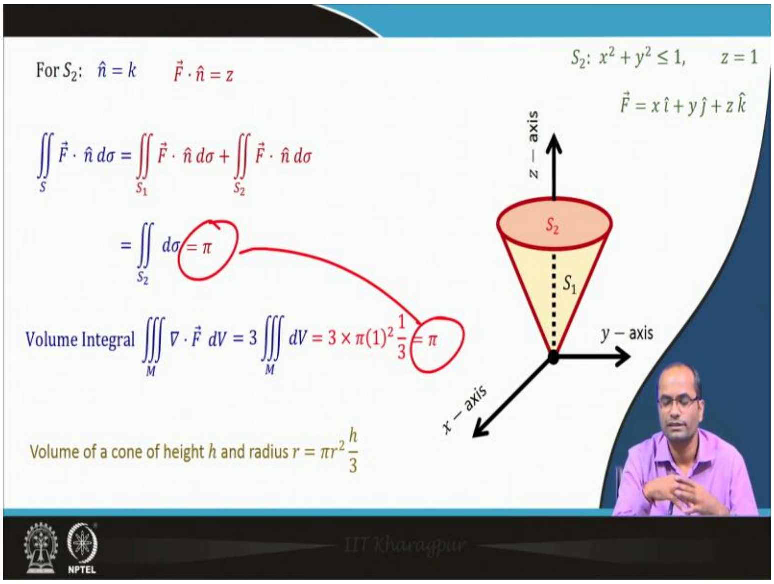

The second integral we will compute now. So for the second integral, so this is our surface which is lying in this Z is equal to 1 plane and we know already without any calculation that what is the normal to the surface. So this is plane z is equal to 1 and the normal will be just in the direction of this K. So, for S2 the normal unit vector is the unit vector this K and then F dot n we can compute because the F is xi, yj and zk. So this is just simply z and then F dot n d sigma we can compute, as we discussed already that this is coming to be 0, the second term here we have the F dot n is z and z is 1 on the surface.

So, this z is 1 on S2. So having this we have this 1 as a integrant and we are integrating over this S2, so that is nothing but the area of this disc, which is Pi R square, R is 1, so we have the value Pi there. Now, we will compute the other integral which is over the volume, over the whole volume here given in this figure. So, volume integral we need to compute the divergence of F. So, the divergence of F if we compute now, so since F is given as here, so the divergence of F, partial derivative of x with respect to 1, here also 1, we will get 1.

So, that is 3 there. So, this is 3 and the volume of this, volume enclosed by the surface. So, that we know already, it is a common and we know the formula that the volume of the cone when we have height h and the radius R, that is Pi R square and this h by 3. So, here we can we have the radius 1 and the height is also 1 here. So, height is 1 and the radius is also 1. So, we can compute the 3 and Pi R square and this 1 by 3.

So, this value is coming to be Pi and here also the surface integral was also Pi. So, we have verified the divergence theorem in this case where the surface was a little bit more complicated. There were 2 surfaces, one was cone and the other one was this disc.

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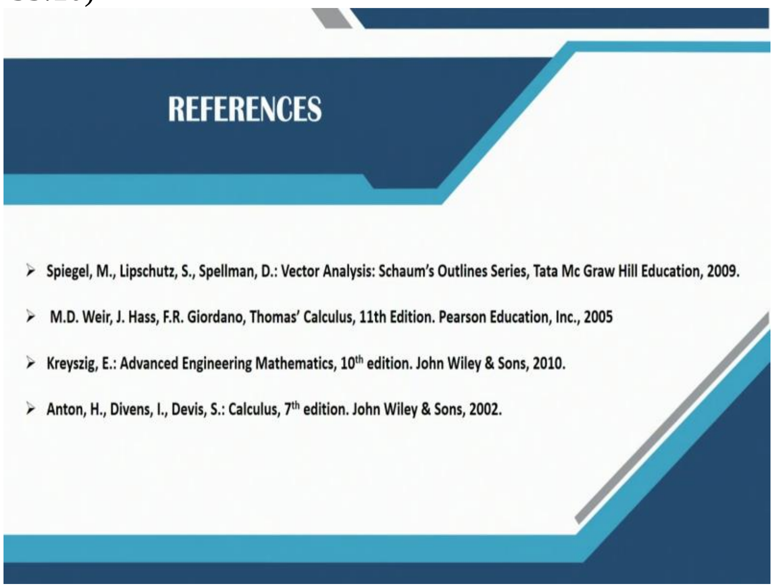

Well, so, this is, these are the references we have used for preparing this lecture.

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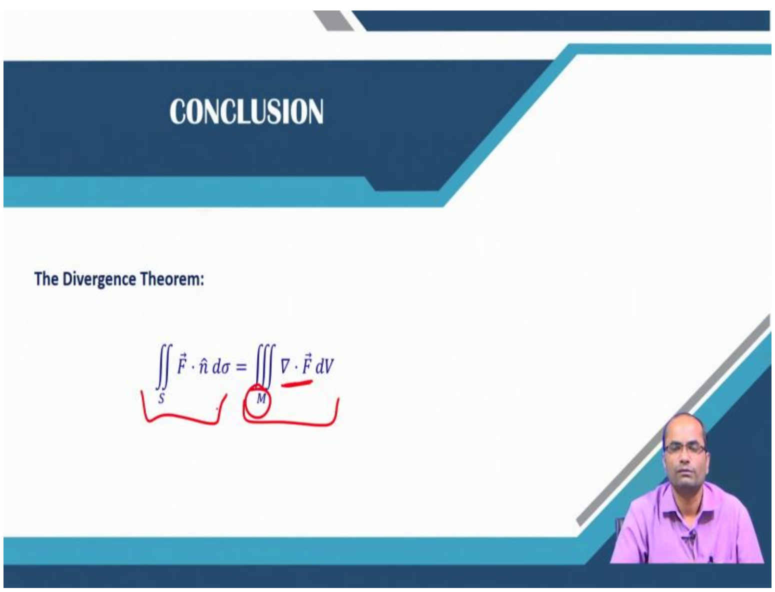

And just to conclude that we have discussed the divergence theorem, which is very useful and we have seen with the help of many examples and it says that the surface integral or the flux across the surface in the direction of normal can be computed with the help of this divergence of F and integrated over the volume covered by this closed surface, S. So, that is all for this lecture. And thank you very much for your attention.