Engineering Mathematics - II
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Lecture - 01
Vector Functions

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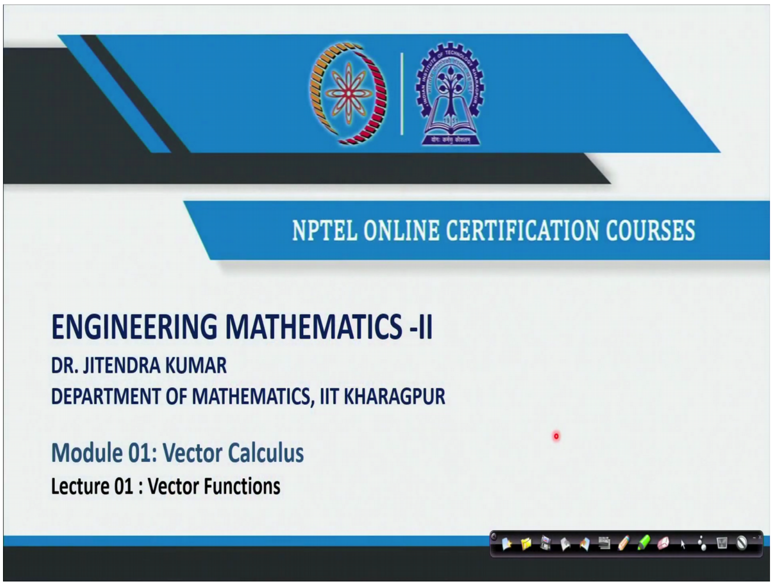

Hi, welcome to lectures on Engineering Mathematics II and this is a sequel course of Engineering Mathematics I. So, this is module number 1 on Vector Calculus and we will go through the vector functions in lecture 1.

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So, we will cover what are the vector functions in this lecture and their limit, continuity and differentiability, also we will be talking about the gradient of a scalar functions. So, these scalar functions are the functions which we have learned in calculus in previous course.

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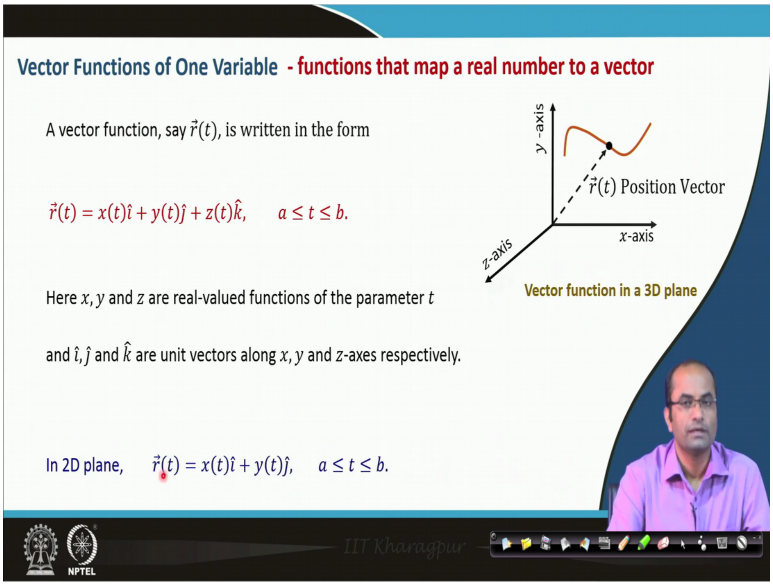

So, what are these vector functions of one variable? So, these are the functions that map a real number to a vector. So, we can define such functions by this vector here r, vector r is given by this component x, component y and component z and this t will vary from a to b. So, if for a given value of t this r vector will define a position vector of a point and as we vary the t there will be another point and so on. And all these, the collection of these points will form a curve in the space.

So, to define this vector function of a single variable, so, here the single variable is t. So, this factor the input is t which varies from a to b and the output is a vector whose components are for instance xt, yt and zt. So, this is the case of 3 dimensions. But in case of 2 dimensions we can have like this vector equal to this xt one component and yt the second component there is no third component in this case. So, this is the situation in 2d plane.

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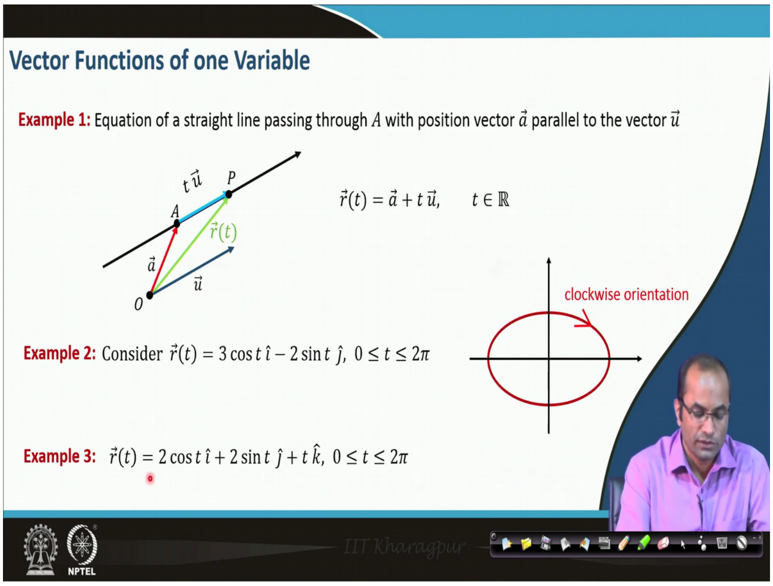

So, vector functions of one variable, these are the examples. So, for instance the equation of a straight line which passes through point A, whose position vector is given by the vector a and the line is parallel to the vector u. So, this is the situation here we have a point A, whose position vector is given by this red vector a and then there is another vector given which is u here. So, we want to have a line which is parallel to this u and passes through this point A.

So, we are interested to find the equation of this line. So let us consider a general point P here on the line whose position vector is given by this vector r. Then, since this direction of this line is u, so, this segment here AP of this line can be described by the vector, some multiplication, some scalar multiplication to this vector, so that the magnitude of this vector will be adjusted to fit in this length AP. So, this is the vector AP which can be described by some t is a real number and this vector u.

So, then we have this equation from the vector addition that this a plus this vector t u will give us this position vector r. So that is this position vector r can describe the equation or the position vector on this line, a general point given by this a plus t, u where t can vary from the set of real numbers. Another example where we can see the function of one variable. Consider four instance is 3 cos t i minus 2 sin t, so the 2 components 3 cos t and minus 2 sin t.

So, if you draw this curve in 2 dimensional plane, then this is basically the ellipse here, because this x component is 3 cos t, the y component here is 3 sin t. So, for instance, at t equal to 0, we have the 3 and this is 0. So 3, 0 point which is given here already. And then if t is for instance pi by 2, so this will become zero and this will be minus 2. So that will be this point. So we are moving from this point in this direction and that is the orientation of the curve.

So in this vector setting, we are not only getting just a curve, but its orientation as well. So, for instance here this curve, the orientation is the clockwise orientation, which is described in the increasing direction of t. So, as we are increasing t we are moving in this clockwise direction and therefore, we call it the clockwise orientation of this curve. Another example in 3 dimensions for instance could be like 2 cos t 2 sin t and then this t in the direction of k.

So, this will be helix here, these are the equation of the circle in 2 dimensions, but we have the third component also which will lift this curve in the direction of z. So, here we have a helix.

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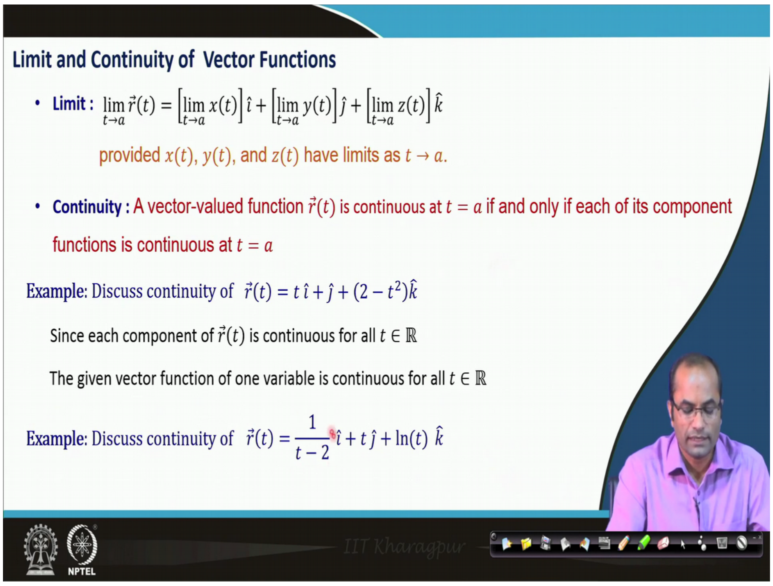

Well, so, coming to the continuity and differentiability later, so, first the limit and continue of such functions. Limit, we can compute for such functions by computing the limit of each of its components. So, for instance, here rt was xt yt zt, then the limit as t approaches to a, we can compute by computing the limit of these scalar functions or this function xt, yt and zt, as t approaches to 0. So, naturally this limit exists, provided these all limits exist.

So, and about the continuity, so a vector valued function this rt is continuous at t is equal to a, if and only if each of its component function is continuous at t is equal to a. So, here if these xt, yt and zt these are continuous, then the given vector function rt will be also continuous. So for instance, if you want to discuss the continuity of this function, which is described by this t i and 1 j and 2 minus t square k.

So, what we observe here that each of its component whether it is t,1 or 2 minus t square, they are continuous for all values of t. So, in that case, this function, the given function is continuous for all t in r. If you want to discuss the continuity of this function which is given by 1 over t minus 2, the second component is t in the third component is this log t. So, in this case we have a slightly different situation because this logarithmic function is defined only for positive values of t.

And there is another problem in this component which is not defined at t is equal to 2, so, we cannot discuss the continuity at t is equal to 2 and also for the negative values of t. Hence this function is continuous for all t except, I mean all t positive, except t is equal to 2.

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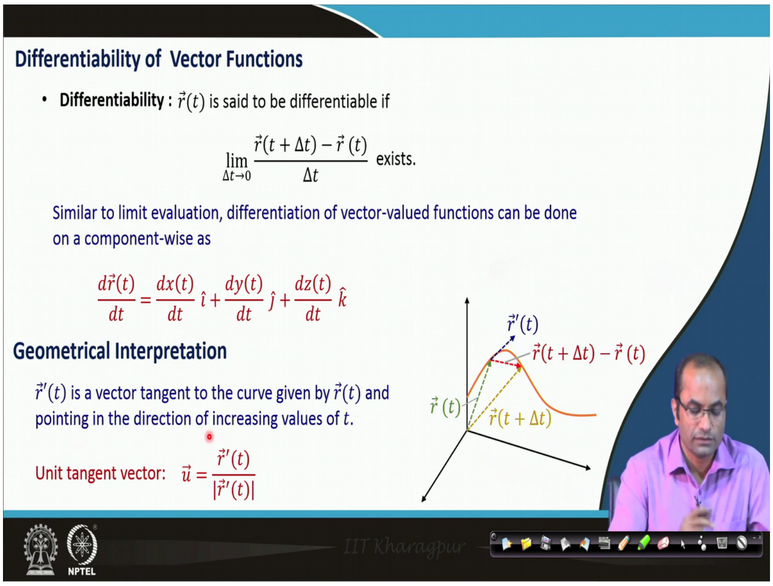

Coming back to the differentiability, a very important concept in vector calculus. So, the differentiability, the function rt is said to be differentiable, the definition is parallel to what we have for the scalar functions. If this limit r, the vector r, evaluated at t plus delta t, so, there is a variation given to its input here, t plus delta t minus the rt and divided by this delta t. If this limit exists, then we call that the function is differentiable.

So, similar to the limit evaluation, differentiation of vector valued function can also be done component wise. That means, we can have the derivative of this vector function as the derivative of this x component plus this derivative of y component and derivative of the z component in this form. Coming back to the geometric interpretation, so we have a curve which is described by this vector function rt.

So, this is the position vector of a point here at t and then this is the position vector ht plus delta t. So, this here vector will be the difference of the 2 vectors, that means the vector r evaluated at t plus delta t and minus this rt. So, if we divide this difference by delta t, the direction will not change, only the magnitude will change because delta t is a scalar quantity, so, we can divide here by delta t.

And then we are looking what will happen when this delta t approaches to 0. So, naturally this line will approach to this point and it will become a tangent at this point here, which was described by this rt. So, this derivative here is exactly gives us the tangent, the equation of the tangent we can also get, but this is the tangent vector r prime t which we have denoted here.

And the direction of this tangent vector will be again in the direction of increasing values of t, because this was rt and this was rt plus delta t. So when we have an increment here in delta t, we are moving to this direction and the direction is given exactly by this one. But when delta t approaches to 0, so this will become the tangent vector. So with the help of this the derivative, we can easily get what is the unit tangent vector.

So that means we can have we can divide by this magnitude of this vector to get a unit tangent vector of a curve at a point P. So, that formula can be used to get the unit tangent vector for a given curve using just this derivative of the vector function.

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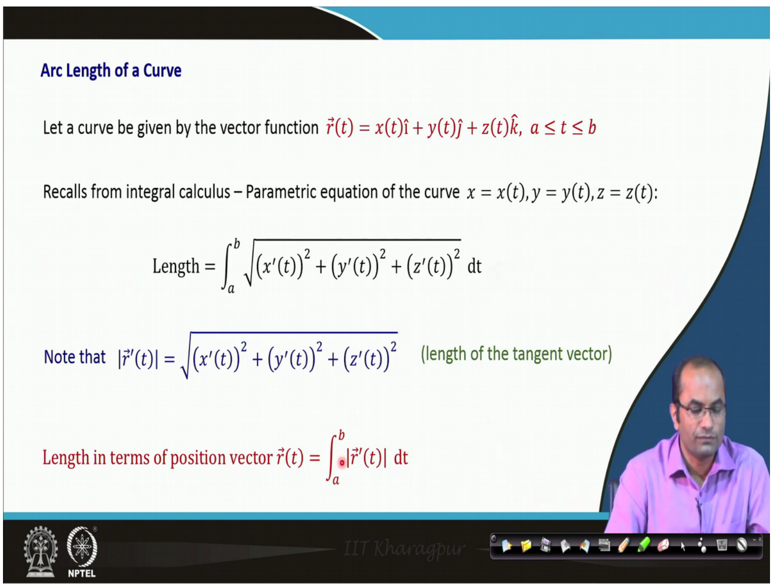

So, just a nice application, which we already know that how to find the arc length of a curve. So in this vector setting, we will see how this formula looks like. So let this curve be given by this vector function rt, where we have 3 components there, the xt, yt and zt, this t again varies from a to b. So if we recall from the integral calculus, the parametric equation of the corresponding curve can be given by like x is equal to xt, y is equal to yt and z is equal to zt.

And the formula for the calculation of the length of this curve was given by this integral a to b and the square root of this x prime square plus y prime square plus z prime square t and then we can integrate over t. Now in this vector setting what we should note here that if rt this curve is given by this equation, xt plus yt plus zt, then its magnitude can be evaluated by this square root of the squares of, sum of the squares of these derivatives.

And this is precisely the integrant of this formula, which is used for the calculation of the arc length of a curve. So what we can, now in the vector setting we can replace this formula by this formula that we can integrate a to b, the magnitude of this r prime t, which is the length of the tangent vector, because r prime is the tangent vector and its magnitude will be the length of the tangent vector. So if we integrate this length of the tangent vector over the given domain then we can get the arc length of a curve.

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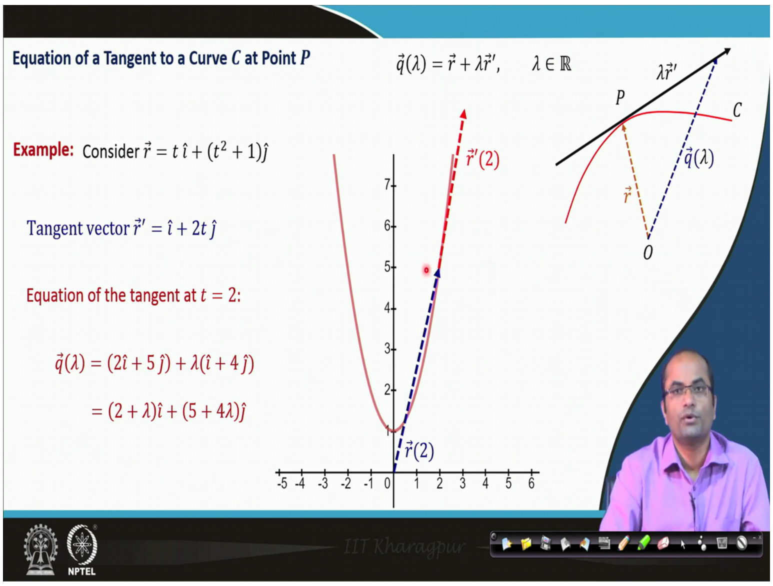

Now, we will look into that how to get the tangent, the equation of the tangent of a curve at a point P. So, for instance, this is the curve given by this arc here. And then we are interested here to find the equation of this tangent line at this point P. We know how to get the tangent vector, but now we want to get the equation of this tangent line. So, suppose this is the r vector here, the position vector of this point P.

And we take another general vector, this q, that precisely will give us the equation of the tangent. So, we have q vector at this lambda and then, so this one P to this distance here, we can have this lambda times the tangent vector, because this was the tangent vector. And we have multiplied here by lambda to adjust the length accordingly, so that it covers from P to this general point.

So, there will be lambda here, such that this lambda and r prime vector will become exactly this vector. And then if we just see the setting that this vector will be r plus this vector. So, we get this equation of a point here, on this tangent line. So for different values of lambda we will be moving on this line. So that is precisely the equation of the tangent line. For instance, if we take the function here, t i plus t square plus 1, this will define the parabola because this is the t and then y component is t square plus 1.

So this is the equation of the parabola. So the tangent vector r prime we can get just by differentiating this. So we have 1 here the derivative of this first component t and then t square plus 1, will give us 2t. So we have the r derivative i plus 2t j. So if we plot this, this is the equation for the parabola given by this, r, vector r. And then r at 2, if you want to get the tangent vector at this point 2 or the equation of the tangent line we want to get at this point t equal to 2.

So this is the position vector for this 2. And then we can also draw from this 1 plus 2 t, the tangent vector at this point. And now to get the equation we have to just add the 2. So, we have the r evaluated at 2 plus this lambda, which is a real number times this r prime again evaluated at 2. So, we can just evaluate the given vector at 2, which will give us here 2 plus 5j plus this lambda and this r prime again evaluated at 2. So, this is the equation of the tangent line, taking different values of lambda we will be moving on the tangent line at this point 2.

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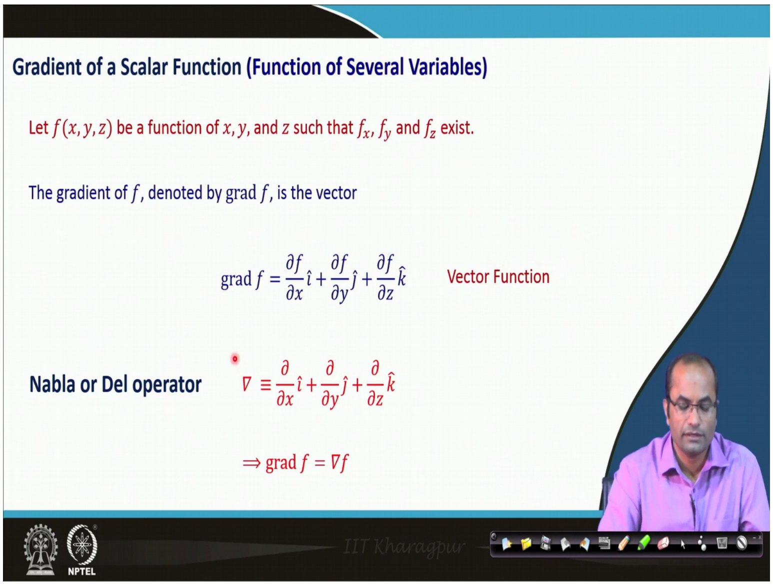

Now we will define that what is a gradient of a scalar function. So these scalar functions are nothing but the function of several variables which were studied in calculus. So let this f x,y,z be a function of x, y, z, such that it its derivative the partial derivatives exist. In that case the gradient of f which is denoted by this grad f is a vector quantity, which is defined by this expression.

So, gradient of f, gradient of a scalar function is given by partial derivative of x in the ith component, then partial derivative of y and then partial derivative of f in the direction of z axis. So, we have, this is a vector function again which we have just studied, so the gradient f is a vector function. And with the help of this nabla or Dell operator we can again define this for our convenience.

So, this Dell operator is defined as the partial derivative with respect to x as the x component, y component and then the z component. If we define this Dell by this vector operator, then this grad f can be written as del f. So, this grad f will be this Dell operated on f. So, we will get exactly the grad f defined here. So, this will be a convenient in future calculations to use this grad f as this dell f.

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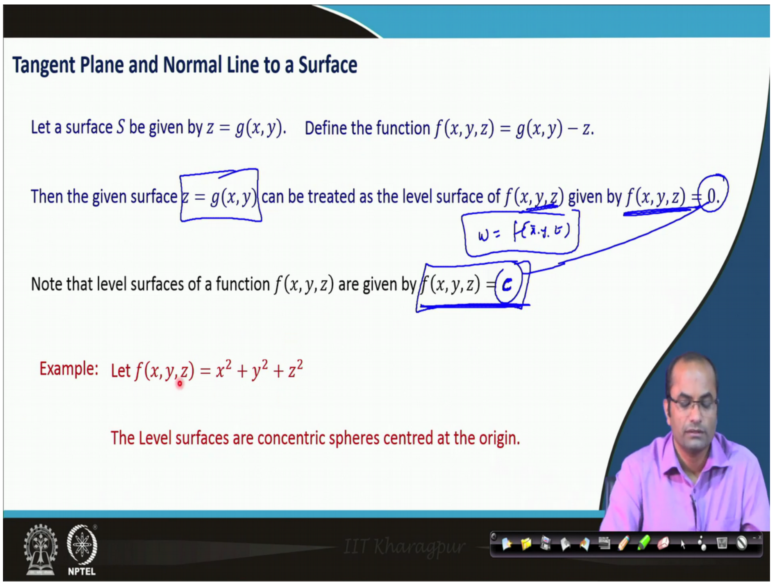

Now, we will come to this, how to get the equation of the tangent plane and the normal line to a surface. So, suppose the surface S is given by z is equal to g x, y. So, we can define a function here, the scalar function f x, y, z, taking the difference, so can bring this z to the other side. So we have g x, y minus z and note that the given surface here z is equal to g x, y this can be treated as the level surface, as the level surface of f x, y, z is equal to 0.

So, note that the level surface of a function f x, y, z is equal to, f x, y, z, are given by f x, y, z is equal to there putting just some constant there. So, if we take this constant precisely at the 0 here, then we will get the given equation of the surface. So, these level surfaces or level curves, we will also mention later, these are very useful for representing for instance the functions which have 3 variables.

So, if we have the W is equal to a function of x y z then the representation in a plane will be very very difficult. So, with the help of these levels surfaces f x, y by putting some constant that means that we are drawing now for fixing the value here the C, the constant and then we are drawing this curve. So, that can be represented by surface which is easy to plot in a 3 dimensional space.

So, here the given surface that is equal to g x, y we can also write down in this form f x, y, z is equal to 0. Now, moving further, so, we are having this for instance, if we take this function f x, y, z is equal to x square plus y square plus z square, in that case the level surfaces. So, by just putting this value equal to some constant, we will get the spheres which are centered at the origin.

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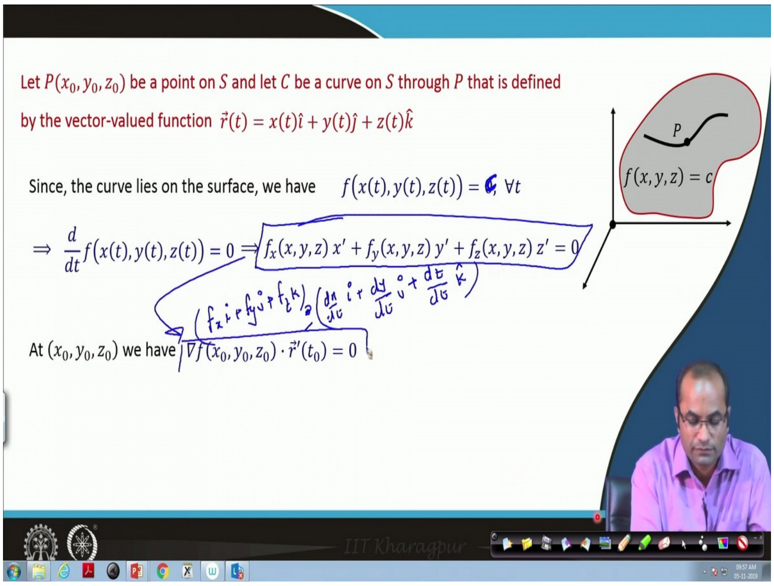

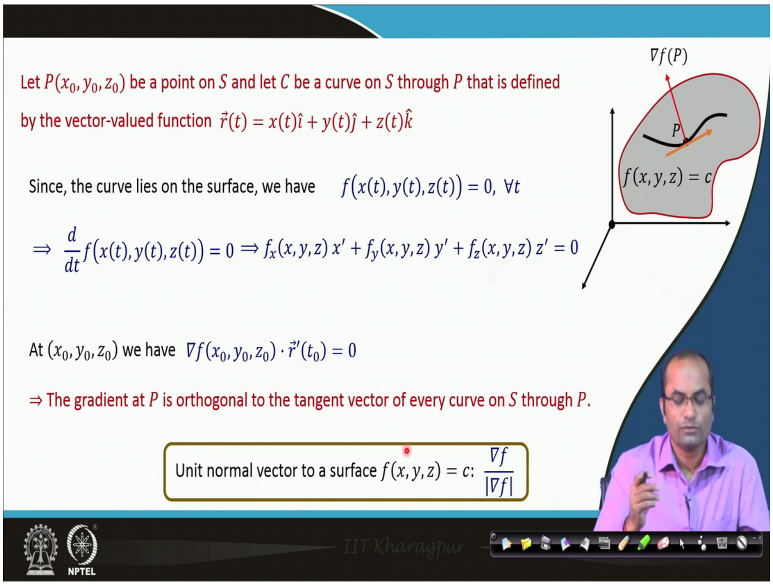

So, let this Px0, y0, z0 is a point on a surface and C be a curve which passes through this point and it lies on the surface. So, c be a curve on S, so the curve completely lies on the surface and it passes through the given point p. So, we will now find out that how to get the normal to the tangent at this point P and later on the equation of the tangent plane. So, the equation of the curve can be given by this vector valued function.

So, this is the curve given which is defined by this vector valued function, the equation of the surface we can take as a more general putting this C. So, f x, y, z is equal to C, is the equation of the surface. So, the curve lies on the surface that means, these point xt, yt, zt for any t as long as we are on the surface this curve lies on the surface, this will satisfy the given equation.

That means, this if we substitute this x y z from this curve, this will satisfy the given equation for all t. So, for more general setting we can instead of 0 we can also work with C. So well, we have now this one we can differentiate both the sides. So taking the derivative left hand side, taking the derivative right hand side, so right hand side will become 0, so whether it is 0 or constant, the right hand side will be 0. Now we can apply the chain rule here.

So the chain rule says that the partial derivative of f at x and then the derivative of x with respect to t, so this is dx over dt, then we have f y dy over dt and f z dz over dt equal to 0. So at this point or at any other point also it is a general point here, we have the setting here. So this expression we can also write in terms of this Dell and the derivative of r. So what is the derivative of r, the derivative of r was dx over dt with i component, and then we have dy over dt, the y component and we have the dz or dt as the kth component.

So this was the r prime, and then its dot product with this gradient of f, which was defined as the partial derivative of f with respect to x, then partial derivative of y with respect to j and the partial derivative of f with respect to z. So this is the gradient, if it is dot product will exactly give this equation. So we have written this equation in this form that partial derivative sorry, the gradient of f, and the dot product with this tangent vector r prime.

So what is the situation now, that we have this r prime which is the tangent vector and with this dell f, the dot product is 0. That means this dell F is perpendicular to this tangent vector. So, and exactly this is the point here that through this point P you take any curve and then this dell f will point in the direction which is perpendicular to the tangent at this point. So, more precisely that this dell f will be the normal to the tangent plane.

So, this can be used now. So if you want to find the normal vector to a surface f, then we can just get this dell f and divide by its magnitude.

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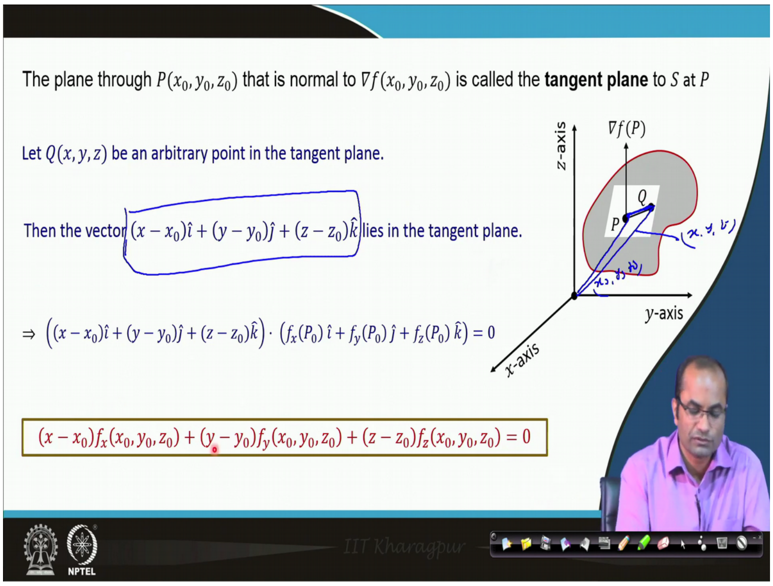

Getting through the equation of the tangent plane we can again consider that this is a surface and then at p we have computed this dell f which points out in the direction of the normal to this tangent plane. So, consider a general point here Q on the tangent plane now, and suppose this P has a position vector which is given by this x0, y0 and z0 and then we have a Q we have taken a general point this Q there, which can be given by this x, y, z.

And now this difference of the 2 we can get by this vector x minus X0, y minus Y0 and z minus z0 which will lie on the tangent plane because we have taken the Q also on the tangent plane and P is also a point on the tangent plane, so this PQ will be on the tangent plane. So, the second consideration here, so having this on the tangent plane what we realize that this line and also this Dell f which is normal to the tangent plane, so, the dot product of the 2 should be zero.

So, this is the line here and then the line vector this PQ and then we have the perpendicular which is dell f. So, the dot product will give us to 0. So, in that case, if you just put this dot product there that means x minus x0 with this partial derivative y minus y0 plus this partial derivative and so on. This is exactly the equation of the tangent plane because this Q was the general point on the plane now, which can be described by this formula, which is the equation of the tangent plane.

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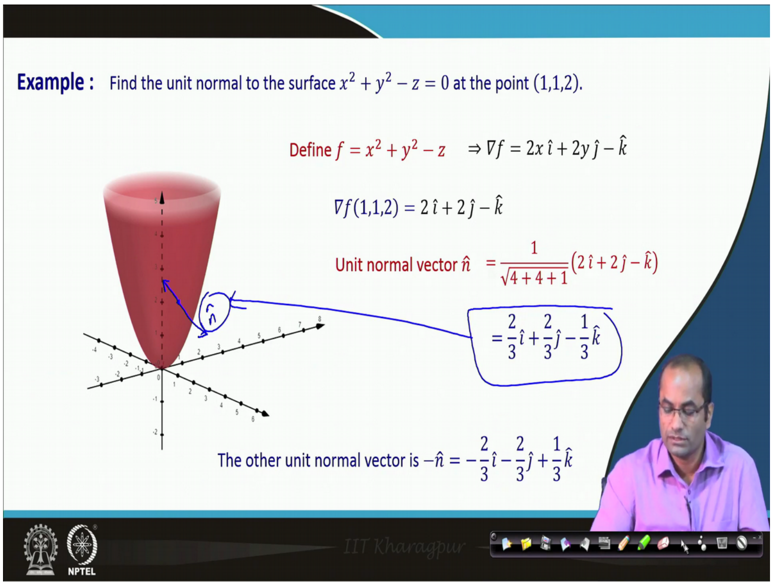

Okay, so now we go through some examples. So find the unit normal to this surface here, we have given x square plus y square minus z equal to 0 at this point 1, 1, 2. So note that this is like z is equal to x square plus y square. So it is an equation of such figure here paraboloid. So we have the x square plus y square minus z. If we get the gradient of this f that means the 2 x this component the 2 y and then minus 1, so this is given by this at 1, 1, 2, we can also compute this so this vector will be 2i 2j minus 2k.

So, we can find the unit normal vector and which we can divide by its length. So, this is the unit normal vector at the point 1, 1, 2. So, if we consider the situation here like one in the direction of x and y. So, there is some point this 1, 1, 2 somewhere there, and then at this we have this normal vector the equation is given by this one. So, we have the unit normal vector on this surface and this is in the outward direction.

So, there will be another normal which will be in the inner direction of this figure. So, we can get both the normals by just computing this dell f, one will be pointing out in the outward direction, the other one will be pointing out in the inward direction. So, we have the other normal vector which can be just given by the minus n so putting minus n we have this vector there. Okay.

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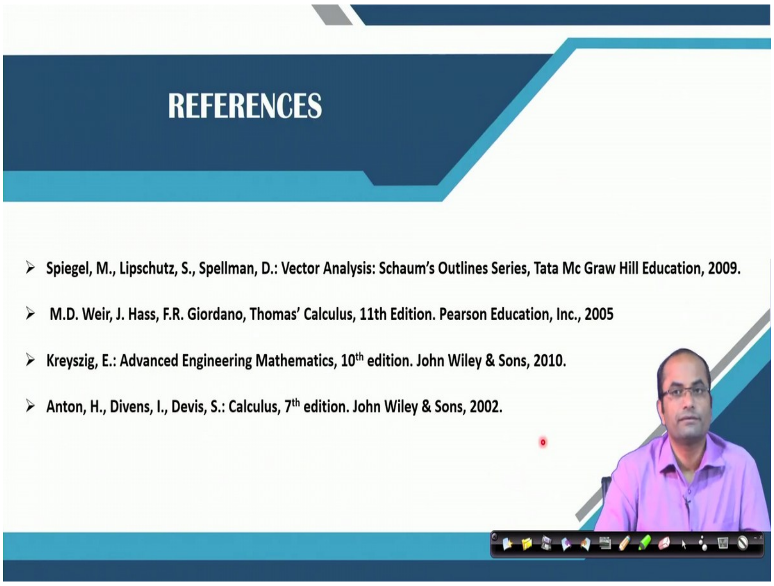

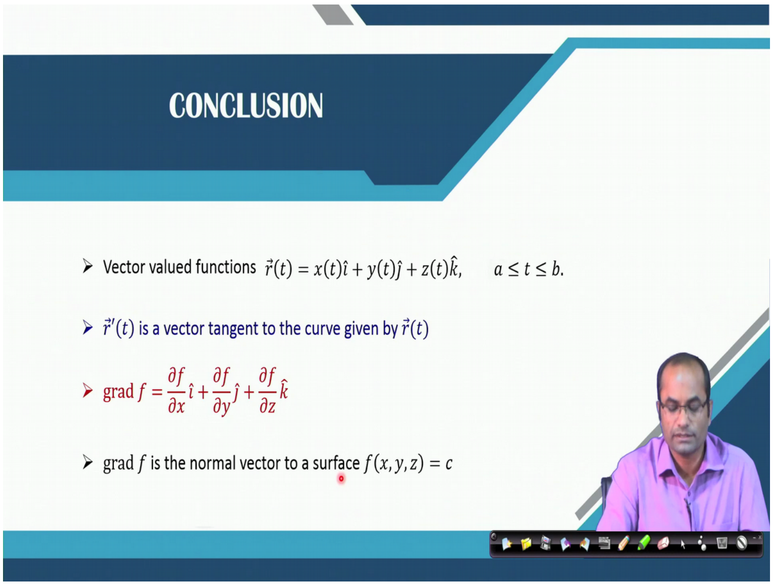

So, these are the references we have used for preparing this lecture. And to conclude this, so, we have gone through the vector valued functions, so that was a new concept which was not covered in the calculus. And second, the most important that r prime, just getting the derivative of this, this is the tangent vector to the curve given by this rt. And the grad f, another important concept we have covered which is defined by this expression, and most importantly, that this grad f gives us the normal vector to the surface f x, y, z is equal to C. So, thank you very much for your attention.