Mathematical Methods For Boundary Value Problem

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Lecture - 09

Numerical Techniques for IVP; Shooting Method for BVP

Welcome back. So, we will be talking about numerical solutions of differential equation in the subsequent lectures. So, first we will start with the Initial Value Problem.

Now, initial value problem as we stated in the very beginning, it has got advantage that if you have an n-th order initial value problem which you can reduce to n first order initial value problem because all the conditions are prescribed at a single point. So, if I know how to solve a first order problem that is good enough for me to handle any n-th order differential equation that is a advantage of the initial value problem on the contrary for boundary value problem the conditions are scattered at two different points at the boundary. So, that advantage is not there.

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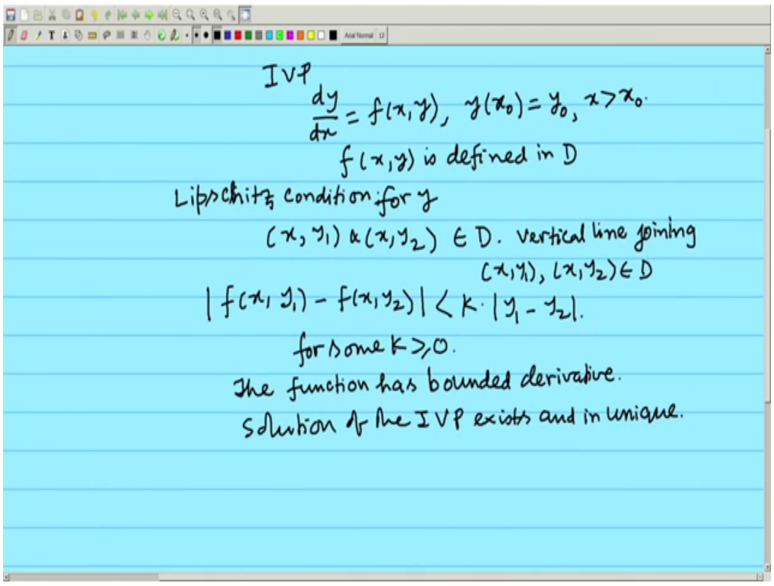

Now, a initial value problem in a simple form can be expressed in this manner dy dx equal to f x, y with condition is given at a single point say y 0 and what we need to find out y at difference x 0 at different x 0. Now, we say that f x, y is defined in a domain D in a domain D over which we are finding the solution now.

Now, this equation will have a unique solution if it satisfies Lipschitz condition. So, Lipschitz condition basically Lipschitz condition which is a somewhat stronger than the continuity of the function f x, y. Basically it says that it is a derivative is bounded.

So, Lipschitz condition has to be for Lipschitz condition for y. So, if we say that there are two points say any arbitrary x and two points x, y 1 and x, y 2 which belongs to D and in such a way that a vertical line if I join between any vertical line joining these two points joining this x, y 1 plus xy 2 is also belongs to D.

Then the Lipschitz condition states that f of x, y 1 minus f of x, y 2 is less than some K y 1 minus y 2. So, for any for all x, y and so, any choice of for some K greater than equal to 0. So, for any choice of x 1, x y 1 and x, y 2 we can find a K such that this happens. So, this is a this inequality holds for every choice of x, y.

So, so, basically this implies that the function have function has bounded derivative. So, if a function if a function given by this way satisfy this Lipschitz condition then we can say that solution exists solution of the IVP exists and exists and is unique. So, this is basically the situation now in our case we will be considering a numerical solution. So, what we assume that the problem is wellposed.

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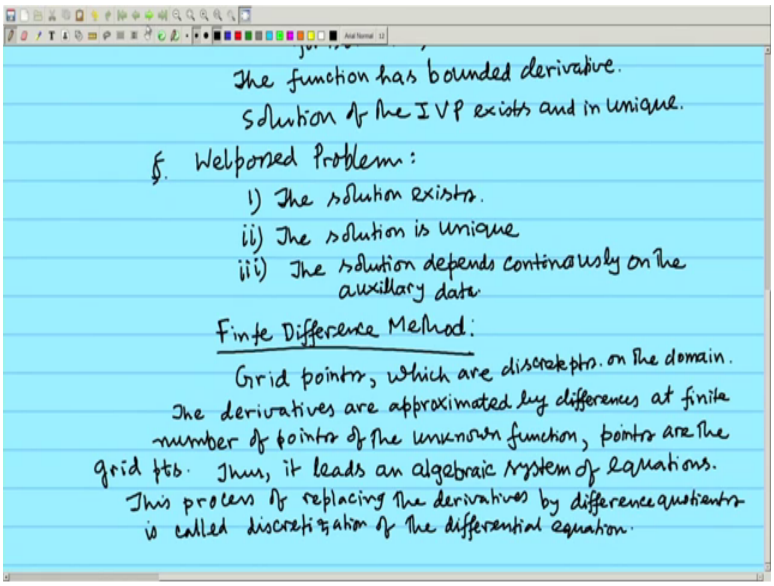

So, that means, a wellposed problem. So, basically we are talking about wellposed problem share in a well posed problem basically it means the solution exist and it is continuously depends on then the we can say that solution is unique and the solution depends continuously on the auxiliary data that is the solution depends continuously on the auxiliary data. So, whatever the auxiliary data is, it can be a it can be initial condition or boundary condition.

Now, we will we will be talking about the numerical solution. Now, in the numerical solution what we do will be doing finite difference solution. So, in the finite difference method what we will be doing is that we find out the solution at certain discrete points not in a continuous fashion. So, the solutions are obtained in a discrete number of points which is are defined predefined within the domain and on that discrete points what we do we replace this derivative by a finite difference formula.

So, a grid points; so, what we do is discrete variables which are defined. So, first we construct some grid points which are discrete points on the domain over which the we are looking for discrete points on the domain on which we are looking for the solution. So, the derivatives are approximated derivatives are approximated by difference by difference at finite number of points. Oops differences at finite number of points of the unknown function. So, this points are finite number of points means these points are points are the grid points are the grid points.

So, basically what we do we are replacing, thus it leads to a an algebraic system of equation. So, basically what we do in these finite difference method is that we have a differential equation which is basic consists of continuous function continuous differentiable functions. So, those continuous differentiable functions we are replacing by a formula or difference quotient. So, which is based on the finite number of points differences and in that process which leads to a system of algebraic equations and algebraic equations.

Now, if it is a linear equations so, we get a if it is a linear equation we get a matrix form or we can solve it directly , but if it is not then we have to do some iterative technique. Now, this process this process of replacing the derivatives by say different quotients is called discretization discretization of the differential equation.

So, basically what we do is for finding the solution we choose some finite number of points over which we are going to solve the algebra over which we are going to solve this equation and then we replace the derivatives by finite difference formula in at those points. And, then we are leading to a algebraic set of equations which will be solved.

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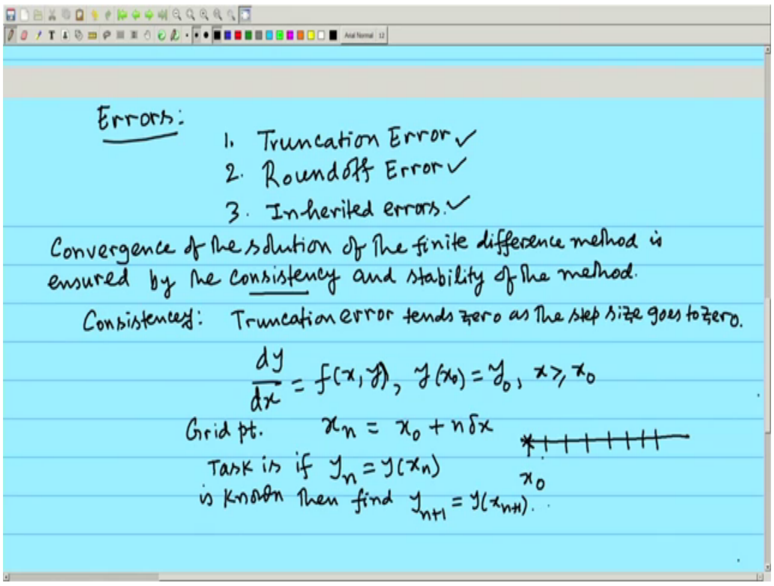

Now, what are the errors we are supposed to come across by this process? So, one error is very apart from the I errors whatever we are excluding the other errors like other calculation mistakes and all. So, those errors are excluded here apart from that.

So, even if I exclude those error the computational errors and all so, what we can have is the one is that truncation error. We call as truncation error this is the one when the formula when the derivatives are replaced by finite number of points difference. So, there will be a truncation error because we are truncating if a infinite series to a finite one.

So, that is one and then this will be the round off error. So, this is a very significant error because every stage we will have to have a infinite decimal places which need to be rounded off. Another is inherited errors. Now, these inherited errors are this inherited errors if I have a step by step process. So, say a time dependent process or one step is depending on the solution of the previous step.

So, in that process we have to suppose if the previous step there are some error so, that will be conveyed to the next one. So, this error will be conveyed. So, that means, this is the error is inherited from the previous step solutions. So, that is one of the error. So, even if I am not doing any error at this stage, but it is already error is inherited. So, these errors are all very important.

Now, if we have a situation now what basically we need to know is that under what condition the solution whatever we obtain by the numerical method converts to the exact solution? Now, we can say that convergence of the solution of the solution convergence of the solution of the or convergence of the finite difference method; convergence of the solution of the finite difference method is also a good one finite difference method is ensured by the consistency and stability there are two new term I bought over here stability of the scheme difference method of the method.

Now, consistency is the one when the truncation error which ever arises due to the replacement of the derivatives into finite number of points discretization. So, this truncation error tends to zero as the step size goes to zero I will illustrate this now goes to zero. And, another is the stability that is little complicated and that will be stability means suppose a slight perturbation is met to a situation if that remain bounded. So, then we call it stability. So, that we will talk little bit elaborate to some extent in subsequent subsequently.

Now, suppose I have the because we are talking about the initial value problem. So, I have the very simple situation dy dx equal to f x y with dy dx equal to f x, y with y 0 say is y 0 and x is rather y at x 0 let us say the first point does x 0 that is how our notation is x greater than equal to x 0.

Now, I suppose I know the solution now first of all before knowing the solution what strategy we are making is we would like to find out the solutions at these discrete points; that means, in some delta x. So, you have say a situation x 0 from where from this x 0 a certain process has started or some process is ongoing. So, and then that process is solve satisfy this equation now we would like to find out that what will be the y in the subsequent value of x. So, to do that what we will be doing is we are considering a discreet number of points which we call the grid points.

So, generally in a generalized manner we can sense we can talk call this as the grid points also note points; in this case if it is a one dimension this is a single point. So, , but later on we will be talking about multi dimensional. So, in that case it will be a grid only. So, these indicated grids we are going to find out the solution. So, our task is these are the grid points. So, task is that if y at n that is y at x n is known then find y at n plus 1 that is y at x n plus 1. So, this is our task.

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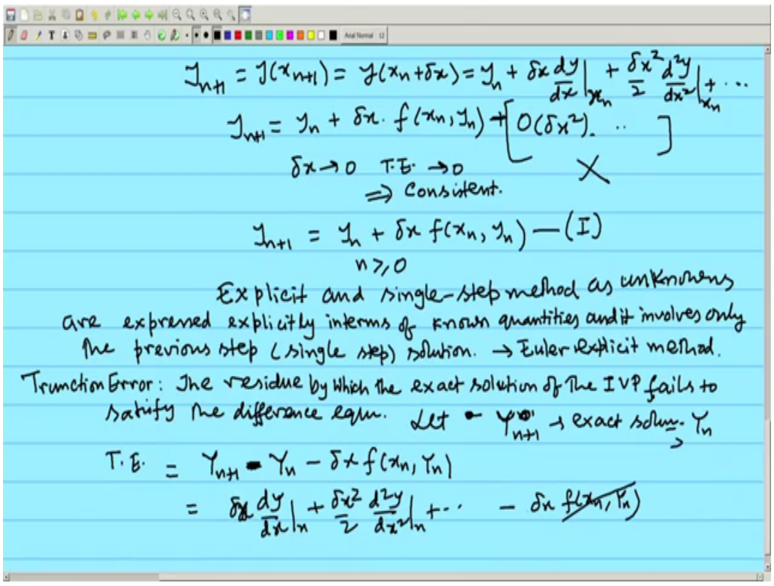

Now, this n can be greater than equal to 0. So, that means, I can start from x 0 is known I want to find out the x 1, then x 1 is known, then x 2; so, in general if I know say solution at x n and how to get the solution at x n plus 1. So, this will be enough for me to obtain the solution in all the subsequent grid points. So, I have the solution y n is known and like to find out the solution at y n plus 1.

Now, what we do say what is y n plus 1 is y at x n plus 1 equal to y at x n plus delta x now if I apply taylor series expansion over here. So, this is y n plus delta x dy dx evaluated at n at x n rather x n plus delta x square by 2 d 2 y dx 2 this is evaluated at x n and so on. Now, I can replace this dy dx by. So, this is y n plus 1 equal to y n plus delta x this is our f x n, y n is given plus all these terms are order delta x square.

So, if I assume that these terms are all neglected that is from this infinite number of terms are all neglected. So, then is the truncation error is we have truncated the infinite series to a finite number of terms and what do you find that if delta x tends to 0? So, this tray T.E whatever these also tends to 0. This implies this is consistent maybe I should elaborate in little better way. See suppose I stop it here, so, I get y n plus 1 equal to y n plus delta x f x n, y n.

Now, so, this is our one method. So, we call this is 1. So, now, I put n equal to 0, 1, 2 etcetera. So, from u solution at n equal to 0, I can find out the solution at n equal to y 1. So, from y 0 I can get y 1, y 1 if once I obtain then y 2 and so on. So, subsequently I can obtain the solutions for all the values of n. So, this is the method. So, now, this method is referred as explicit and single step method; single step because the solution depends only on the previous n 1 as unknown unknowns are expressed explicitly in terms of known quantities and it involves only the previous step ; that means, single step solution.

Now, the truncation error; so, this method this is a method and. So, it as a truncation error. So, this method is referred as the Euler explicit method this is the method is the very basic one is termed as the Euler explicit method. So, the truncation error we define this way, when the exact solution exact solution of the IVP is or I would say the truncation error I think this is the better way to say this the residue by which the exact solution of the IVP fails to satisfy fails. So, fails to satisfy the satisfy the difference equation. So, this is called the truncation error.

So, let capital Y let capital Y n plus 1 capital Y n plus 1 with the exact solution and capital Y n is the exact solution capital Y and capital Y n they are the exact solution. So, I substitute there. So, I get Y n plus 1 so, and this is Y n. So, the residue is delta x f x n, Y n and this is the truncation. So, if something left over this is the truncation error. Obviously, it does not convey anything. So, what we do I expand by Taylor series this Y n plus 1.

So if I expand by Taylor series, so, first term is y n plus 1 the second term is this one and ah. So, what I get is first term is Y n plus 1. So, delta x dy dx at n; then second term sorry delta x delta x square by 2 d 2 y dx 2 n etcetera minus delta x. These f x n, Y n that is basically the. So, these get cancelled. So, we have an infinite series. So, this is the truncation error.

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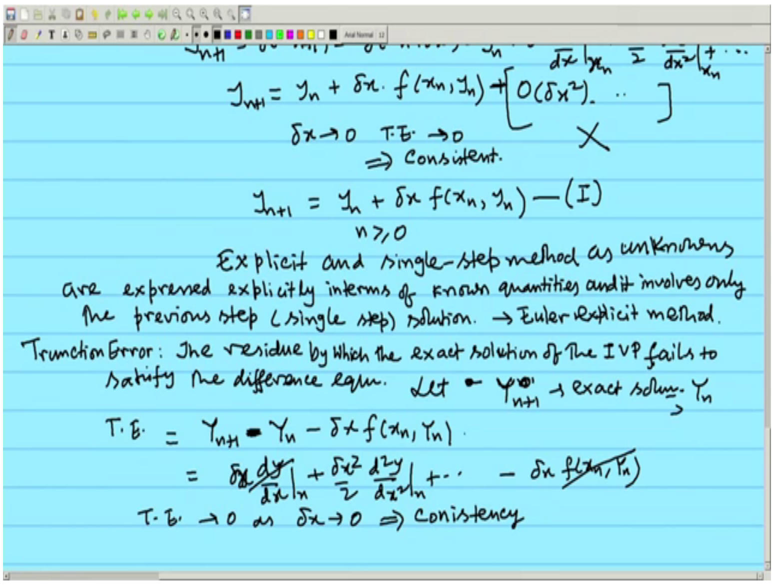

And T. E. goes to 0 as delta x tends to 0, so, this implies that the method is consistent consistency. So, this is the one Euler method are. Now, to check the stability what does it mean by stability.

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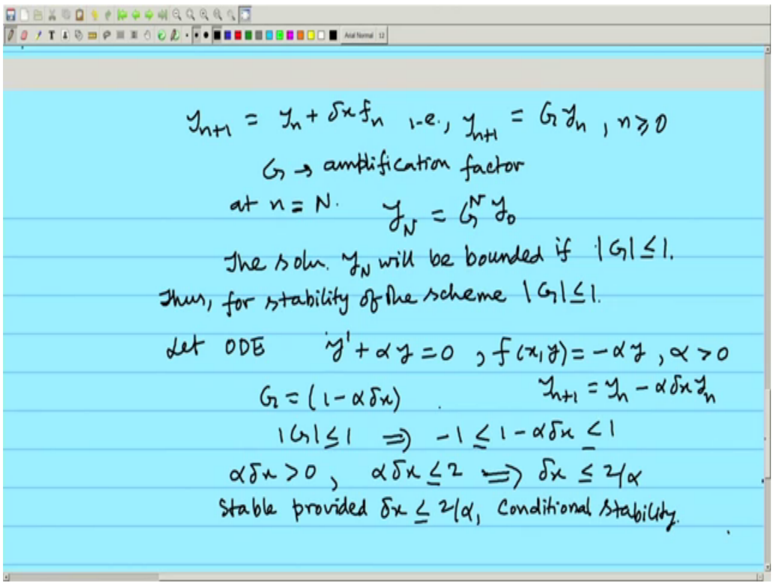

Now, to check the stability what we do is now we consider a very simple equation. Now, so, in the Euler method so, what we have is every step y n plus delta x f n. So, this can be written as in other words I can write it as y n equal to G into some y n, where G we can call as the amplification factor. Now, at say some at a certain stage n equal to some capital N, I can write this y capital N is nothing, but G N into y is 0 because this is n greater than equal to 0, n is moving from.

Now, the solution y N will be bounded the solution y N will be bounded if bounded means as a intense to infinity the it will have a finite value. So, G N less than equal to 1, so, that is the geometric series. So, this G N remain G to the power N remain a bounded number. So, G to the power N tends to 0. So, that is possible only when mod G less than equal to 1. So, what we need is thus for stability of the scheme of the scheme or this method mod G should be less than equal to 1.

Now, let us see what is the situation over here by considering a simple example. So, let the ODE this given by say y dash plus y alpha y equal to 0 y dash plus y dash plus alpha y equal to 0. So, y dash plus alpha y equal to 0. So, f means f x, y is nothing, but minus alpha y. So, our G is so, y n plus 1 y n plus 1 equal to y n minus alpha delta x y n. So, G equal to 1 minus alpha delta x y n.

Now, alpha delta x y n I suppose. Now, what we need is for stability so, sorry not G equal to y n. So, G is alpha 1 minus alpha delta x. So, mod G less than equal to 1 implies minus 1 less than equal to 1 minus alpha delta x less than equal to 1. Now, alpha is we have taken alpha greater than 0 alpha greater than 0. So, alpha delta x is greater than 0. So, this part is satisfied the other part gives you alpha delta x should be less than 2..

So, so, alpha delta x should be less than equal to 2. So, that means, this implies that the step size should be choose in such a way that it is less than 2 by alpha. So, that means, stable provided delta x is less than equal to 2 by alpha. So, that is called a conditional stability. So, this Euler explicit method what we have here is a conditional stability.

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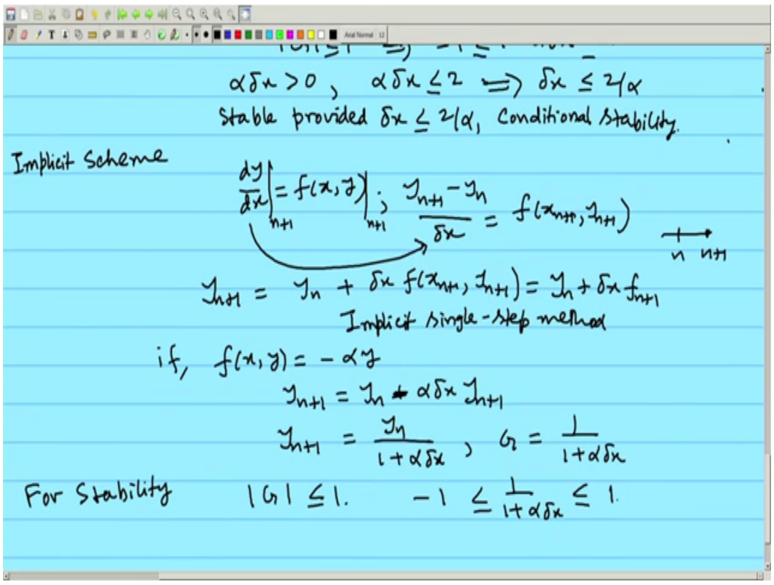

Now, if I would have taken a scheme like this way. So, I considered as scheme a different scheme which I call as implicit scheme. What we do? We have dy dx equal to f x, y. So, choose x equal to x n plus 1 at the point x n plus 1 discretized both. So, y n plus 1 minus y n by delta x equal to I take it as x n plus 1 and y n plus 1.

So, that means, we are substituting at n plus 1 this is at n plus 1 and these dy dx are replacing by these first order forward difference. No, this is backward difference because we are at n plus 1. So, we are at this point n plus 1 and we are using n.

So, this is a. So, I will get it y n plus 1 equal to y n plus delta x f x n plus 1 y n plus 1 or one can say y n plus delta x f n plus 1. So, which is a implicit and of course, single step because a single step method because it is only depend on the previous step solution, but implicit because we are not able to write the unknown directly in terms of known quantities.

Now, what about the stability the same example, so; that means, f is f x, y we are taking as minus alpha y. So, if I substitute so, y n plus 1 equal to if f x y equal to this. So, y n plus or rather minus alpha delta x y n plus 1 alpha delta x y n plus 1. So, what I get y n plus 1 equal to y n by 1 plus alpha delta x 1 plus alpha delta x. So, this is the n to y n. So, this is the amplification factor is governed by this way now. So, I can say G is nothing, but 1 plus alpha delta x.

Now, for stability what I need is mod G should be less than equal to 1. So, that means, minus 1 less than 1 plus alpha delta x less than equal to 1.

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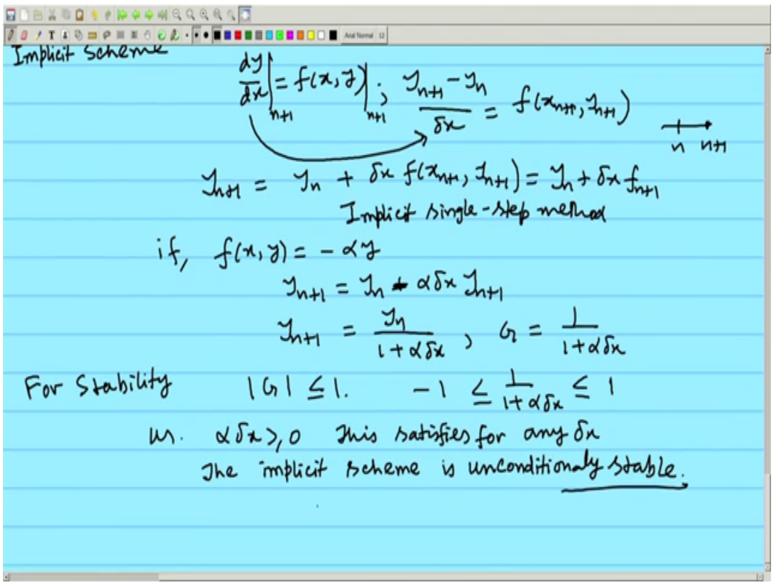

Now, since alpha delta x is greater than 0 so, this is happening as alpha is positive. So, these satisfies for all delta x for any choice of delta x for for any delta x. So, that means, the implicit method is unconditionally stable.

So, this is the advantage of using the implicit scheme over the explicit scheme. So, always we will have we will see that implicit scheme is advantageous, but implicit scheme one of the difficulty is that the implicit scheme a handling the implicit scheme is quite difficult. It will be difficult compared to the explicit one ok. So, we will continue in the next class.

Thank you.