Mathematical Methods For Boundary Value Problem
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Lecture - 08
Green’s Function for BVP and Dirichlet Problem (Contd.)

Welcome back. So, now, we will talk about the homogeneous equation Green’s function that we have constructed are the previously. So, which was this is the form; now we will go by an example a simple example.

So, basically what we did is we have a just to revise our previous one we have a non homogeneous boundary value problem and then we have two independent solution for the homogeneous boundary value problem. So, and based on that, we have constructed the Green’s function.

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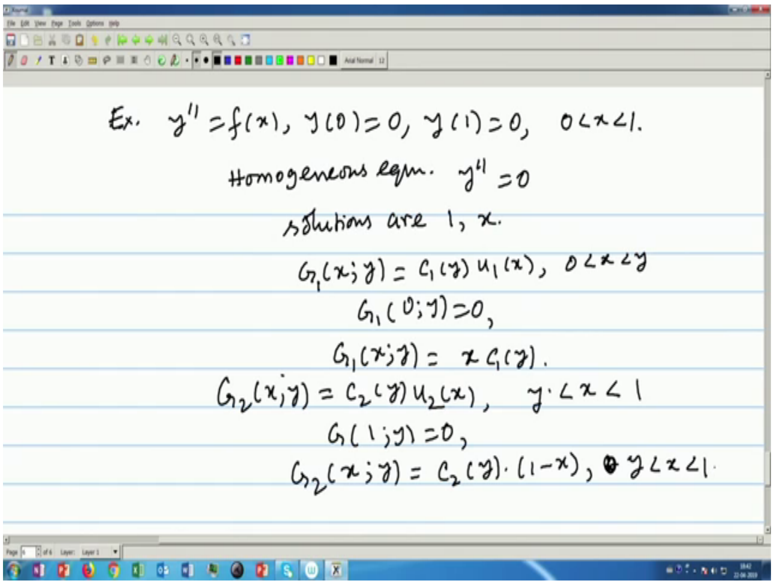

Now, let us take a example simple example say given by this way say y double dash equal to f x, y 0 is 0 and y 1 equal to 0. So, our domain is 0 less than x less than 1, now this is the domain. So, this is a very simple one. So, the homogeneous one homogeneous equation first of all we have to find out the fundamental solutions y double dash equal to 0.

So, we can have solutions are 1 and x any combination of that. So, we construct a Green's function. So, G 1 x y say C 1 y u 1 x for 0 less than x less than y and it has to be in such a way that G 1 it should be G 1 0 y should be equal to 0 so. So, now, I if I choose, so, for that I choose G 1 x y I choose as x into C 1 y x into C 1 y and the other part G 2 x y. So, this will be C 2 y u 2 x 0 less than y less than x less than y, y less than x less than 1. So, what I need is G 2 should be G 2 1 y should be equal to 0.

So, we choose u 2. So, we choose the G 2 x y as G 2 x y we choose as C 2 y into 1 minus x which is valued for 1 0 y less than x less than 1 ok. So, now, we need to find out C 1 C 2.

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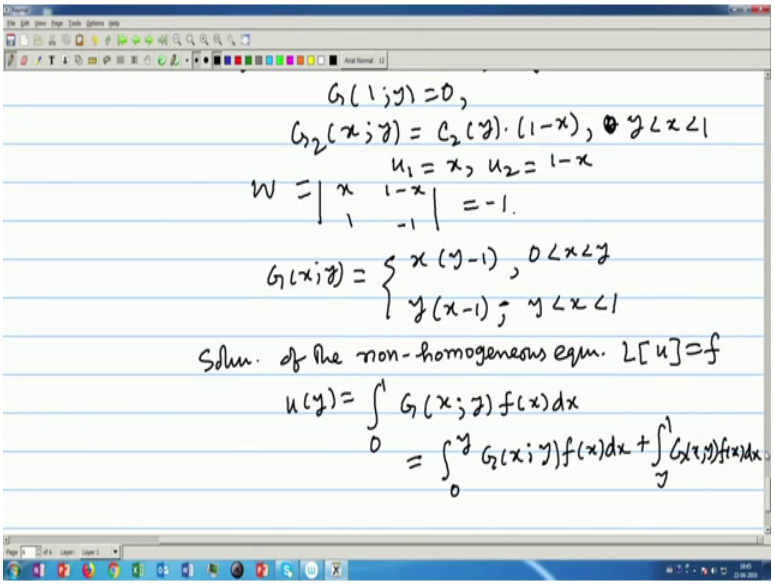

So, now w for this case, so, u 1 is our in the situation is x and u 2 is 1 minus x. So, w is x 1 minus x and 1 minus 1. So, this becomes minus 1 ok. So, Wronskian is minus 1 and what we need to find out.

So, G x y if I apply that formula, so, that becomes x into y minus 1 for 0 less than x less than y and y into x minus 1 when y less than x less than 1. So, the solution of the non homogeneous equation L u equal to f is u x equal to integral 0 to 1 G. So, whichever way either if I go by this manner. So, better I should do this way I write u y G x y f x d x this is either way I can take because of symmetry. So, this becomes 0 to y G x y f x d x plus y to 1; so, G x y f x d x.

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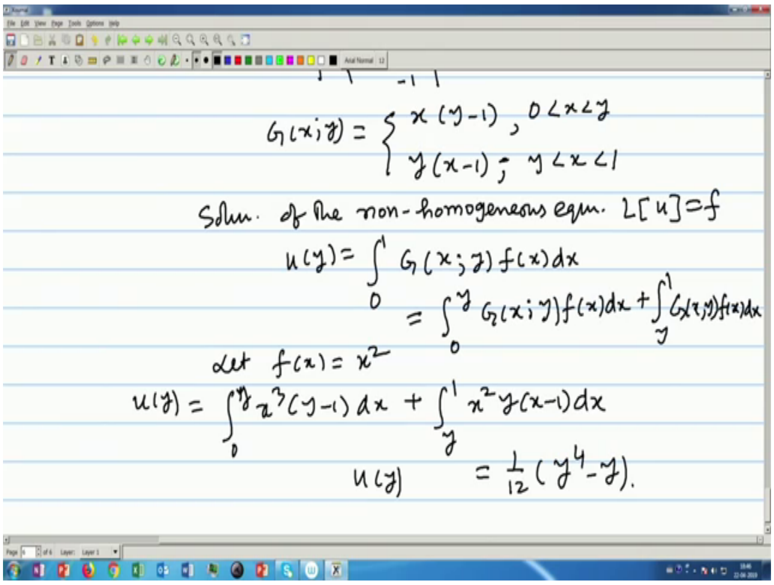

So, for example, if I choose let f x equal to x square. So, in that case u y will be 0 to 1 x cube y minus 1 d x plus sorry 0 2 1 9 0 to y and y 2 1 x square y into x minus 1 d x. So, this gives you 1 by 12 x to the power 4 y to the power 4 minus y; so, y to the power 4 minus y. So, this is the simple way to find out the solution of this non homogeneous equation. So, this is one of the example of course, one can solve it straight away.

So, this Green's function basically what we transform the differential equation now to a integral equation. So, some cases integral equation is easier to solve.

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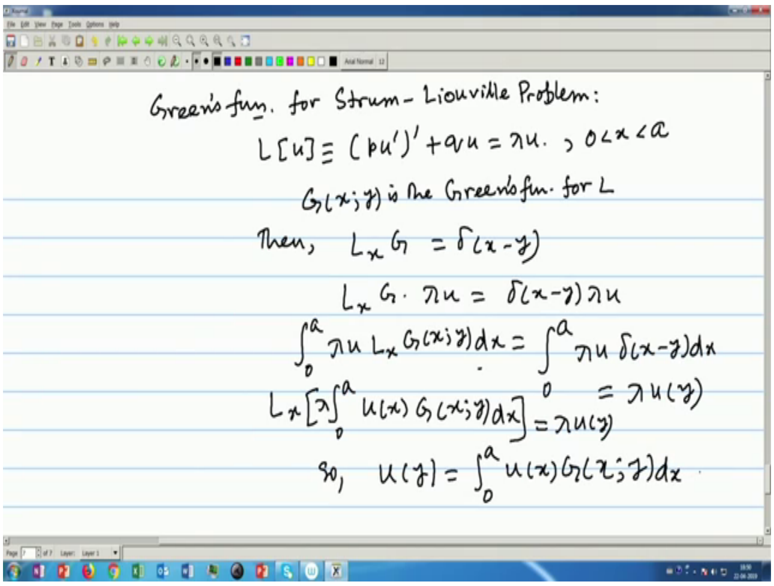

Now, the Strum-Liouville problems the Green's function for Strum-Liouville problem can also be defined like this way. So, we have this Strum-Liouville problem given by p u dash dash plus q u equal to lambda u, say 0 less than x less than a I have frequently used either a 0 less than x less than a or a less than x less than b both are same I can transform the coordinate instead of x equal to a, I can make it x equal to 0 there is no harm in. So, it should not be confusing.

Now, if G x y so if this is a homogeneous problem Strum-Liouville problem. So, if G x y is the Green's function of the operator L u say G x y is the Green's function for L then what we will have is L x G equal to delta x minus y and so, if I now multiply by lambda u both side lambda u delta x minus y lambda u and integrate and then integrate between 0 to a. So, lambda u l x g equal to d x lambda u l x g x y d x equal to integral 0 to a lambda u delta x minus y d x and that gives you lambda u y and this side will be L x can be taken away. So, 0 to. So, lambda is out is a constant.

Say 0 to a u x g x y d x equal to lambda u i. So, in other words the solution is. So, I get a solution u y equal to 0 to a u x g x y d x. So, this is a (Refer Time: 11:26) type integral equation, but the problem here problem here is. So, there will be a lambda inside.

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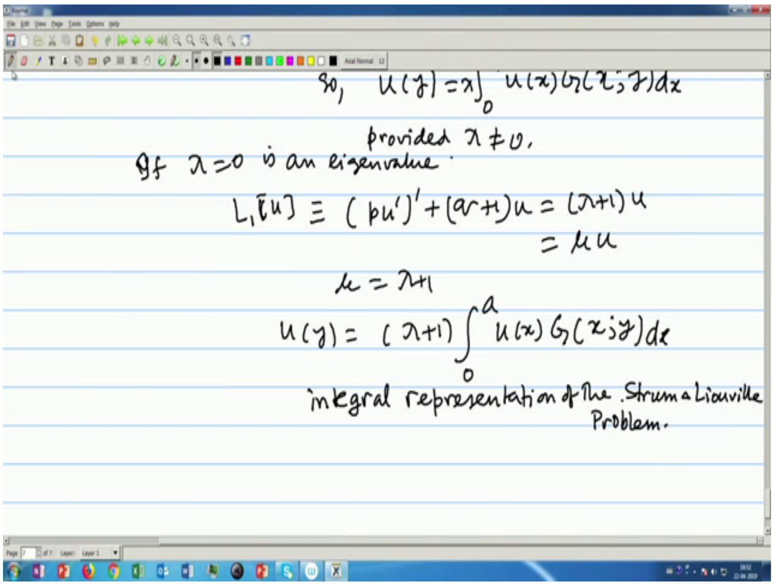

So, provided lambda is not equal to 0. So, a integral equation representation is given by this way. So, if lambda equal to 0, we can do little modification is an Eigen value of L of this Strum-Liouville problem.

So, we can modify this way in (Refer Time: 12:12) like is an Eigen value then what we do is we can modify we just write the modified form this way pu dash dash plus q plus 1 u equal to lambda plus 1 u. So, this I call as mu u. So, in that case mu is nothing, but lambda plus 1. So, I can do the sum operation and get a integral representation as 0 to a u x G x y d x in this case the G x y is the Green's function of L 1.

So, this is an equivalence with the Strum-Liouville problem integral representation integral representation of the Strum-Liouville problem. I am sorry I am now Green's function is not a very simple procedure as we could see that it leads to a ordinary differential equation a differential equation to an integral equation.

So, now, all the time it may not be a very helpful 1, but in some cases only the boundary value problems and all the Green's function is found to be little useful particularly the heat conduction Laplace a Parisian type of equation of course, Parisian type of equation with the Laplace operator basically what I mean. So, now we will construct the Green's function for the desolate problem or the Laplace type of equation.

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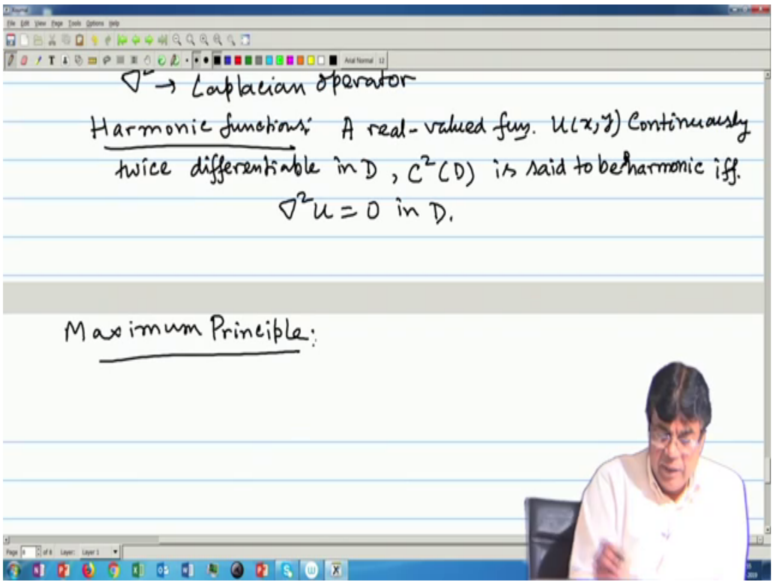

Now, before that; that means, we will be talking about these situations del 2 Laplacian operator. Now define a harmonic function. Basically the harmonic functions are which solves this del 2 u equal to 0.

So, we define in this way a real valued function u x y continuously twice differentiable in a interval a b in D say in a domain d we will also write as C to D is said to be harmonic if and only if said to a harmonic even only if del 2 u equal to 0 in d. So, harmonic function has to satisfy the Laplace equation; now there are certain important property of this harmonic function one is the maximum principle this says that the all the maximum or minimum of u will attend at the boundary of the domain.

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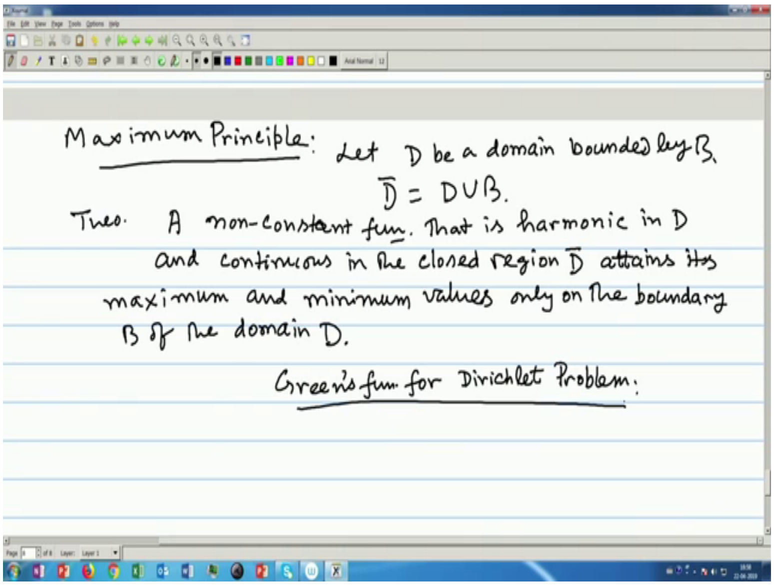

So, let D be a domain bounded by B say it can be rectangle or a circle or spherical or whatever and we call D bar equal to D union B. So, theorem is the principle of the maximum principle is a non constant function that is harmonic in D and continuous in the closed region D bar which is nothing, but D plus the boundary attains its maximum and minimum values only on the only on the boundary B of the domain D.

So, all these maximum and minimum whatever the extremum values will be attend it by the harmonic function on the boundary of the domain. So, this is referred as a maximum principle. Now our intention is to construct the Green's function for Dirichlet problem.

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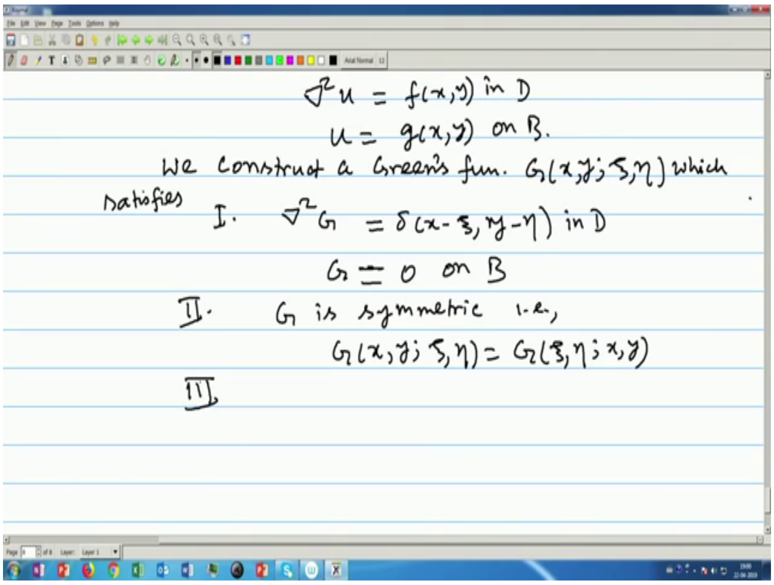

Now so; that means, we what we have is del 2 u equal to say f x y in D.

So, it is not a homogeneous boundary condition because we are talking about Dirichlet problem so; that means, u is prescribed on the boundary. So, maybe it can be also 0 on B. So, we consider we construct a Green's function, we define this way G x y xi eta which satisfies following condition I delta G is delta x minus xi y minus eta in D so; that means, it is harmonic everywhere etcetera xi eta, I would say I will not say the harmonic in that sense because harmonic means it has to be 0.

So; that means, it is satisfying the Laplace equation everywhere except at xi zeta and G equal to 0 and G is 0 on the boundary on the boundary B, then second condition is G is symmetric that is what we can say that G x y xi eta is G xi eta x y.

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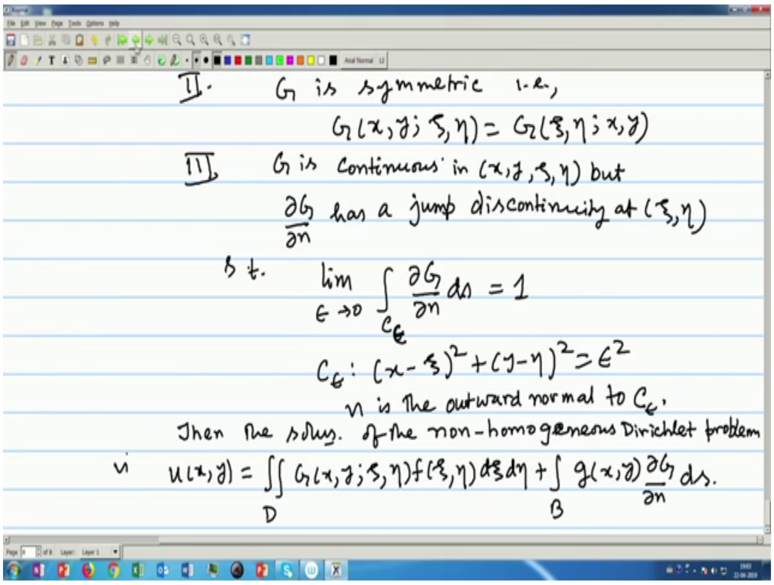

Third it is continuous at x y xi eta G is continuous in x y xi eta, but del G del n has a discontinuity has a jump discontinuity at xi eta such that limit epsilon tends to 0 C epsilon del G del n d s equal to 1 where so; that means, if I enclose this xi eta point by a circle C epsilon is this circle plus y minus eta whole square equal to epsilon square of radius epsilon. So, if I and in is the outward normal to C epsilon.

So,. So, this has a jump discontinuity a c surface area. So, you have a circle C epsilon encircling the point xi eta. So, as I contract the radius of this circle and approaching x y 2 xi eta then you have a jump discontinuity in the gradient or the derivative normal derivative given by this way. So, if I can construct a G based on these 3, then the solution then the solution of the non homogeneous homogeneous problem Dirichlet problem is given can be given this way u x y equal to double integral G x y xi eta f xi eta d xi d eta is the area integral plus on the boundary G x y del G del n d s.

So, this is the this is the construction of the solution can be expressed by this manner now. So, provided all these conditions are satisfied by G. Now to show that so, what we have is G satisfying this it is the Laplace equation homogeneous equation except an homogeneous boundary condition and symmetric and it has a jump discontinuity at the point xi eta. Now, to prove that we have to use the Green’s second identity.

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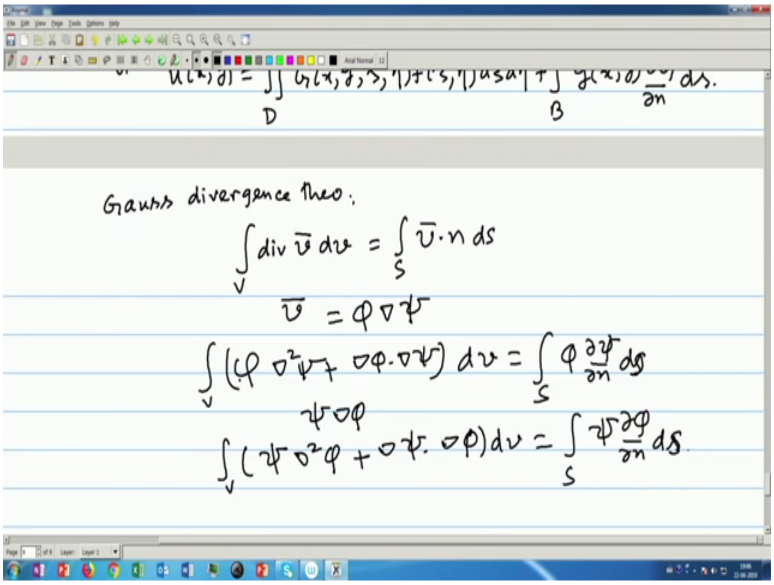

Now, recall that the Gauss divergence theorem what it says? The Gauss divergence theorem is integral over v any area or any region if I integrate. So, a function say v. So, that it will be v dot n ds over the surface.

Now, if I replace this v bar equal to some phi grad psi. So, what I get is phi. So, phi del to psi minus plus of course, phi del 2 psi grad phi dot grad psi d v equal to phi del psi del n d s s. Now same way if I write now if I construct this way psi grad phi. So, what I get is same way I just substitute here. So, psi del 2 phi plus grad phi dot grad phi dv equal to psi del phi del n d s. Now if I this is capital s or whichever way you can write.

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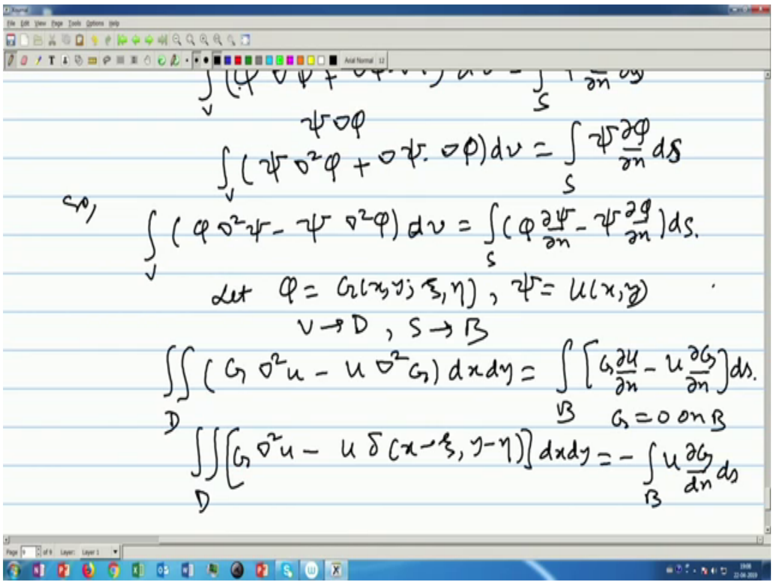

Now, I will subtract this two. So, what I get is, phi del minus to psi minus psi delta phi equal to dv equal to integral over s phi del phi del n minus phi del phi del n d s. Now let phi equal to G x y xi eta and xi is u x y and the domain v is our D and S is the. So, D is the area bounded by S, S is B. So, in that case what I get this is double integral because we are in the 2 dimension. So, this is phi G del 2 u minus u del 2 G d x d y equal to integral over B phi. So, d G del u del n minus u del G del n ds the line integral.

Now, you remember that G is 0 over B G equal to 0 and B and also G del 2 u minus u del x minus xi y minus eta. So, d x d y equal to minus B u del G del n ds. So, we have this u G del 2 u.

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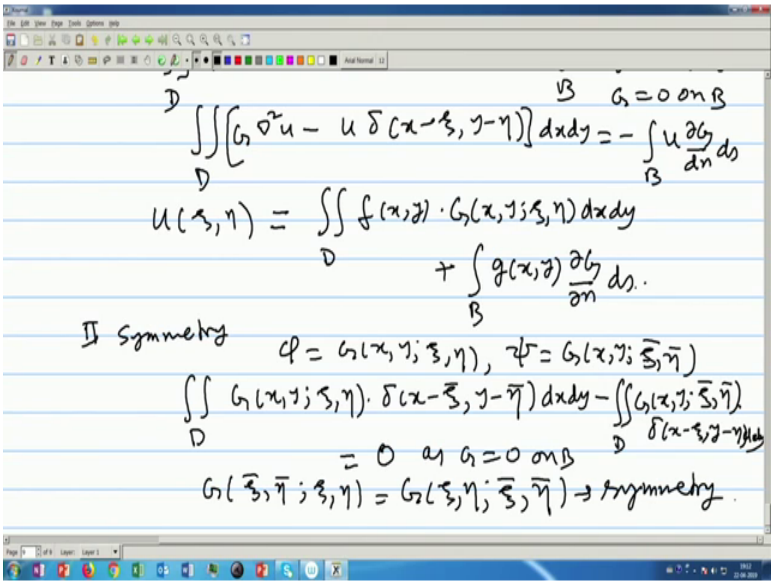

Now del 2 u is given to be f x y and this one so; that means, u what we have is u xi eta if I now integrate this one. So, u xi eta. So, I take out the minus. So, u xi eta my equal to I transfer to this side. So, double integral d del 2 u means h x y what was the what was our notation was equation was del 2 equal to f x y.

So, del 2 u equal to f x y f x y into G x y xi eta d x d y plus u is given to be G. So, G x y del G del n d s. So, this is what the yeah. So, I have notation was all correct. So, this is the solution of the non homogeneous equation in terms of the Green's function. So, this is also a very easy to show that the symmetry condition symmetry can be shown is that. So, this is a solution now, now symmetry say this second property the symmetry. So, the symmetry what we do is in that this is called the gauss or Green's second identity this one. So, here what we put phi equal to say G x y xi eta and psi equal to G x y xi bar eta bar.

So, double integral D G x y xi eta and del 2 psi. So, del 2 psi I can write as del x bar xi bar y minus eta bar d x d y minus G. So, this is G x y xi bar eta bar into del x minus xi y minus eta d x dy equal to 0 as G equal to 0 on B. So, from here what I can find that this will be is G xi bar eta bar xi eta equal to this is G xi eta xi bar so; that means, symmetry. So, now, what we left here is to show that other one that is the jump condition.

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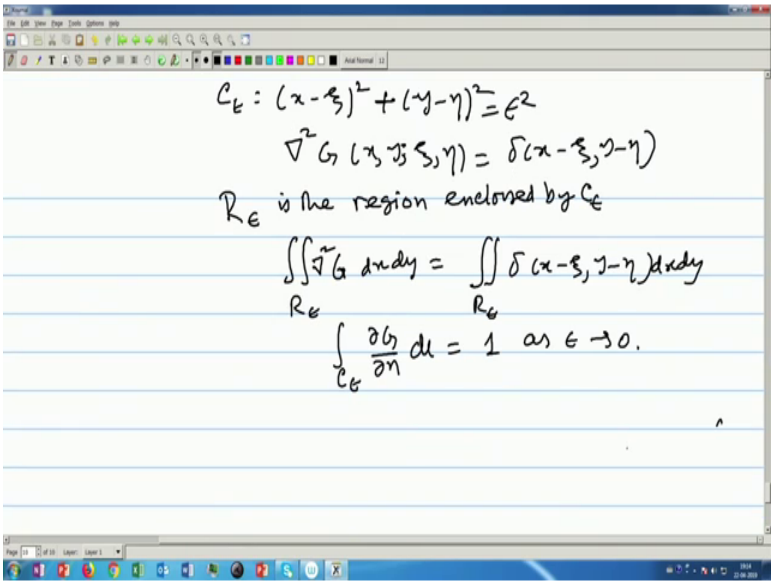

Now, to show the jump condition, so, we consider that C epsilon as x minus xi whole square plus y minus eta whole square equal to epsilon square. Now del 2 G x y xi eta equal to delta x minus xi y minus eta. So, if I consider R epsilon is the region enclosed by C epsilon and then if I integrate over R epsilon del 2 G d x d y is a R epsilon eta d x d y.

So, this I can write as transformed to surface one area C epsilon or the curve. So, this is del G del n d l and this is becoming 1 as epsilon tends to 0. So, the discontinuity of jump discontinuity of this equation is satisfied. So, I will show 1 example 1 or 2 example.

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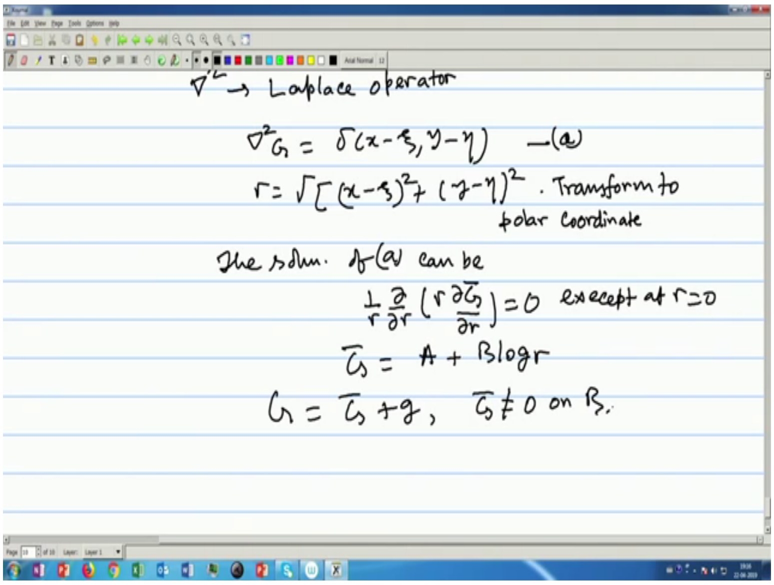

So, now, if we have a Laplace operator del 2 is a say Laplace operator. So, if I choose. So, del 2 G if you to construct this Green's function. So, delta x minus xi y minus eta, say if I choose say R equal to a coordinate a polar coordinate if I choose with y minus eta whole square; so, transferred to polar coordinate with pole at with pole at xi eta. So, what I can find that if a solution of these. So, let us call this is a equation a the solution of a can be like this way I can write as if I transfer to polar coordinate.

And if I consider that this is a del G bar a del r because it cannot be may not be Green's function; because Green's function means it has to have a homogeneous boundary condition. So, G bar can be taken to be some A plus B log r. So, now, if I say that Green's function G equal to some G bar plus small g.

So, that this G bar may not be may not be 0 on boundary B. So, in that case I can find out the B by this manner.

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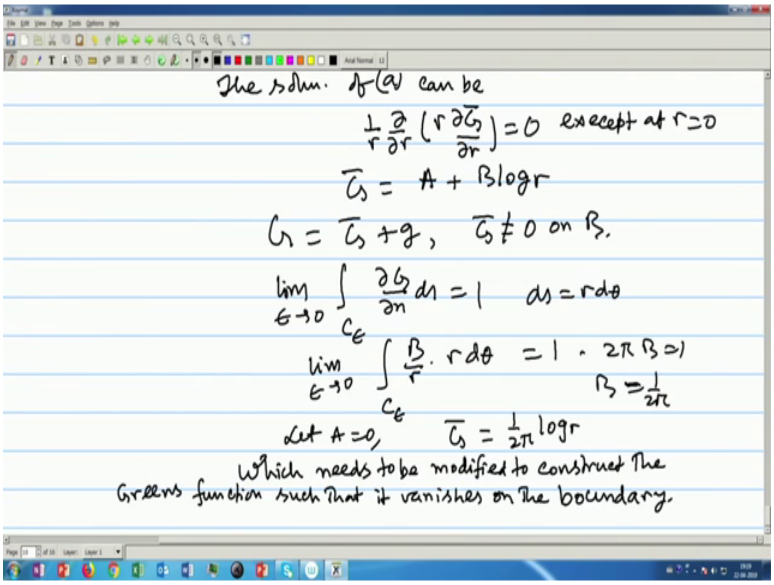

Now, we know we can use this condition that C epsilon del G del n d s is equal to 1 now if it is a circle. So, d s can be r d theta. So, this is a I can write as limit epsilon tends to 0. So, B by r r d theta C epsilon is equal to 1 so; that means, 2 pi B equal to 1. So, B equal to 1 by 2 pi. So, 1 can write and let A equal to 0.

So, G bar can be taken as 1 by 2 pi log r, but which needs to be modified to construct the Green's function to construct the Green's function] so, that such that it vanishes on the boundary. So, it is not a very furnaces on the boundary. So, it is not a very trivial task as such and there are quite a bit manipulations need to be done.

So, even for this kind of linear equation linear elliptic equation or Laplace equation linear Parisian type of equation we are finding is quite difficult. So, that is why the Green's function is only for typical cases one consider. So, most of these are all were very popular when before the super computer computers and all computing power technology are not that much developed. So, today I stop here my discussion on the Green's function. So, next class we will continue.

Thank you.