Mathematical Methods For Boundary Value Problem

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Lecture - 07
Green's Function for BVP and Dirichlet Problem

Welcome back. We will talk today on construction of Green's Function. Basically what we will be doing here is that we know say for a linear operator; linear boundary value problem if we know the solutions for the homogeneous part. And based on this homogeneous equation homogeneous solution we will construct the non homogeneous equation solution of the non homogeneous equation that is all about the Green's function construction.

In that process what we will do is we will convert the boundary value problem to a equivalent integral equation; so that means, now from the differential equation you will get a integral equation. So, that can be done for the ordinary differential equations or also for the partial differential equation also. So, this Green's function technique is quite popular for boundary value problem and or in a differential equations and particularly Dirichlet problem.

So; that means, the heat transporting heat conduction equation; that means, the Laplace equation or rather in this case it is a Poisson equation because we are talking about non homogeneous. Now before that let us talk about the solution of homogeneous equations.

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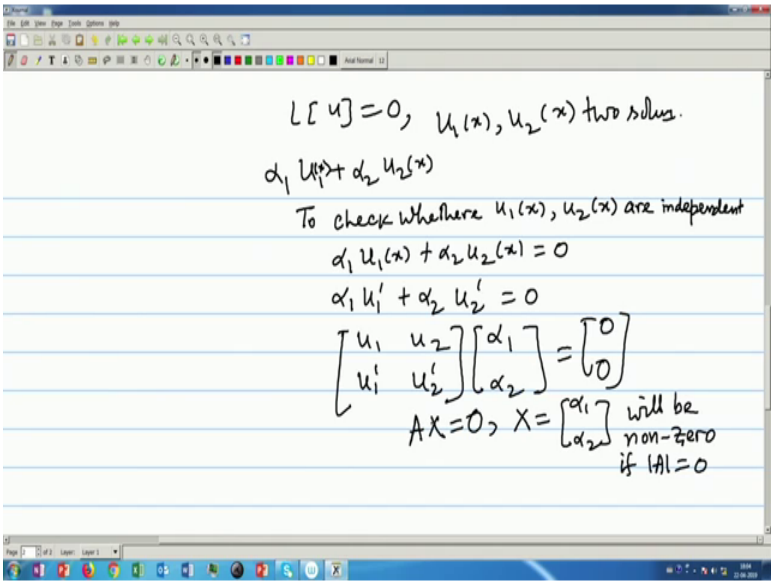

So, consider a linear operator say L u equal to 0 L is a linear operator, second order BVP say for example, it can be a any other orders also. So, suppose v 1 or u 1 x and u 2 x are the two solutions. So, since it is a homogeneous equation; two solutions say. So, any combination u 1 x alpha plus alpha 2; u 2 x is also a solution; so this is also will satisfy the linear equation. Now this will be a new solution provided alpha 1 alpha 2 are any number; any constant it can be real if we are talking about the real functions real solutions.

Now, this will lead to a new function provided this u 1 and u 2 are independent. Now to check whether to check whether this u 1; u 2, u 1 x and u 2 x are independent; so, very simple procedure is that you choose a combination u 1 x plus alpha 2 u 2 x equal to 0. If there is some nonzero both alpha 1 alpha 2 are nonzero then they are dependent; if they are only possible if alpha 1 alpha 2 equal to 0 then they are independent.

So, if I differentiate I get another equation this alpha 2 u 2 dash equal to 0. So, I can write the matrix equation for alpha 1 alpha 2 as u 1 u 2; u 1 dash u 2 dash alpha 1, alpha 2 equal to 0 0 0; so this is a kind of homogeneous equation. So, this is a homogeneous equation it will have x will be non trivial; x we are calling as alpha 1 alpha 2 will be nonzero if determinant of A is 0. If determinant of A becomes 0; then we can have a non trivial solution.

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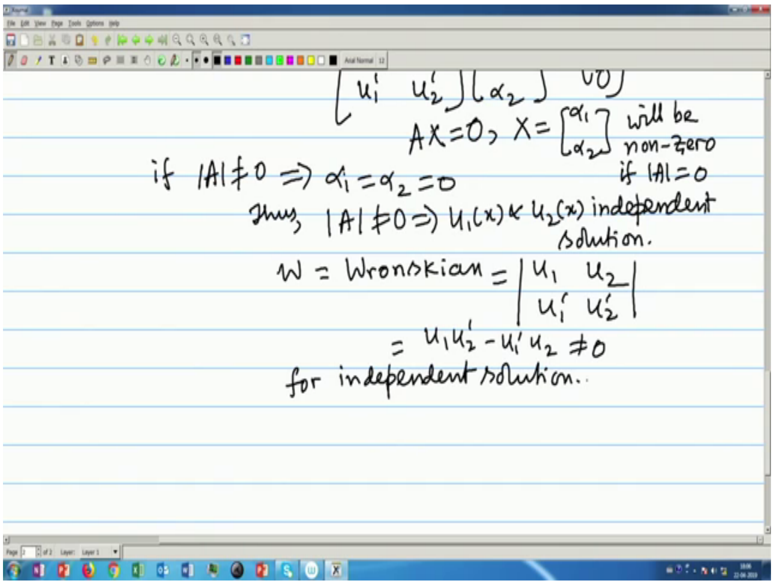

And if determinant A is not equal to 0; this implies alpha 1 alpha 2 is 0 or in other words thus determinant A not equal to 0 implies u 1 x and u 2 x are independent. So, this is one way of independent solution. So, the combination of this u 1 u 2 is a another solution which is not coinciding with the because any solution a one is a solution then any scalar multiplication or any constant multiplication of that solution is also forming a solution.

So, we have to check whether it is a new solutions is forming. So, new solution is provided they are independent. So, this is the one is referred as the Wronskian W. So, this is Wronskian and it is the determinant u 1, u 2; u 1 dash u 2 dash. So, this is becoming u 1 u 2 dash minus u 1 dash u 2 dash.

So, this has to be the Wronskian has to be nonzero to for a Wronskian to be nonzero for independent solution. So, this is one independent solutions this is one way of checking whether we have a independent situation. Now another thing I want to bring it here; maybe we have discussed little bit in the introductory lecture; that is the Dirac delta function.

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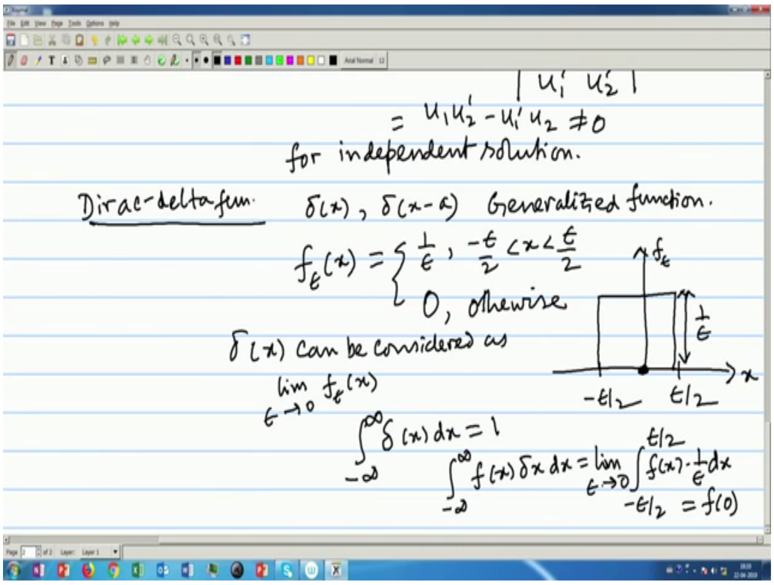

Now, this Dirac delta function is denoted by delta x or delta x minus a; this notation will be using several cases in the talking about the Green's function. So, Dirac delta function how it defines is it is a generalized function. So, it is not exactly the same as we talked about the ordinary function. So, suppose I have a sequence say if I denote let epsilon; if epsilon x if I define this way; it is equal to say 1 by epsilon; when minus epsilon by 2 less than x less than epsilon by 2 and 0 otherwise.

So; that means, suppose in the x domain minus infinity to infinity. So, if epsilon is something like this between minus epsilon by 2 to plus epsilon by 2; if is ranging like this so; that means, this is your 1 by epsilon. Now, so it is also a kind of say any function which is centered at say origin; so, at the point the concentration or mass at a particular point; if we want to find out. So, to define that way; so then this Dirac delta function or this is the process we can adopt.

Now, one way this delta x can be considered as considered as limit epsilon tends to 0; if f epsilon x. So; that means, as I contract, so this point is or the value of epsilon becoming unboundedly large. So, this is the way one can define the delta x. So, normally these delta x or the generalized functions are operated with another function and cannot be set independently.

So, say for example, if we have this integration from minus infinity to infinity delta x; dx that I can say as 1 in the same way if I apply this limit. So, we get this way; so and this is the delta x i am sorry this will be delta x. So, delta x also minus infinity to infinity if it is operated with another function dt dx.

So, this becomes limit epsilon tends to 0; this way we can show the value of for this integral fx into 1 by epsilon dx because that is how we have defined. So, it becomes f 0 if I apply the integral mean value theorem because this one by epsilon is constant comes out of the integral; so this becomes f 0.

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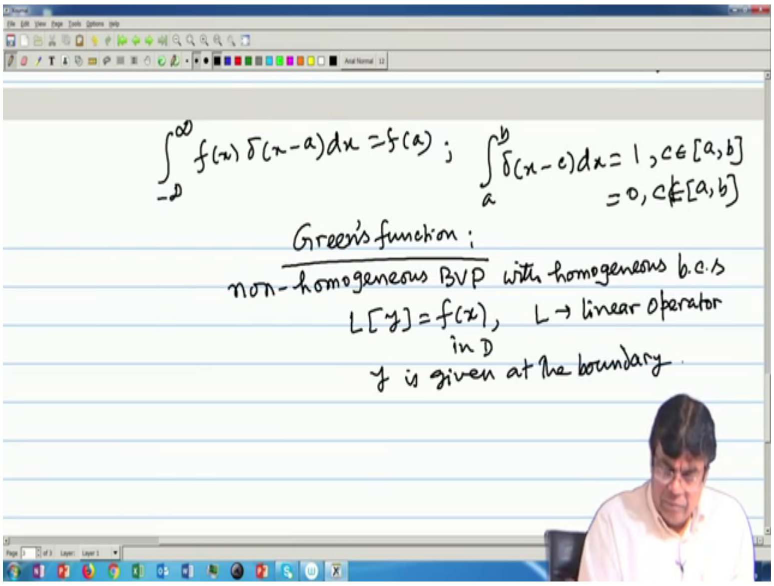

So, another also interesting things comes is that if I take minus infinity to infinity f x delta x minus a can we shifted to some other point this is equal to f a. And another important property of this Dirac delta function is a to b delta say x minus c is d x equal to 1, if c belongs to a b. And it is 0; if c does not belongs to a b so; that means, this is a singularity it shows that it becomes unbounded at as we approach to this point c or a or 0.

So, that is how the Dirac delta function is defined. Now we talk about the Green's function construction. So, this is the Green's function that we will be talking now; say we have a non homogeneous boundary condition; boundary value problem. We have a non homogeneous BVP with homogeneous boundary condition homogeneous b c; b cs. So, let L y equal to f x; again L is a linear operator.

Say L y equal to fx in certain region D and y is prescribed; y is prescribed is y is given at the boundary. I was thinking that probably u should have been better and because x y;

here x is one dimension, but it can be generalized to other dimension. So, in that process it could have been y could have been the other variable. So, let me put this as u is the dependent variable instead of y.

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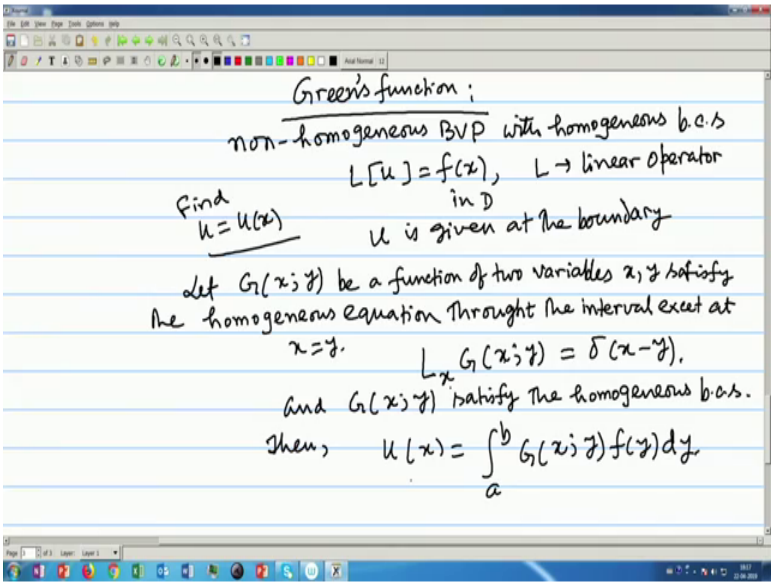

So, this is u; u is given at the boundary. So, u equal to u x is the basically our find u equal to u x; which satisfy this condition. That means say non homogeneous boundary value problem with homogeneous boundary condition. Now even if it is not a homogeneous we could not transform it to homogeneous for some cases, but what we need for simplicity that we have considering u has a homogeneous boundary condition is imposed.

Now, we define a Green's function a function let G x y be a function of two variables. Let us consider a function x y of two variable which satisfies these following criteria; satisfies the it is a satisfying satisfies following criteria. Before that let us say satisfies satisfy the homogeneous equation; homogeneous equation throughout the interval.

If it is a one variable; so this is will be a interval except at; except at x equal to y. So; that means, L x G x y is delta x minus y this is the way we defined so; that means, this is equal to 0 everywhere except at x equal to y which is remain undefined and this subscript L sorry; x is denote that derivative is taken with respect to x.

Now, there are two variable x and y; now when we write because we are giving it as a semicolon why is that one variable is fixed why we keep fix and x is varied. So, we vary x and at x when x becomes y; so that is becoming the one which is unbounded. Now and G x y satisfy the homogeneous boundary condition; homogeneous b cs ok.

So, we have this condition; now if this is the case, then what we can show is that then u can be expressed as u x can be expressed as integral a to b G x y f y d y. So; that means, you have a; you have a differential equation L u equal to f x and now this you have construct a Green's function G. And then determined a integral equation given by this equation a integral equations are result developed.

So, that is the conversion of a boundary value problem or differential equation to a integral equation. So, some cases it is advantageous to solve that; now our question now comes how to get this G? Now to get the G; one condition what we said is G satisfies the homogeneous equation. Now first of all let us prove that if this is the; what condition we have imposed that G satisfy the homogeneous equation.

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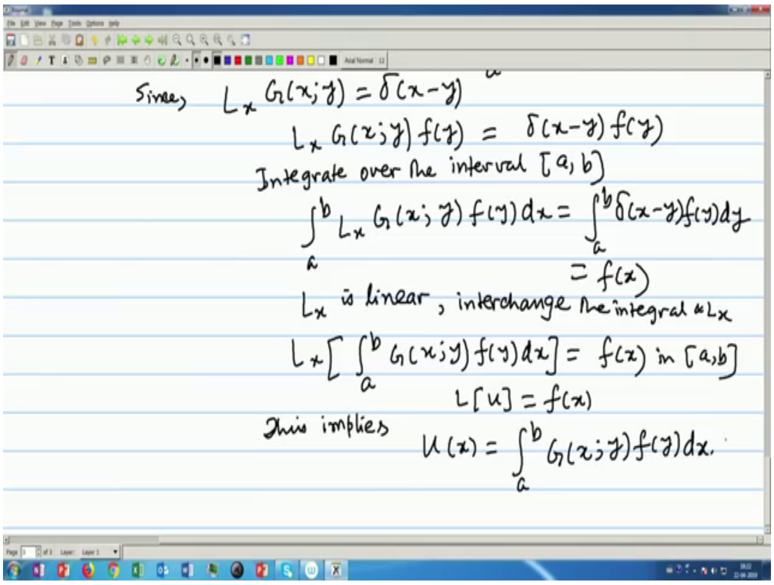

Now, since L x G x y is delta x minus y. So, multiply both side by f y and integrate with respect to y. So, what we do is L x G x y f y equal to delta x minus y f y. Now integrate say our domain is a between a to b we have not we have defined here; so the domain is a to b. So, integrate over the interval a b. So, they get is a to b L x G x y f y dx equal to integral a to b; delta x minus y, f y; d y. Now this x y are within the domain of a b.

So, this can be written straight away as f x; this can be written as fx itself because that is how the Dirac delta function we have already defined. And this site since L x is a linear operator is linear. So, interchange the; interchange the integral and the linear operator integral and L x. So, what I do is we can take out this L x integral a to b G x y f y d x equal to f x in a, b because x belongs to a b.

Now, if I compare with the given boundary value problem. So, our problem is this L u equal to f x; this implies u the solution can be written as a to b; G x y; f y d x. So, in other words if G is constructed then we get the solution given by this way.

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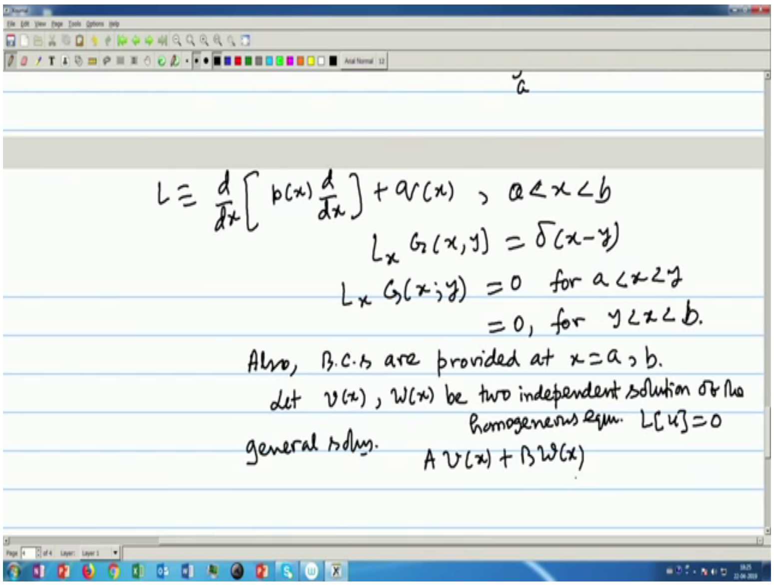

Now we consider a; we consider a situation where we consider a say a linear operator L is given by say d d x of general way p x d dx plus q x in a; well I can take 0 less than x less than a or a less than x less than b that is how we have started doing.

So, what we have here these L x G x y equal to delta x minus y. So, that can be written as L x; G x y equal to 0 for a is varying is x less than y and is also 0 for y less than x; less than b the other part. So, we have broken into two sub intervals and also we have the conditions; also B cs are provided at x equal to a b. Now let v x and w x be two solution; independent solution of the homogeneous equation; be two independent solution of the homogeneous equation.

Homogeneous equation means here L u equal to 0 they are the two independent solution. So, general solution is; general solution can be written as any combination A v x plus B w x; where they are independent. So, any two constant can be expressed.

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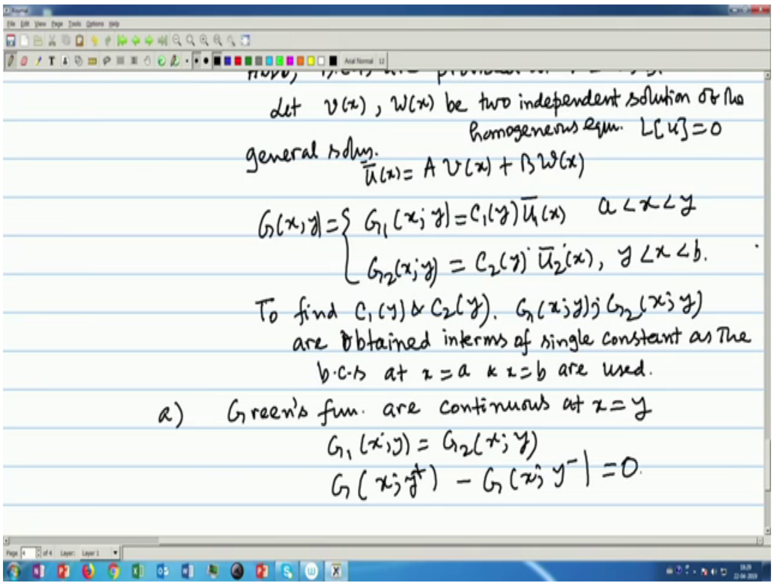

So, in one part I can write; so thus I can write the say let us call this is as this solution we call as u bar x say. So, thus what we have is G if I call this G 1 x y which is equal to G say; let us call this is the way say G x y which is equal to G 1 x y equal to say some; this can be a function of ok. Let us take this as some C 1; C 1 say v x for which is valid for a less than x less than y and this is C 2.

So, this will be a function of y C 2 function this can be a function of y and w x in I am sorry. So, what we will have here because general solution we are writing. So, it can be; so it can be any combination of this two. So, we can write as C 1 y u bar 1 x and this another one is C 2 y u bar 2 x or y less than x less than b. So, this is the way it forms the; so now, our task is; so this is we call as G 2 x y.

So, we need to find C 1 y and C 2 y because it will come in terms of one constant because G 1 x y r of G 1 x y G 1 x y and G 2 x y are obtained in terms of single constant as single constant; as the boundary conditions b cs at x equal to a and x equal to b are used. Now we need to impose other conditions two more condition need to be imposed to get. So, we impose the conditions a; Green's functions are continuous at x equal to y.

So, in other words G 1; G if I say so; that means, G x y; G 1 x y equal to G 2 x y. So, now G 1 x y means this is a left way. So, G x y minus; minus G 2 G x y plus rather this is the customary is 0. So, y plus y minus means we are approaching from left side or right side.

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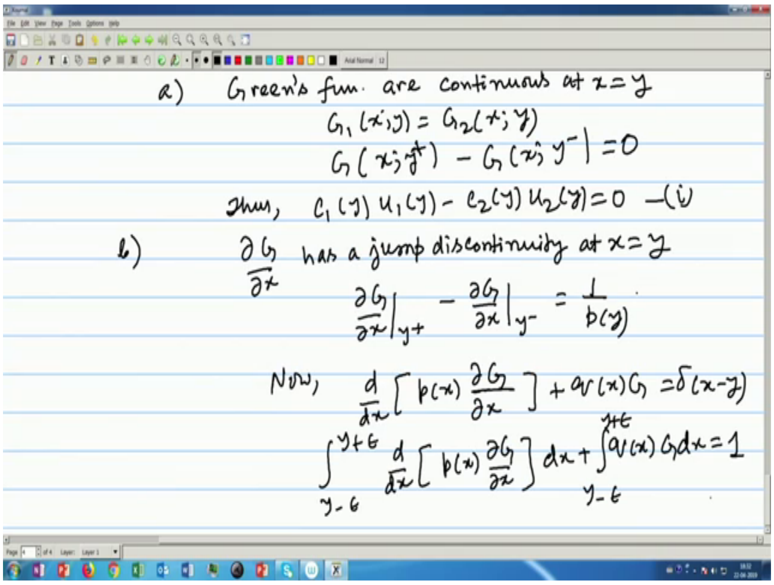

Now, Green's function continuous; so what I can write is thus we have a condition will be C 1 y; u 1 y minus C 2 y u 2 y equal to 0. So, we have a condition like this way; then another condition that G x y; del G del x a jump discontinuity at x equal to a; at x equal to y. So, how to measure that? That means, what we have is del G del at x equal to y plus minus del G del x at x equal to y minus is equal to some constant. And we will find this will be equal to 1 by p y; now how to prove that.

So; that means, there is a discontinuity to show that now what we know is d d x of p x d G d x or you can say del G del x because we are talking about two variables here. So, del G del x plus q x G equal to delta x minus y. Now if I integrate both sides between y minus epsilon 2 y plus epsilon d d x of p x del G del x plus d x plus q x G y minus epsilon to y plus epsilon d x and that is equal to 1. Now that is equal to 1 now because the delta function now this part is continuous.

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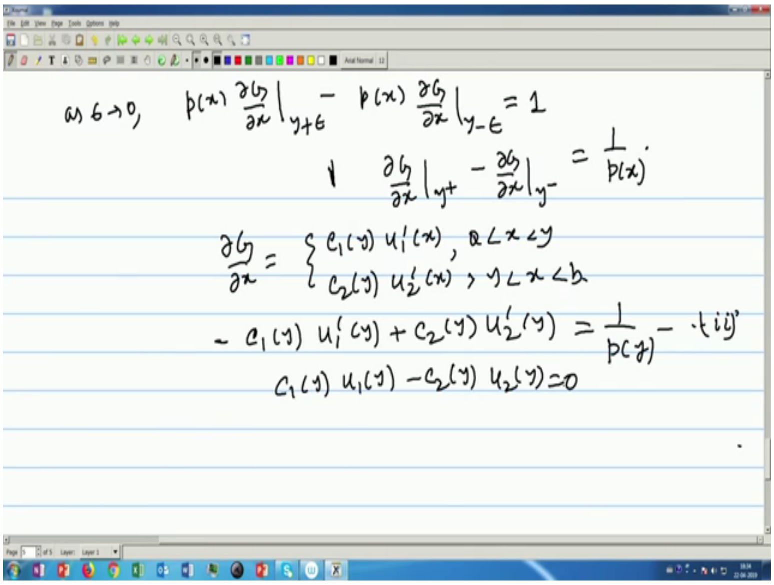

So, what we have is. So, p x; p x del G del x at y equal to y plus epsilon and minus p x del G del x at y minus epsilon and what we have is here q x and G x they are continuous. So, if I make as epsilon tends to 0; so this part will be vanishing. So, what we will have is this is one. So, in other words; so what we will have is del G del x at y plus minus del G del x at y minus equal to 1 by p x; so we have this condition.

So, now if I apply this condition; so del G del x is becoming C 1 y; u 1 dash x and C 2 y; u 2 dash x for 0 less than x less than y; y less than x less than sorry a less than x less than b. So, if I apply this condition; so what we have is C 1 y; u 1 dash y this is what is so this will be minus plus C 2 y u 2 dash y equal to 1 by p y.

So, this is our second condition. So, already what we know is C 1; y u 1 y minus C 2; y u; 2 y equal to 0, already we have shown that.

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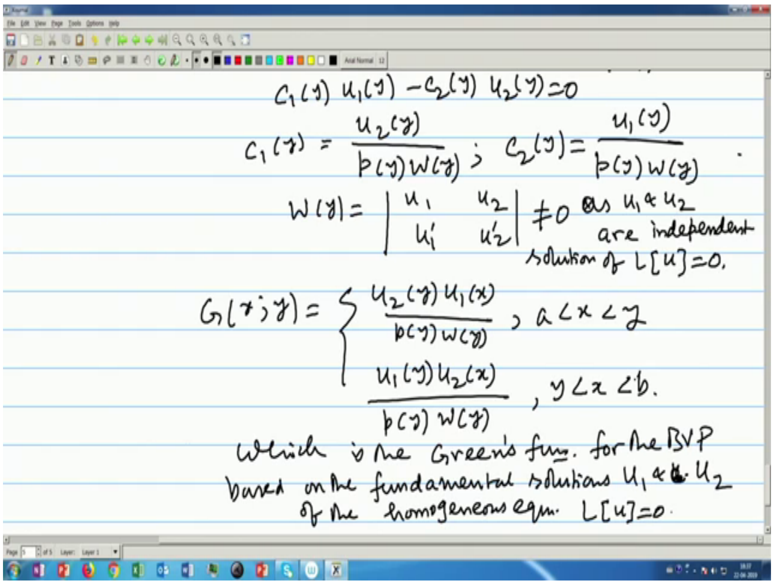

Now this two can be used to get that C 1 y; so C 1 y will be equal to u 2 y by p y into w y; w y is the Wronskian and C 2 y equal to u 1 y by p y into w y. Now this w y is the basically the Wronskian. So, u 1 u 2 and u 1 dash u 2 dash which are the which is not equal to 0 as u 1 u 2 are independent solution; independent solution of L y; L u equal to 0.

So, very quickly I can write the Green's function. So, G x y our required Green's function is nothing, but u 2; y; u 1 x by p y, w y when 0 less than x less than y. And u 1 y u 2 x by p y w y when y less than x less than a sorry; this is a; this is a; so this is a and this is b. So, one thing from here; so this is the Green's function which is the required which is the Green's function for the BVP; based on the fundamental solution solutions u 1 and u 2; u 2 of the homogeneous equation L u equal to 0.

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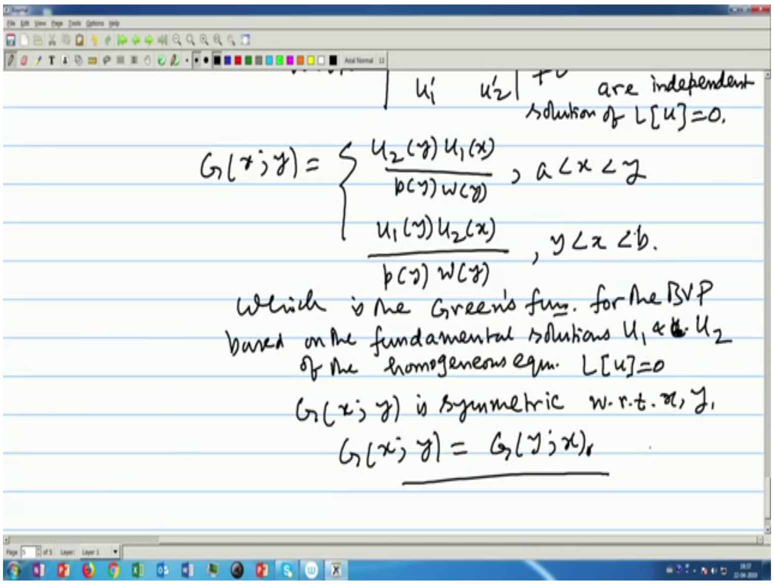

So, one observation is that G x y is symmetric with respect to x, y. So, in other words G x y I can write as G y x. So, we stop here you, now we will continue next one.

Thank you.