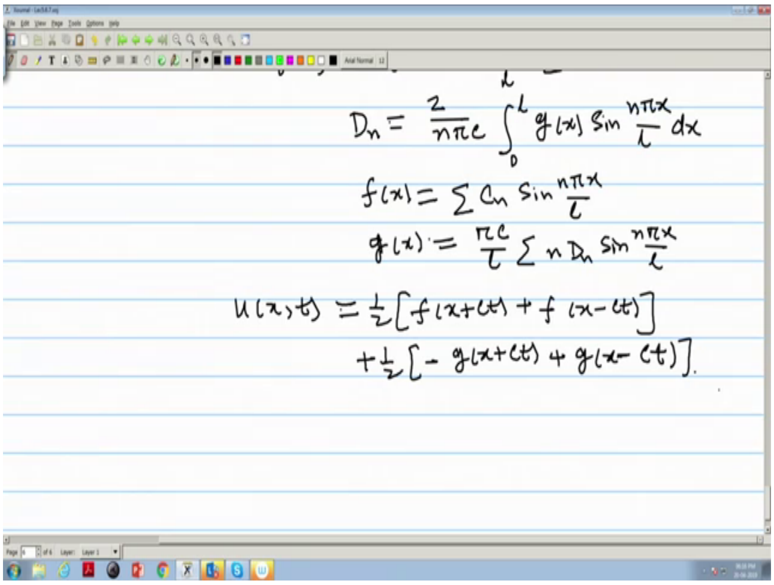
Mathematical Methods For Boundary Value Problem

Prof. Somnath Bhattacharyya
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 06
Solution of Linear Parabolic, Hyperbolic and Elliptic PDES with Finite Domain by Eigen Functions (Contd.)

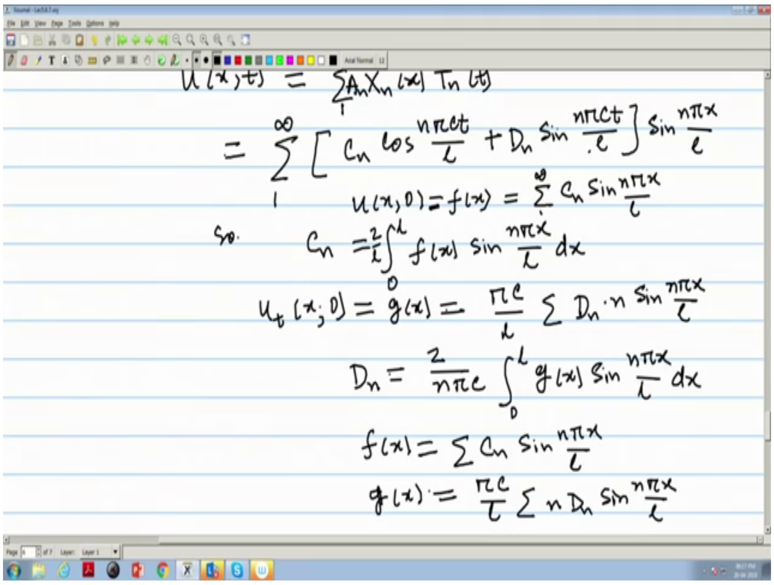
So, to continue with our previous discussion on solving by Fourier series expansion of the linear wave equation, we are considering homogeneous equation with homogeneous boundary condition.

(Refer Slide Time: 00:47)



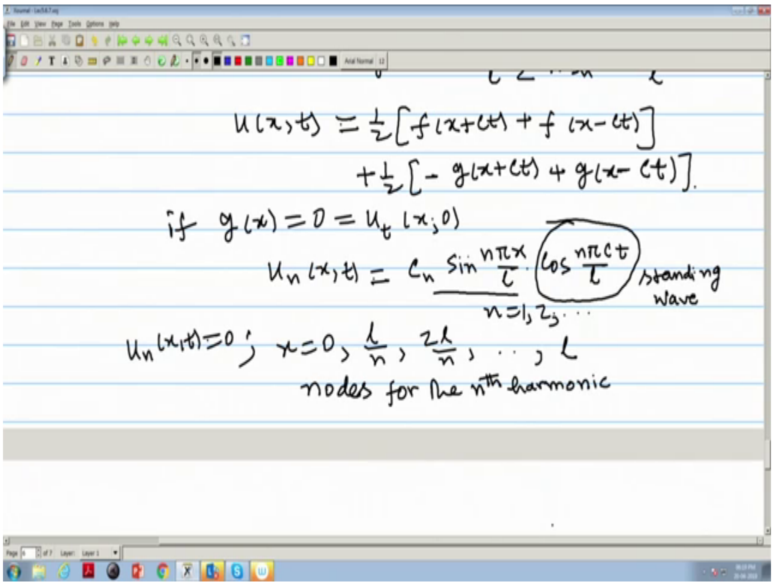
So, we get a solution of this form as we discussed before.

(Refer Slide Time: 01:04)



Now, some characteristics of the solution we can discuss now. So, one of these is say for example, if I have a situation say suppose a string which is attached at a fixed at the two ends and it was deformed to a initially a certain form as f x and then without any velocity initial velocity is 0 that is what I wanted to say.

(Refer Slide Time: 01:24)

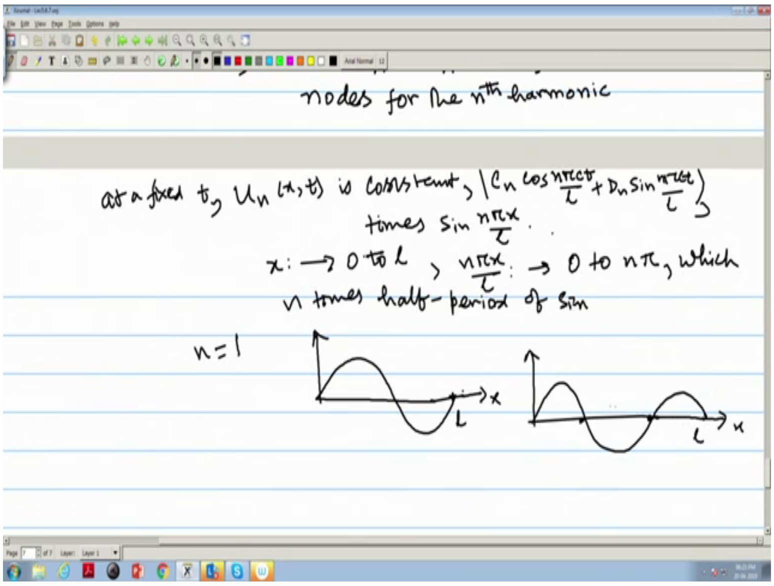


See, suppose if g equal to 0, if g equal to 0 so, what we have is that is u t x 0 is 0. So, in that case what we have is u n x t each of the solution will look like some constant C n into sin n pi x by l into cos n pi c t by l. So, n is varying from 1, 2, etcetera.

Now, this is a representing a standing wave with amplitude as given by this way and frequency is pi c by l. So, and say for a for n equal to no, I can say this way that x equal to 0, u n x t is 0, l by n, 2 l by n etcetera up to l u n is 0 u n x t is 0. So, that means, displacement is 0 for x equal to 0 x equal to n which we call as the notes for the n th harmonic. So, each of this we can call as a n th harmonic.

Now, so, this is a standing wave represents. Now, at a fixed t at a fixed t, so, this is constant this part is constant and their displacement is measured by this way. So, this is the one we can call as a amplitude.

(Refer Slide Time: 04:19)



Now, if we have say so, if we have say not g not equal to 0. So, u n; so, u n, so, each of this mode we can say as the u n x, t is at a fixed t; at a fixed t; u n x t is constant which is equal to C the whole C n cos what was that n pi c t by l plus D n sin n pi c t by l way c t. This whole times constant times sin n pi x by l.

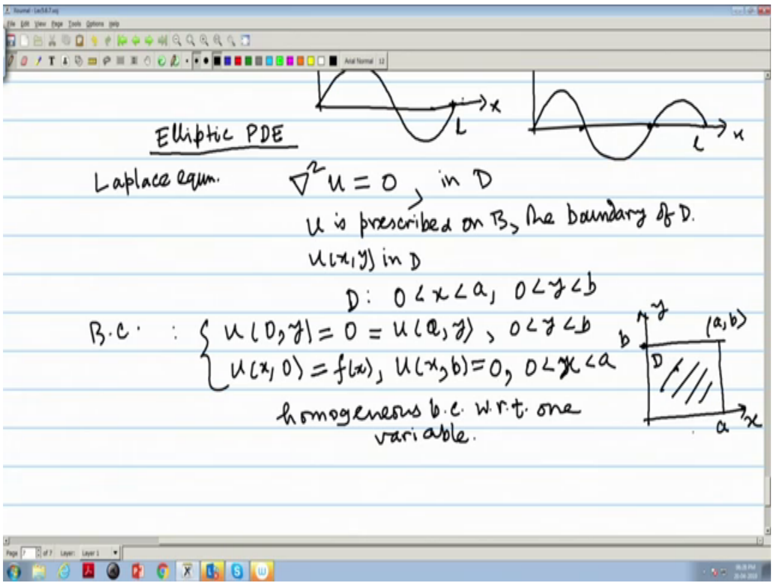
So, x range from say 0 to l x is ranging from 0 to l. So, this u n is ranging. So, n pi x by l; n pi x by l range from 0 to n pi. So, this is the n times the half period of sin, n times the so, which is n times half period of sin. So; that means, say n equal to 0 that is the usual position; n equal to 1 you have a situation if you have n equal to 1 so, if you have a situation like this way n equal to 2 you have this is x l like that right.

So, this is the; so, this is the form of different form or different step the vibrating sting we will assume as the time progresses. So, it can have different mode of vibration and these are the form or say sin curves will take. So, depending on that harmonics we have number of nodes how many it will within the interval can be talked about.

Now, the same idea or same technique will illustrate for another important type of equation which is the elliptic type of equations. And is also some time it is very important this like Laplace equation or Poisson equations which provides the heat transfer heat distribution or electric potential or force potential any potential equation is covered by either a Laplace equation or Poisson equation and these equations these type of problems are purely boundary value because this all the boundary conditions are given.

So, for example, you have a sphere whose temperature on the surface of the outer surface of the sphere we know the temperature distribution and there is a heat source and if you want to find out the temperature at any point within the sphere metal sphere so, that is governed by the equation the Laplace equation.

(Refer Slide Time: 08:27)



So, let us first talk about a solution of the Laplace equation. So, let us take u is the 1. So, now, we talk about elliptic PDE and say this is the Laplace equation if it is a homogeneous. So, first we restrict to homogeneous equation del 2 u equal to 0 and in D certain domain and u is prescribed on B, the boundary of D boundary of the domain D.

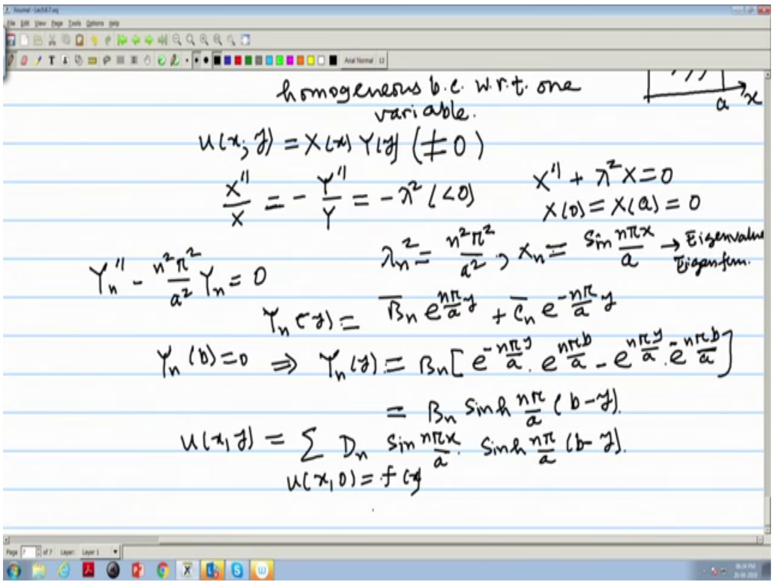
So, our task is what is u x, y to obtain in the in D inside the D. Now, first we let us talk a simple situation. So, D let us take a rectangle; D is given by a or instead of a let us take is 0. So, origin is the first point 0 less than x less than a 0 less than y less than b, this is the rectangle. And so, we have the condition BC boundary conditions are say are prescribed one condition should be homogeneous condition in one variable for our way of simple way of solving.

So, 0 to y say; that means, this is for 0 less than y less than b. Another BC is about the x that can be non homogeneous that has to be no homogeneous otherwise if you have a homogeneous equation homogeneous boundary condition you will get a zero solution; that is not possible to have a nonzero solution. So, this can be a homogeneous boundary condition. So, this is for 0 less than; 0 less than x less than a.

So, basically what we are looking for really have a domain say like this way this is say x this is y and we have a domain these. So, this is say a, b this is a, 0 and this is b, 0. So, the solution to be find out inside the domain D; this is we are calling as domain D and the four sides are the boundary and u is prescribed on the boundaries. So, you need to find out u at the interior point.

Now, one condition as I said is that the one of the boundary condition should be homogeneous B C with respect to one variable without that also is possible, but there is some tricks to be applied. So, for the time being we assume that we have a situation given by this way elliptic PDE given by this way is also referred as a temperature distribution equation say certain slab we of certain heated to a certain temperature boundary are maintained with a fixed temperature and heat diffusion is in progress. So, at any position the temperature inside the domain is governed by that Laplace equation.

(Refer Slide Time: 12:48)



So, we look for a solution u x, y as X x, Y y. Now, if I substitute I get X double dash by X equal to minus Y double dash by Y. Again, this has to be negative less than 0, you can challenge this and you will find that it will be a non-trivial solution. So, for the sake of non-trivial solution we should have it is negative.

And if I solve X double dash plus lambda square X equal to 0, because if we have homogeneous condition with respect to x then we should take up the solving the X capital X; that means, the function with respect to X. And if we have the condition homogeneous condition is given with respect to Y, so, then we have to take the first solve this Y. So, if I solve this so, what I get is lambda n in the same way lambda n square is comes out to be n square pi square by a square and corresponding X n is sin n pi x by a. So, this is the eigen value and eigen value and eigen functions respectively.

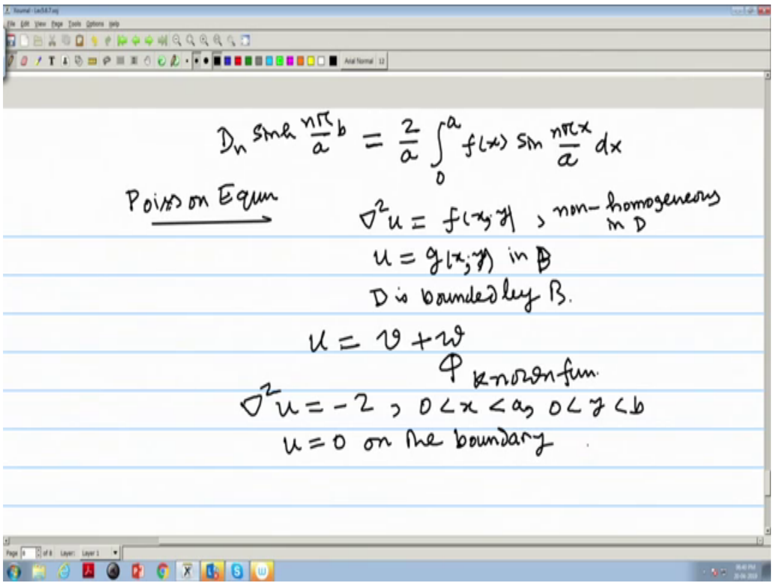
So, with that we find out the Y n double dash minus n square pi square by a square Y n equal to 0. So, we get Y n, this is Y n y is comes out to be B n e to the power if I do this after doing some manipulation n pi and using the boundary conditions. So, if I use the boundary condition that say it will be B n (Refer Time: 15:41) B n if we write; so, B n n pi and a by y plus say C n e to the power minus n pi a by y.

And what we have is y not y capital Y Y n because we have homogeneous condition which is at b so, Y n b is 0 this implies I get a relation between B n and C n. So, using this I can write as y sorry capital Y. So, capital Y n y can be written as some constant say let us call some B n bar and C n bar. So, let us called B n e to the power n pi minus n pi y by a into e to the power n pi b by a minus e to the power n pi y by a into e to the power n pi b by a minus n pi b by a. So, this gives you B n sin hyperbolic n pi by a b minus y.

So, now we have used both only one conditions left both conditions were x is used and one homogeneous condition for y is used. So, we can write now u x, y in terms of a single constant this order is called that D n because X n is also having a constant; so, C B n and that constant because all this u n will be multiplied with the constant.

So, combinedly I can consider only one constant. So, D n sin n pi x by a into sin hyperbolic n pi by a into b minus y. Now, which condition we left to use or left is u x, 0 equal to f x. Now, as usual again we use the condition for orthogonality of the eigen function.

(Refer Slide Time: 18:49)



So, one can do this way that D n and if I multiplied by so, x equal to 0, here y equal to 0 if I put so, D n sin hyperbolic n pi by a into b this becomes and this is 2 by a. So, 2 by a 0 to a f x sin n pi x by a dx. So, this gives you the solution requisite solution. So, from here we get the D n kind of all the constant D n's are obtained. So, this gives the series solution. So, this is from the 1 to infinity in terms of the eigen functions.

So, we have discussed so far all the particular type of equations parabolic hyperbolic and now this is the elliptic and these are all very important in the context of say transport equation that is governed by the parabolic equation that is without any dissipation only rather which is only by dissipation. So, that is the parabolic PDE only which is governed by the diffusion mechanism. Another is that convection mechanism without any diffusion occurrence that is the hyperbolic one and this is the elliptic which governed say any potential equations or heat distribution in a steady state and all these things.

Now, some cases when we have a non-homogeneous situation we have already discussed one situations. So, non homogeneous means where we have to expand the non homogeneous term in a form of a series expansion eigen function expansion technique that is the thing we have utilized. So, for example, even the Poisson equations can also be solved by the same method.

So, Poisson equation means what we have is now one thing to remember is that the elliptic equation cannot have a infinite domain in the sense that it is a boundary value problem. Because x and y they are corresponding to or r theta of whichever way so, the space variable. So, there is no time derivative involves. So, the space variables which will have a finite range, but one can have in several situations also infinite semi in finite or infinite cases. So, in this case the technique whatever we have discussed here is no longer applicable or straight away cannot be applied.

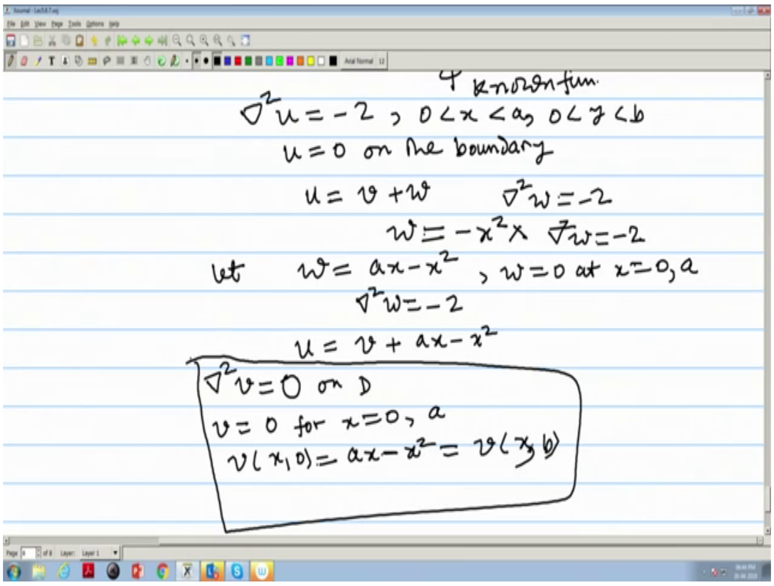
Now, consider these Poisson equation means when we have non homogeneous situation so; that means, you have a situation like this way. So, if we have a situation like this. So, this is a non-homogeneous equation non-homogeneous and we can have a boundary condition say u equal to maybe g x, y in b sorry b which is the boundary non-homogeneous in D. So, D is a B is a boundary of D; D is bounded by B.

So, as the previous one we discussed there what we required is the boundary condition for one variable must have to be homogeneous. So, now we have if we want to apply the previous case. So, we have to do some trick and the trick will be that if I write that u as some function v plus w. So, w some known function a simple form should be chosen in such a way that w takes care of the non-homogeneous f x y and also w makes the v 2 assume homogeneous boundary condition.

Now, say we have a say equation like del 2 u equal to minus 2 for example, which is a Poisson equation del 2 u equal to minus 2 and in say 0 less than x less than a and 0 less than y less than b and u equal to 0 on the boundary; rectangular boundary in this case.

So, it is a Poisson equation and it is the rectangular boundary, but homogeneous. Now, it is a non-homogeneous equation, homogeneous boundary condition, so, nonzero solution is possible.

(Refer Slide Time: 25:01)



Now, if I write u equal to some v plus w; w to be determined. Now, w should be in such a way that del 2 w equal to minus 2 and can be obtained very easily. So, if I choose if I do not want to disturb the non-homogeneous boundary condition for y. So, if I choose say for example, w equal to for example, if I choose say minus x square. So, if I choose w equal to minus x square that satisfies.

So, delta to w equal to minus 2 that is happening, but what is not happening is that the boundary condition. So, at x equal to a, this is becoming x square minus x square. So, that means, it was homogeneous, now for v this is becoming a non-homogeneous. So, this way it does not work. It may not it would be a good choice if I simply take w equal to minus x square.

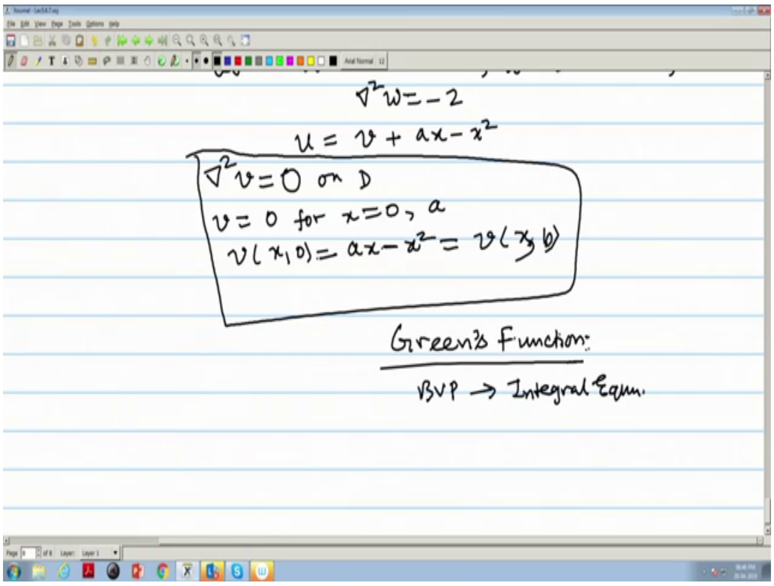
So, one way to one remedy is that if I choose say let w equal to ax minus x square. Now, if I would have taken a only or a square minus x square then the problem was at x equal to 0, it will be again a non-homogeneous, but if I now take this one. So, what will happen is w equal to 0 at both x equal to 0 as well as x equal to a and delta 2 w equal to minus 2. So, both the requirement are satisfied. So, instead of choosing these I choose w equal to ax minus x square.

So, now, our u equal to v plus ax minus x square and this v is now satisfying all the conditions that is v equal to delta v equal to 0 in domain D and v equal to 0 on for x equal to 0 and a in the entire domain of y, but v is v x, 0 is x ax minus x square equal to v x, b. So, we have non-homogeneous boundary condition for y x, b.

So, now this can be solved the way we have discussed before. So, these tricks are possible for solving the situation where we have a Poisson equation, but many cases we may not have a simple type as we have here we may not be that lucky if f x, y is a complicated one. So, but in many simple situations so, our requirement should be or our objective will be that if we have a Poisson equations so, that has to be to a homogeneous equation by a substitute substitution like this way by using a simple function x and a boundary condition with respect to one of the variable must be homogeneous otherwise then the situation creates a difficulty.

But, in many complicated or practically applicable problems we may not have those readily available solutions. So, in that case we have to rely on the integral transform techniques which will be discussed next.

(Refer Slide Time: 29:52)



And prior to that what we will be doing is the Green's function; Green's function technique so, this will be our next topic. So, in the Green's function technique what we do is we reduce this BVP a Boundary Value Problem we reduced to a integral equation. So, this is the objective for the Green's function technique.

But, again the Green's function technique also has several drawbacks and all because the integral equations is also in it another complicated. So, the subsequently we will talk about the integral transforms and the most useful things for all the complicated situation is the numerical methods so that we will discuss in the subsequent lectures.

Thank you for today.