Mathematical Methods For Boundary Value Problem
Prof. Somnath Bhattacharyya
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 05
Solutions of Linear Parabolic, Hyperbolic and Elliptic PDES with Finite Domain by Eigen Function

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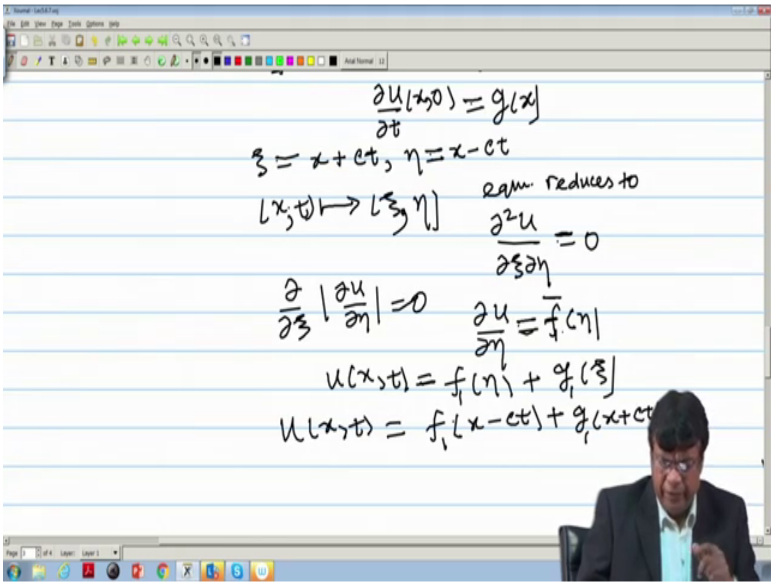

So, I will come back again now what we have is u x t is given by f x minus ct plus g x plus ct. Now; that means, it is now we made a confusion because this was the initial condition was written in terms of f and g; obviously, these are different. So, we call as something like f 1, g 1 and f 1, g 1 like this way.

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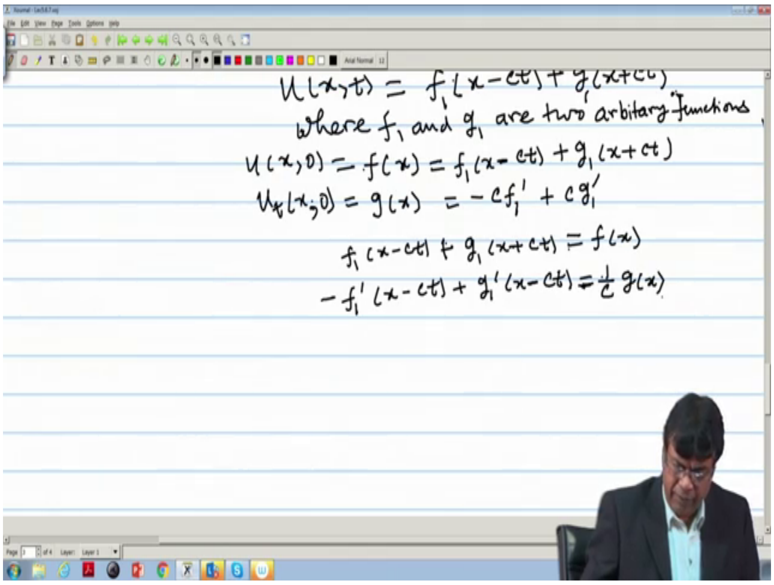

So, they are obviously, different from where this f 1 and g 1 are two arbitrary functions; arbitrary functions yet to be determined. Now, if I use the conditions what we have here; so what we have is u x 0 is given to be f x. So, I can write u x 0 equal to f x equal to f 1 x minus ct plus g 1 x plus ct that is one condition another condition we have is del u del t that is u t x 0 is given to be g x and this lead to del u del t. So, minus c f 1 dash plus c g 1 dash.

Now, remember that this f 1 and g 1 are single valued function of x minus ct and single variable function of x minus ct and x plus ct. So, here del f del x or del f del t or del del t or del del x implies an ordinary differential equation or ordinary derivative not differential equation ordinary derivative. So, what do you what are the two relations we have is for the unknown functions f 1 and g 1 is one is f 1 plus g 1 rather I should write in with the argument, so that is very important. So, what we have now f 1 x minus ct plus g 1 x plus ct equal to f x and f 1 dash x minus ct minus plus g 1 dash x dash ct equal to 1 by c g x.

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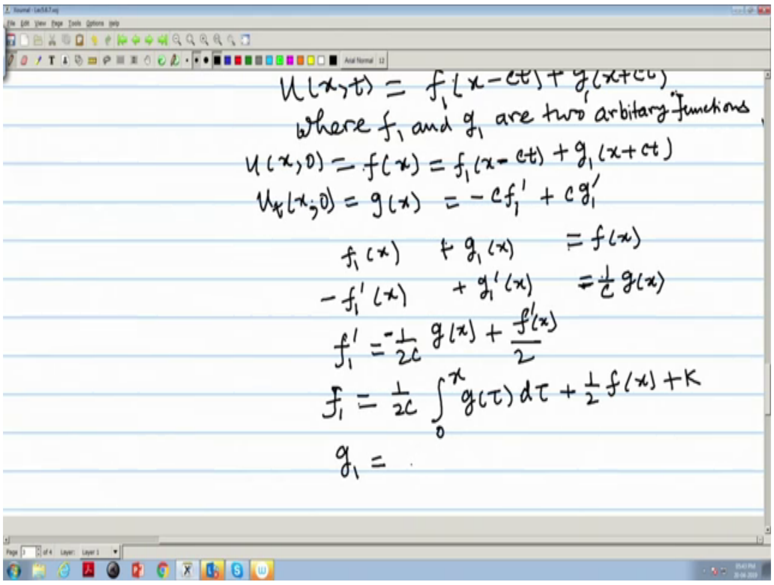

Now, if I eliminate and one can write this as f dash f 1 dash I can write as 1 by 2 c g x plus g 1 dash by 2. What I did is basically I have differentiated; I have differentiated this one with respect to x, so f 1 dash plus g 1 dash equal to f dash. Now, if I so this is x plus ct. So, if I subtract these two; so f 1 dash x minus ct is comes to be subtract if I subtract. So, that means, what I should have here is this will be minus and this is plus, so subtracting.

So, this is minus and this is plus, but this one is nothing, but f dash by 2 f dash x by 2. So, sorry here it is t equal to 0, so since it is t equal to 0 this is the confusion we made if it is t equal to 0; so this is t equal to 0. So, this is nothing, but function of x only t is out because if I put here t equal to 0. So, this is f 1 dash and g 1 dash x only so; that means, f 1 dash x g 1 dash x equal to f x and this 1. So, and similarly, so I can write as f 1 equal to 1 by 2 c, if I now integrate between 0 to x. So, 0 to x; so this is a g tau d tau plus half this is f x plus some constant integrating constant. So, f x is integrated f dash x is integrated between 0 to x.

So, this becomes f x plus a constant and all together all this constant we call as; we call as. So, this is small f f 1; f 1. So, if this is the one then g 1 equal to f x e; so g 1 equal to comes to be f x minus small f 1 x. So, what I get here half if I substitute it here.

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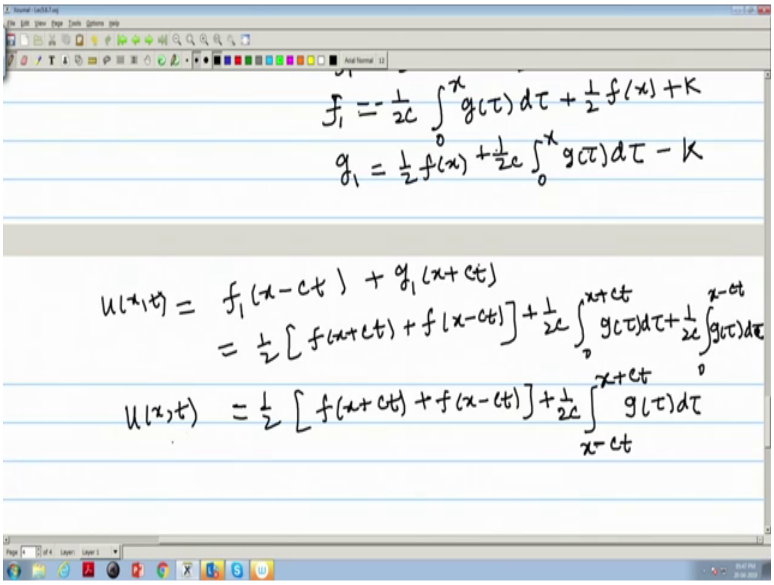

So, half f x half f x there is a minus this is a minus. So, 1 by 2 c 0 to x g tau d tau and minus k. Now our g is x 1 ok, so there is one minus here. So, what so these are the expansion for f 1 and g 1. So, what I can write is now we need f 1 x minus ct am I right u we writing as x t equal to f 1 x minus ct plus g 1 x plus ct. So, u x t we can write now.

So, f 1 and f 2 are rather f 1 and g 1 which was arbitrary at this stage arbitrarily and unknown functions, so they are no longer unknown. So, we got f 1 and g 1 in terms of the initial conditions small f and small g which is provided in terms of the initial condition which is as provided in the initial condition, so f 1 x minus ct plus g 1 x plus ct.

So, if I substitute this what I get here is half of f x plus ct half of x x plus ct plus g sorry f x minus ct it s not g; g will come later f x minus ct. So, this is one term and what we have is plus 1 by 2 c 0 to x plus ct g tau d tau plus 1 by 2 c 0 to x minus ct g tau d tau k got cancelled out. So, these together I can write as f x plus ct plus if x minus ct plus 1 by 2 c this I can write as x minus ct to x plus ct g tau d tau.

So, this is the solution of the wave equation when we have a domain boundary is infinitely long, so; that means, u is varying from minus infinity to infinity. So, in that case the solution it is given in terms of the initial conditions f and initial velocity this is the initial form of u and this is the initial velocity.

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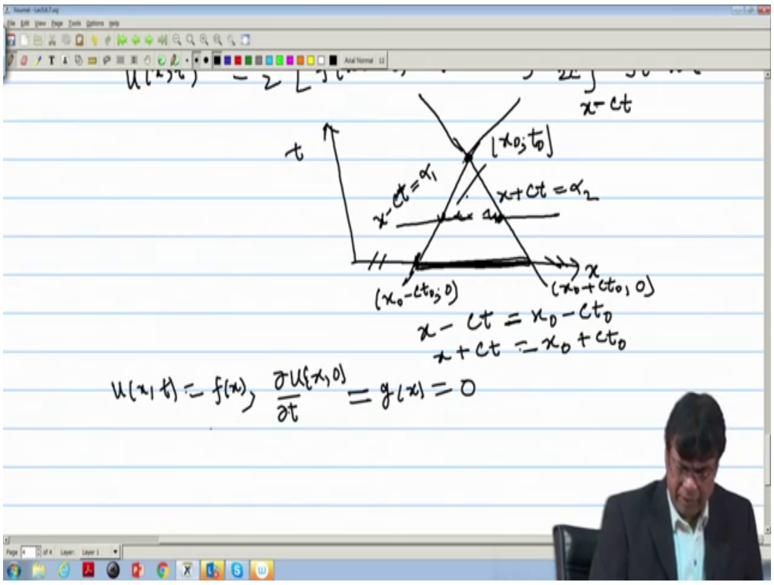

Now, what it means that in the x t domain. So, if I draw in the x t domain, so this is the line say x, so; that means, this is the initial line t equal to 0 and this is the line say t in the x t domain. So, if I drop two straight line say x this will be x minus ct I think; x minus ct equal to sum alpha 1 and x plus ct equal to alpha 2 two straight line of slope plus minus c. So, the solution at this point say x t is depending, so this is the one is intersecting the line t equal to 0. So, say so; that means, if I call this is the point x 0. So, this is the point as x 0; so the solution are depending on; so say.

So, this is at x 0 t 0, so x minus ct equal to say this is let us write as x 0 t 0 because x t are the variable. So, any point say x 0 t 0 and from there I draw this two straight line. So, this is equal to x 0 minus ct 0 this is the line another line is x plus ct equal to x 0 plus ct 0. So, these two straight line are intersecting the t equal to 0 line. So; that means, if I substitute t equal to 0 this is the point as x 0 minus ct 0 and this the point is x 0 plus ct 0; 0 is the time is the other point is, so x 0 minus ct 0 ok, so, x 0 minus ct 0 0 and x 0 plus ct 0 0.

So, the solution over here if I put x 0 plus x 0 t 0 if I substitute here x 0 t 0. So, what you find that solution does not bother about how the value of u outside this domain. So, it is depending on how the solution is prescribed within this interval x 0 minus ct 0 and x 0 plus ct 0 interval itself. So, it does not care about how the solution is initial solution is prescribed of outside of this interval. So, that is why its called the domain of dependence and then similarly if I mean if I have taken these point instead of t equal to 0, so, some other above these.

So; that means, this interval is influencing the solution above this. So, whatever the solution you have here is influencing the solute solution for u beyond this region within this triangular domain and another way also you can take that this point is influencing the solution along these two extended straight line. So, that is called the domain of influence.

Now, another interesting thing is suppose I have a situation that g equal to 0, so; that means, the initial velocity. So, it say suppose you have is a special case you have a situation this is the equal to g x equal to 0; that means and u x t equal to f x. So, in that case the solution which u satisfy a wave equation delta v u u t t equal to c square u x x equal to 0.

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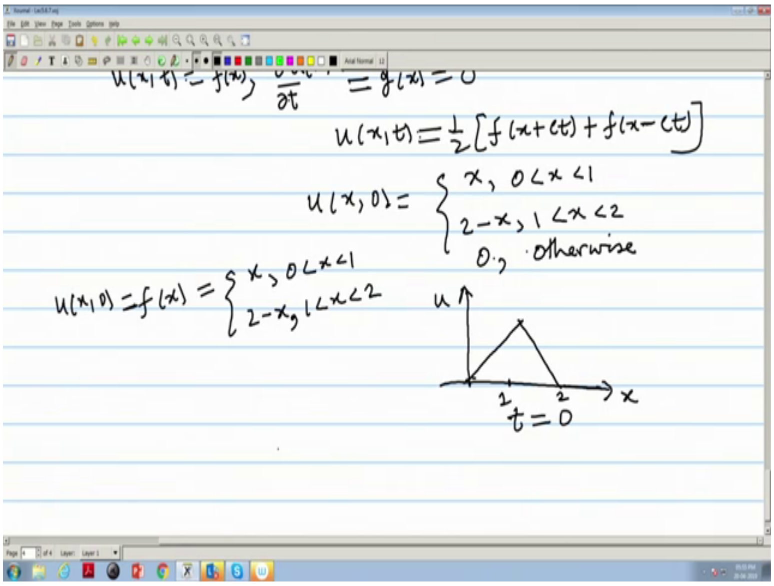

So, the solution is nothing, but u x t equal to half of f x plus ct plus f x minus ct because the other part is out g is not present, so this is the one. Now, so this is the solution f is the initial condition. Now, suppose your initially u x 0 is a triangular shape say for example, so suppose u x 0 is like this x when 0 less than x less than 1 and say 2 minus x when 1 less than x less than 2 and 0 otherwise ok.

So; that means, you have a function u x t which was taking which is satisfying these equation u t equal to u t t equal to c square u x x and initially it has the form like this way. So; that means, you have a at t equal to 0, you have this u is given by this is x and 2 minus x, so this is 1 this is 2; so u is 0 in the remaining portion.

Now, so now, if I substitute it here, so what I find is f x if I put t equal to 0 over here. So, f x is nothing, but x f x which is equal to u x 0 equal to f x. So, I have to find out this f x, so f x is given to be given 0 less than x less than and 2 minus x 1 less than x less than 2. So, f x plus ct, so u x t is a combination or superposition of two functions f x plus ct plus f x minus ct.

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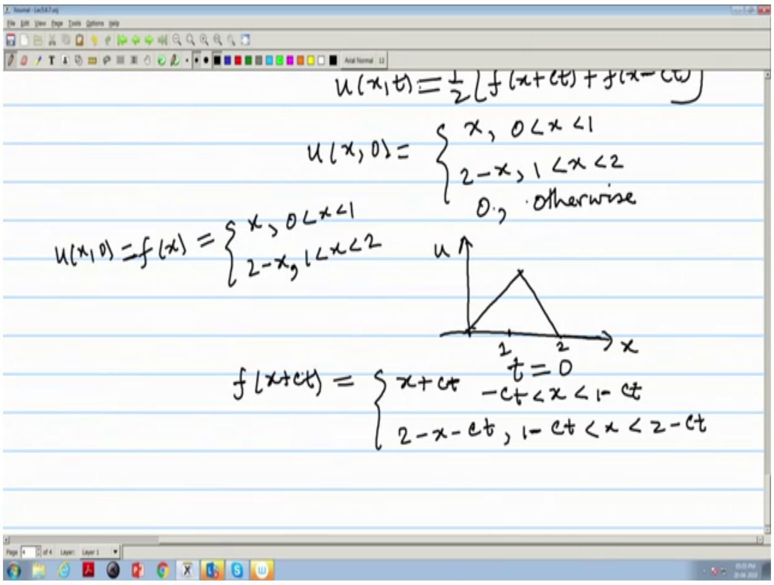

So, how this f x plus ct looks like f x plus ct because it is a single variable function I just replace x by x plus ct. So, I get x plus ct when minus ct less than x less than 1 minus ct and 2 minus x minus ct is less than 1 minus ct, less than x, less than 2 minus ct this is one part, another will be another is f x plus ct his is f x plus ct.

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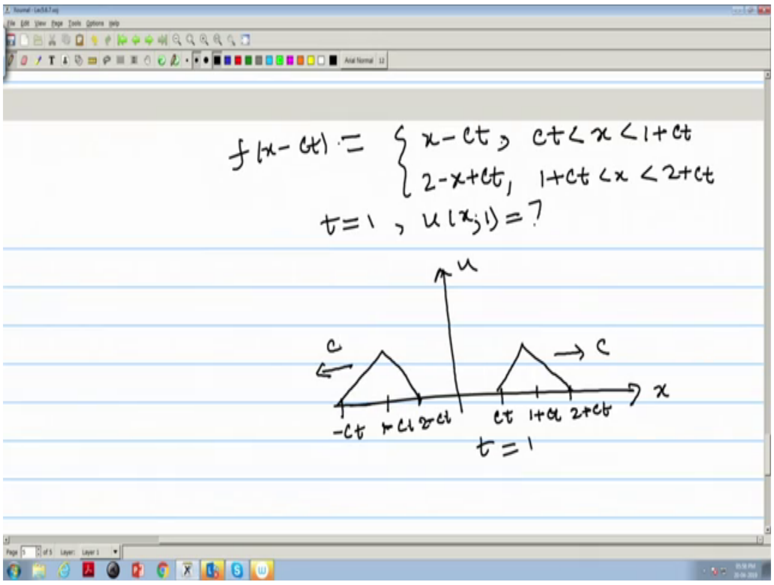

So, the other part f x minus ct is given by x minus ct when ct less than c into t less than x less than 1 plus ct and 2 minus x plus ct correct because x plus ct is 1 plus ct less than x less than 2 plus c t. So, if I want to find out say at t equal to 1 u equal to how much, u x 1 equal to what? So, this will be the addition average of this f x plus ct and f x minus ct.

So, what it shows that in the x at t equal to 1 you would looks like this; this is u; this is x say this is minus ct. So, I have a domain is minus ct 1 minus ct and say 2 minus ct. So, something like this 1 minus ct and this is say 2 minus ct. So, one way is like this way another say ct 1 plus ct and 2 plus c t.

So; that means, one part one wave is moving in the positive x direction with the velocity c another is moving with the negative velo x axis with the velocity same velocity c without deforming, this is the important part to know because also if I differentiate what is the rate of change. So, this is moving with a velocity minus c and this is with a velocity plus c.

So; that means, one part is going to this direction positive x axis with a velocity c itself another with a velocity c the same velocity, but propagating in the negative direction and the important thing is that no deformation is occurring which was whatever the initial f x there is no damping or there is no change or deformation of the form. So, that is why it is referred as the wave equation. So, though the because of my drawing it looks deformed, but it is not it is the same replica of the initial situation itself.

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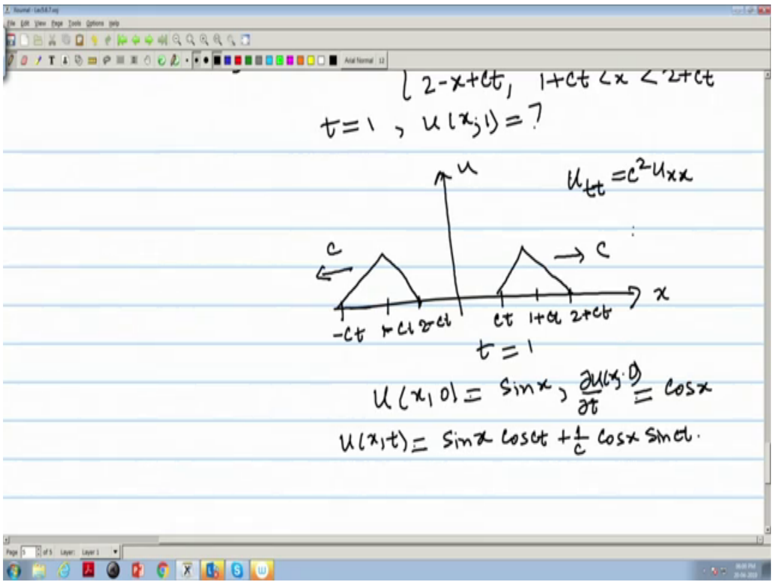

So, this is the property the solution of this kind of equations u tt equal to c square u xx satisfied. So, that is why it is referred as the wave equation itself. So, for example, if you have instead of del u del t at x equal to 0 which is also can be stated as the velocity at t equal to 0 if instead of 0 if you some other value say u x 0 is say sum sin x and del u del t; del u del t x 0 equal to say cos x for example.

So, in that case u x t one can write a form very simple way if we do this little simplification the sin a cos v formula. So, sin x cos ct plus 1 by c cos x sin ct. So, this kind of situation and appears. So, basically this shows a superposition of two waves standing waves. Now, this was the solution what one can find out if the domain is infinite. So, this solution is the solution got as we just discussed is referred as the D Alembert solution because D Alembert was the first person to obtain this solution like this way D Alembert solution.

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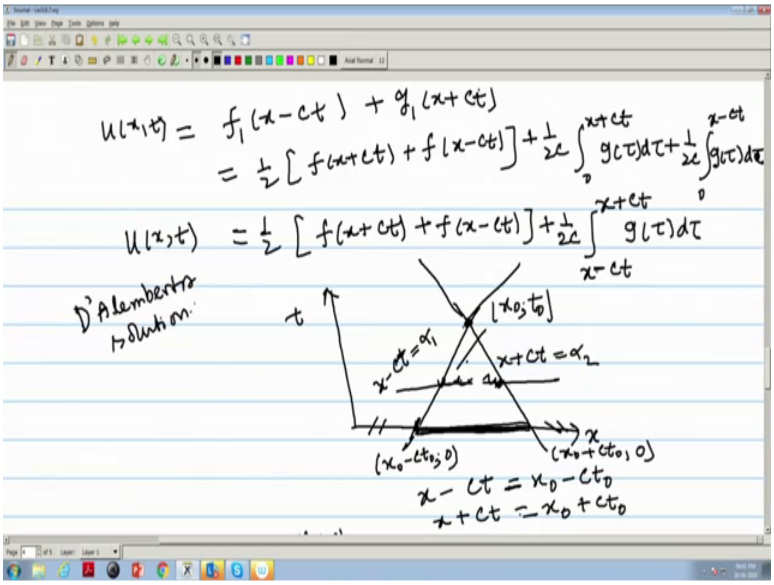

Now; obviously, the D Alembert solution is possible to obtain when we have a infinite domain that is one important thing another thing is that it involves a integration. Now, many of the complicated functions we may not have a integration proper integration.

Now, we consider because we are in a stage of talking the Fourier series expansion solution. So, now, we consider the case when the domain or the boundary is finite. So; that means, you have a situation hyperbolic equation or wave equation in finite domain. Let us write wave equation that is the better way oops wave equation in a finite domain.

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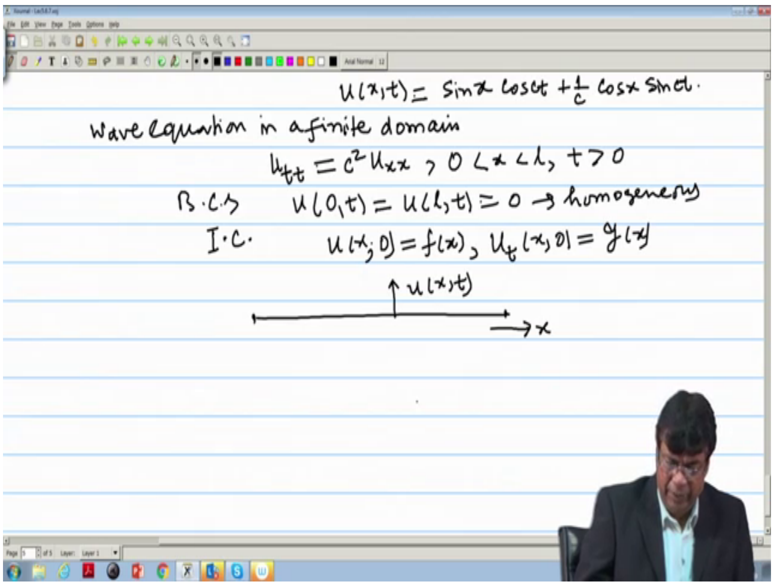

So, we have a equation as u tt equal to c square u xx and what we have here is say x is varying from 0 less than x less than l, t greater than 0. So, you have the homogeneous equation with homogeneous boundary condition has to be prescribed u l t equal to 0 this is our BC Boundary Conditions homogeneous do not. If not homogeneous we can some time not all the time we can do some trick to make it a homogeneous case and we have the I C; I C again has to be a two conditions to be given because it is a second order in time.

So, you have u x 0 equal to f x and u t x 0 equal to g x. So, sometime this equation also can be stated that you have a string which are fixed at this two end and you pluck the string with certain velocity and release the string with certain velocity g x and the initial form of the string is f x t equal to 0. So, any displacement of the string from the stable form a reference for place is a u x t. So, this displacement; so this displacement now follow the equation referred as the wave equation.

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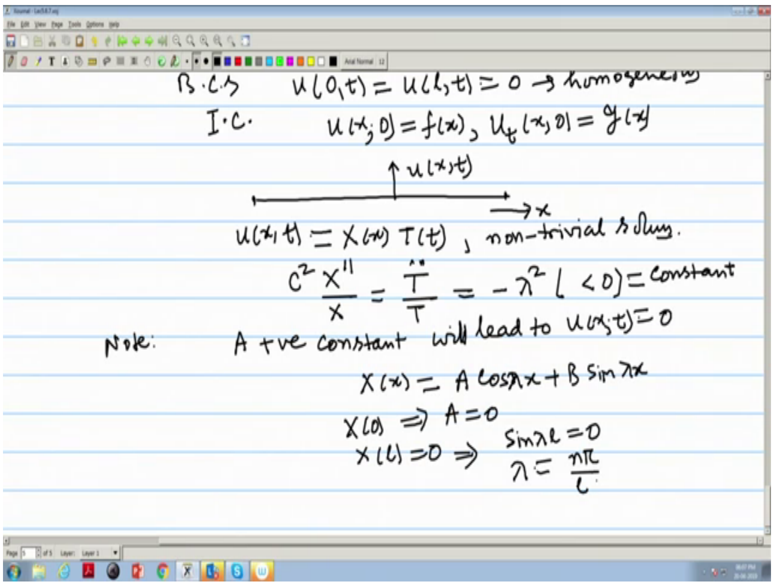

Now, again we look for a solution like u x t in this form I was writing X first X X, T t. So, if I substitute I get X double dash by X equal to T c square T double dot by T and that should be a negative constant, it has to be a negative. You can try if it is a positive as I think we have discussed in the previous one; if it is a positive because we are looking for a non trivial solution if the constant, so this is a constant basically. A positive constant will lead to note is and you can verify this; a positive constant will lead to a trivial solution u x t equal to 0. So, 0 is a solution, but we are not interested in 0, so we are stick to nonzero solution.

So, now, if we solve we get X n you can sum quickly write this X n equal to sin and lambda n better we should say this way that if I now solve this. So, X x equal to A cos lambda x; A cos lambda x plus B sin lambda x. So, now, what we have is homogeneous condition, so x equal to 0; so this implies x is equal to 0 x at x equal to 0; so this implies A equal to 0.

So, the x l equal to 0 implies we cannot say be equal to 0 because we are looking for a nonzero solution. So, what we should say is that sin lambda l equal to 0. So, this gives you a lambda which is nothing, but n pi by l and we refer this as a eigen value, so; that means, for a particular choice of lambda; for a particular choice of lambda.

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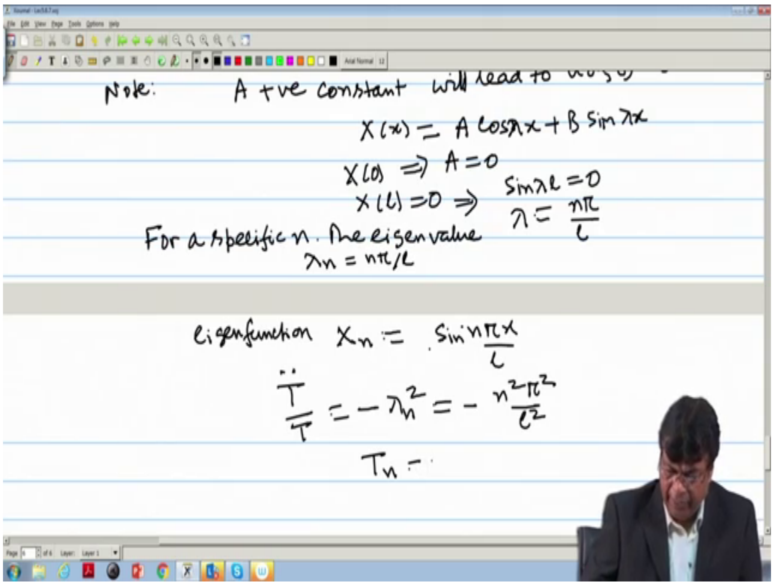

So, for a specific lambda, so for sum n, for a specific n we call that n the eigen value is; eigen value lambda n equal to n pi by l and eigen function is X n equal to sin there is a B n constant n. This eigen function of cos need not have to be constant need not have to multiplied, so you can call this sin n pi x by l. So, we get a solution like this way and once we have then we find out the other one that is t double dot by t equal to minus lambda n square that is minus n square pi square by l square. So, that T n, so as a for a given n; that means, given value of n.

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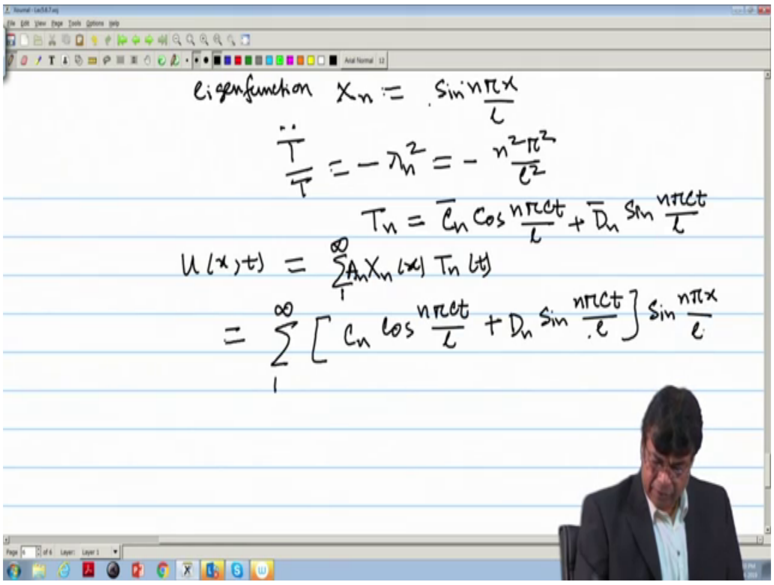

So, I find T n equal to the same way one can find out that T n to be C n cos n pi ct by l plus D n sin n pi ct by l. So, one can find out the solution u x t which is nothing, but sigma sum constant say A n into X n x T n t because its the homogeneous equation each of this X n into T n that is even is a solution.

So, general solution is the linear combination of all independent solutions. So, we get a solution u x t in a summation form given by sigma 1 to infinity. So, this, so I can write this as C bar n and D bar n. So, I can write An into C bar n Cn, so Cn cos n pi ct by l plus Dn sin n pi ct by l into sin n pi x by l, n pi x by l. So, now what we have two more conditions and two constants are there C n and D n.

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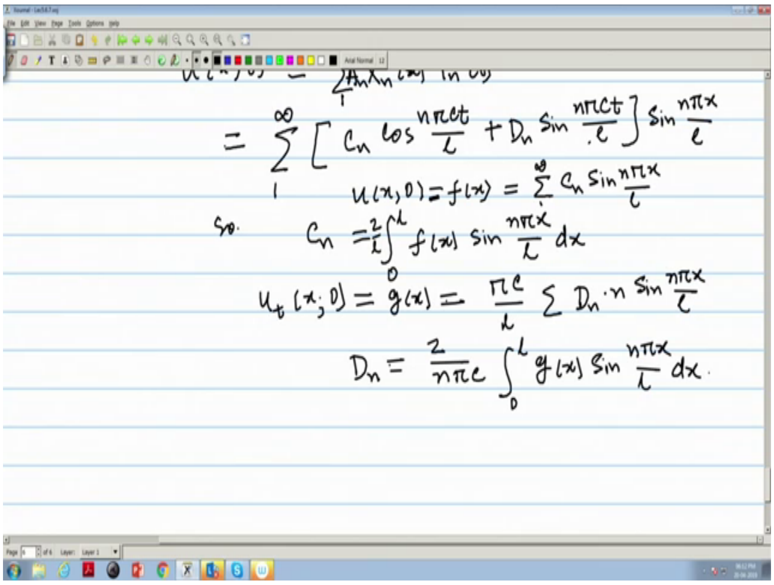

So now, if I apply the conditions, so one condition is u x 0 equal to is given to be f x. So, if I apply this condition f x is then that is nothing, but c n if I put t equal to 0, so we get sin n pi x by l. So, again we take the orthogonality property of sin function. So, one can write Cn; so Cn equal to integral here what is the domain 0 to l, so 0 to l.

So, this will be 2 by l and f x sin n pi x by l dx. So, this is the way we can find out the C n itself and that is the orthogonality. So, other terms will get cancelled 1 to infinity and similarly u t x 0 is given to be g x and this comes out to be if I differentiate. So, pi c by; pi c by l sigma D n in sin n pi x by l.

So, again the same operation, so we get D n equal to 2 by n pi c 0 to l; l get cancelled out for 2 by l get canceled out, but g x sin n pi x by l dx. So, these are the two coefficients or the constant are obtained by this manner. So, this is the Fourier series expansion solution. Now if I one can be simply write for away the or rather correspondence with the D Alembert solution can also be presented like this way because these are nothing, but the Fourier series sine series of f x and g x.

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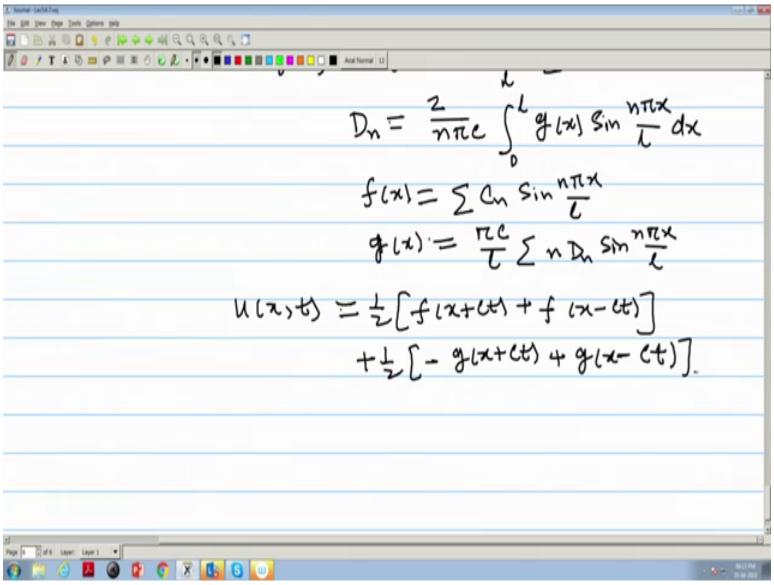

So, if I call a function f x as f x is the way we have written as C n sin n pi x by l and g x is if I write as pi c by l sigma n D n sin n pi x by l these are the eigen function for this case. So, we can write as u x t is half of f x plus ct plus f of x minus ct plus half of minus g x plus ct plus g x minus ct. So, this is the form of the Fourier series solution at when we have a finite domain ok. So, the a little bit discussion on these will continue in the next lecture.

Thank you.