Mathematical Methods For Boundary Value Problem

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Lecture - 04
Solution of BVPS by Eigen Function Expansion (Contd.)

Now we resume our discussion on the non-homogeneous parabolic PDE the heat transport equation or any transport equation which is governed by the diffusion mechanism only.

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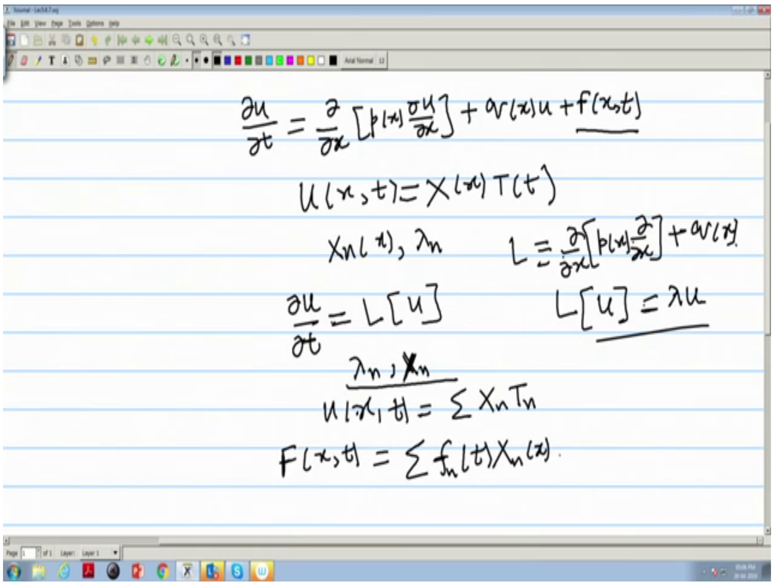

So, extra term we have is F x t which creates the non homogeneity. So, basically what we need is a Fourier series expansion or eigen function expansion method this is also referred as. And is also termed as Fourier series expansion, basically this is for the name of Fourier the French engineer basically, he is started this analysis finding the solution of heat transport equation in terms of series expansion.

So, from that point onwards it was named after that Fourier. So, what we are looking for is u x t in this form; u xt in this form as X x T t. So, in this X x T t form and the separation of variable. So, if X as we discussed before is a purely function of small x and we get the eigen value for X n we call the X n x is the eigen function and corresponding to these eigen value as lambda n.

So, which satisfy the homogeneous part. So, we define this L the homogeneous equation these L as del del x of p x del del x plus q x. So, this is the linear operator; linear operator in this case and basically this is the L u equal to lambda u is the corresponding to that this is the lambda is eigen value and u is eigen function.

So, what we are solving now is, first we get a solution for the homogeneous situation or I would say this situation L u. So, once we obtain the solution. So, corresponding to that is lambda n is the corresponding eigen value and we have a pair this is eigen function X n and lambda n. Now so, u x t is u x t sorry u x t is basically is u x t is sigma of X n and T n any constant; X n is the function of X and T n is corresponding to that eigen value lambda n whatever the solution comes that we call as the T n.

Now, what we do is this F x t which is the non homogeneous part we make an expansion of F x t. So, F x t is now expressed in terms of a eigen function expansion in terms of the orthogonal eigen functions x n. So, we call this as f n can we function of t and X n x; X n x. Now, if this is the situation x f n x n x n equal to 1 to infinity.

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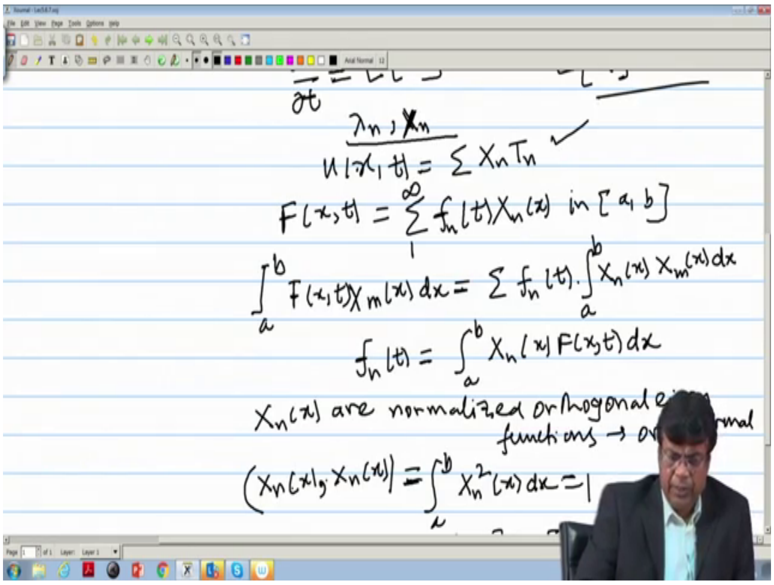

So, you are looking for a expansion of f x t which is valid in a b in the interval over which these u is defined that is the boundary, now since X n x they are orthogonal.

So, what we can do is orthogonality property; that means, if I multiply by a X n x both side and integrate between a to b, F x t and X n x d t. So, sorry dx and what will happen is here f n t and this is will be all. So, let us take this is X m or if I take this is X m.

So, X m we have multiplied we have multiplied by X m. So, this is X n x and X m x dx between a to b. Now if we integrate all the term will vanish except when n equal to m. So, from there what we find is f n t is basically becoming a to b X m or X n x F x t dx.

Now, here we have taken the we have divided by the length of this vector so; that means, what we have done is we are normalized these X n x are normalized orthogonal vectors. So, basically they are forming a orthogonal eigen functions; eigen functions. So, basically they are forming a orthonormal set.

So, in other words it is a orthonormal. So, this is nothing, but we have just if the inner product with itself is not one so; that means, the this is the one, so, a to b. So, X n square xdx is taken to be 1 in this case. If not we have divided by this inner product again I think I have to rub this. So, if not then we have to divide by this inner product. So, we get the coefficients for fx t by given by f n, now if I substitute in the governing equation.

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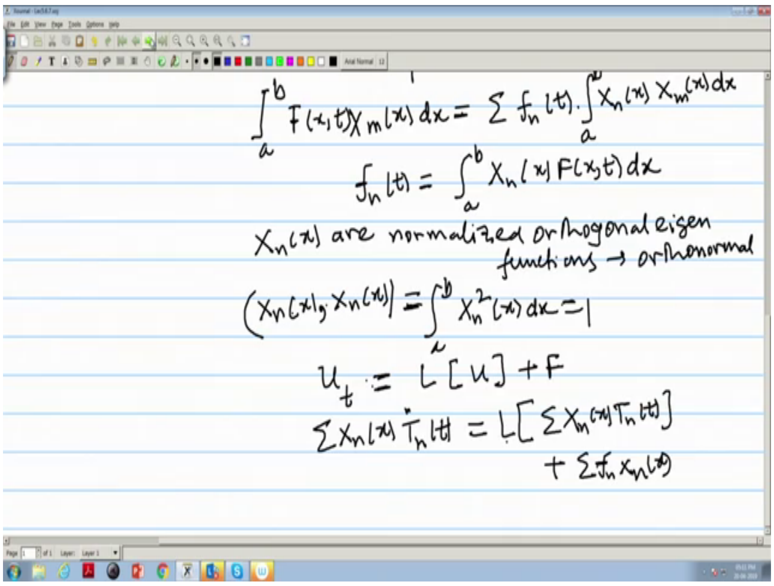

So, what equation we have is u t equal to L of u plus F. Now if I substitute u equal to sigma X n T n. So, this is the one if I substitute in this equation. So, what I get is sigma X n x T n dot t equal to. So, L of this summation X n x T n t plus sigma f n x what was the f n xn x f n t x n x this is the expansion for f x t this is the one we are substituting in this equation. So, and L is a linear operator.

So, we can take inside the summation. So, we can interchange the L and the summation. So, what I get here is if I interchange the summation and; summation and this L operator.

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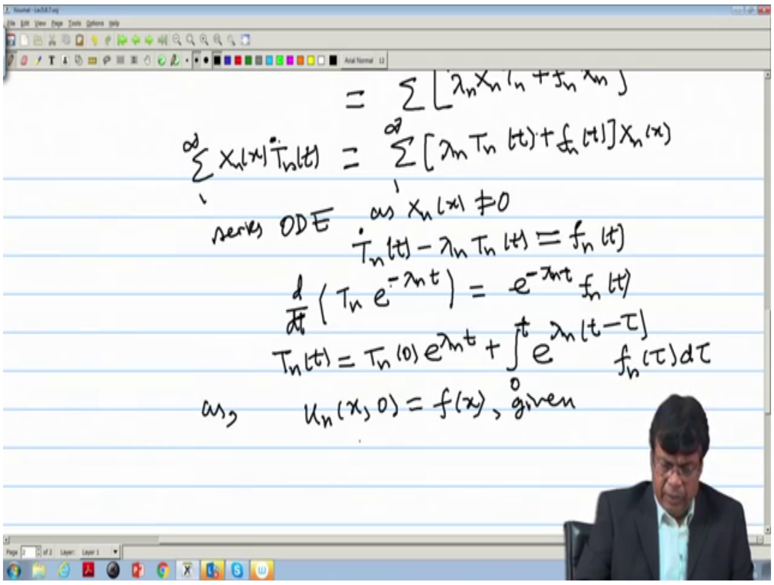

So, what we get is we can take out L sorry we can take out the summation. So, summation now L of x n is nothing, but the lambda n. So, this is becoming L lambda n X n T n plus f n X n this summation we get. So, in other words this is nothing, but sigma lambda n T n t plus f nt X n x.

And this side we have is summation X n x T n dot t. T n dot t and this summation is between 1 to infinity 1 to infinity. Now if I equate so, the summation left side and right side. So, we get a ODE a series of ODE of this form a series of ODE which leads to the series as T n dot this is the derivative T n dot time derivative because as X n x is not equal to 0 we are not looking for a non trivial we are looking for a trivia non trivial solution.

So, T n dot equal minus lambda n T n t equal to f n t. So, is a first order equation which is a exact form which is of exact form. So, we can solve this by integrating factor so; that means, basically what we do is we multiply with both sides with e to the power minus lambda n t so; that means, this side we can write as T n e to the power minus lambda n t equal to e to the power minus lambda n t equal to e to the power minus lambda n t f n t.

Now, if I integrate both side. So, what I get between 0 to t. So, we can write this as T n t equal to T n 0 e to the power lambda n t plus 0 to t 0 to t lambda not lambda e to the power. So, e to the power lambda n t minus tau because this t is positive will go to that side, tau is the integrating variable f n tau ok.

So, we have integrated between 0 to t both side and then if this e to the power lambda n t is multiplied in either side. So, now what we have is as we know that u n x 0 is f x which is given this is given this is a given initial condition. So, this even x 0 equal to f x. So, from there what we find is X n x and T n 0 is given by f x again X n x is orthogonal.

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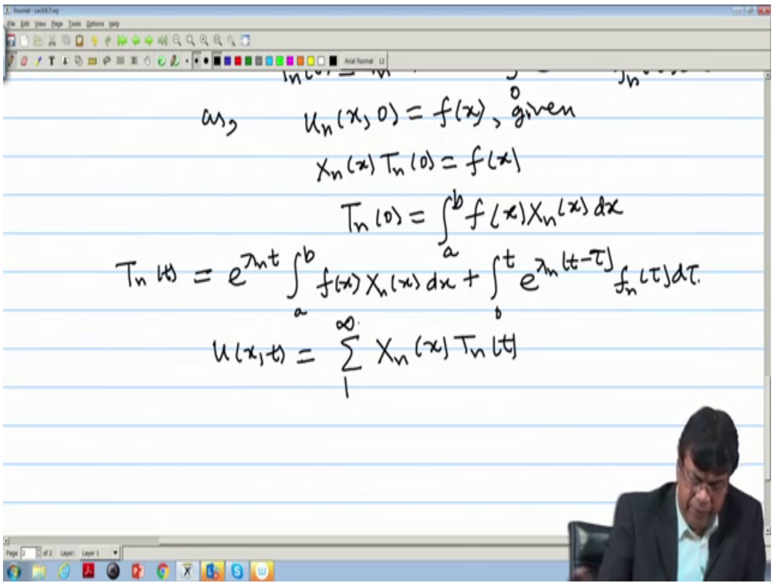

So, from there I can write as T n 0 is I can write in this way a to b a multiplication with X n x dx, again this orthogonality condition for X n x we multiply with both side as X n x and this is again orthonormal in that case; that means, the inner product of X n x with itself is 1. So, if we substitute this. So, we get the T n t if I now substitute this T n 0 to this form.

So, this is becoming e to the power lambda n t a to b if fx X n x dx plus as it is 0 to t e to the power lambda n t minus tau f n tau d tau the usual equation, usual term we have not done any manipulation in that term. So, this is the corresponding non homogeneous equation expansion.

So, ux, so, this is the T n t already we have the un x we have already obtained which was the I mean x n x which was the eigen functions. So, u x t is the summation of all these X n x T n t. Now X n x T n t X n x is already determined as the eigen functions n equal to 1 to infinity. So, now if we substitute it there, so, this is becoming.

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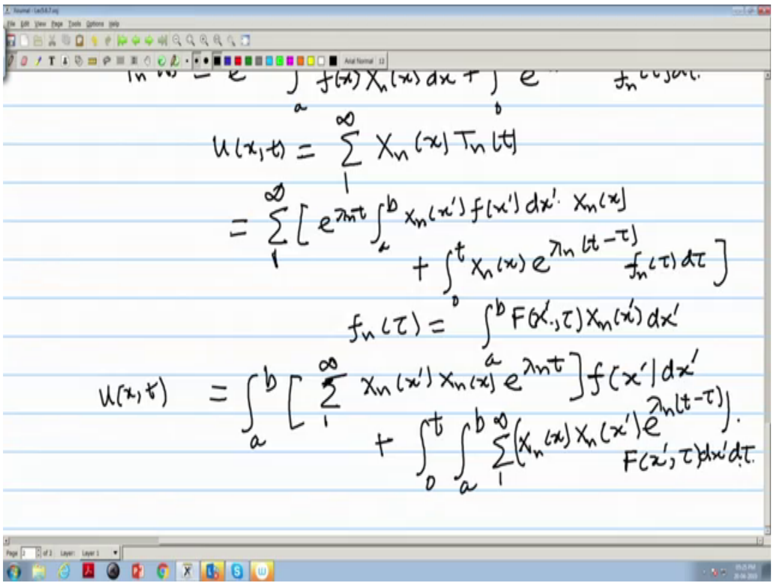

If this is substituted; so, this is sigma e to the power lambda n t e to the power lambda n t a to b X n x dash let us take this integration variable as x dash f x n x dash in to x d x dash this into X n x plus we have this term plus we have this term 0 to t this is X n x; X n x e to the power lambda n t minus tau f n tau d tau this summation.

Now, f n small f n tau which is nothing, but the Fourier series expansion or rather eigen function expansion. So, basically f n tau as we recall is nothing, but integration a to b F x tau X n x dx now if I substitute back to this. So, what I get a instead of f n if I now substitute it in terms of capital F which is the function to is given already. So, we get a form as a summation I can interchange the summation and integrations because if it is a convergent series it is a infinite summation.

If it is a converging absolutely then we can have the term by term integration. So, if we interchange this what I get is sigma 1 to infinity; 1 to infinity X n x dash X n x e to the power lambda n t. So, this is with the n and remaining one is fx dash fx dash oh there is a fx dash which is the condition and it is independent of summation. So, this is the integration and the second one will be a double integration if I substitute n interchange the integration. So, you have first was with respect to tau and x.

So, now I interchange the integration so, first one with respect to t tau and second with respect to x, so, a to b summation 1 to infinity X n x Xn. Let us put here this integration variable as x dash because this will not create any confusion. So, X n x dash e to the power lambda n t minus tau. So, they are independent and the term which is not independent is the one is if x dash tau d tau and then d tau dx dash. So, basically first is dx dash and then this one will be d x dash d tau. So, this is the general form for the non homogeneous case u x d, finding the Fourier series expansion or a non homogeneous parabolic differential equation.

Now, the Fourier this expansion is easier come particularly when you have a situation where this is x dash. So, particularly the non homogeneous term that what appears in the differential equation, this is the term non homogeneous term if it is a complicated function, then the series expansion in this form of a smooth functions this eigen values X capital X n.

So, in terms of the smooth function this is more advantageous. So, later on in this course itself, we will show some simpler situation; that means, if f x d is a simple type of equation. So, we can use many other technique like integral transport technique, Laplace transform technique or Fourier transform technique.

So, those can be used for finding the solution, but one thing we have to remember here that these Fourier this eigen function expansion or the integral transform in most of the cases we can evaluate only if it is a linear boundary value problem. So, here we have the boundary value part this part should be linear. So, if there is a non-linearity, then this methods ceases to work.

So, and for that we have to depend on the numerical solutions. Now there can be several other complicated things, another important thing which we must note over these at this stage is that here we have a finite boundary. So; that means, x is within a finite domain that is either a to b or 0 to a or whatever. Now in many cases what we may come across is x is semi infinite; that means, x is greater than 0 and we do not as x tends to infinity, some process is terminating because it is a constant decay process. So, it has to terminate some or it will diffuse to some constant value or 0 after a large value of x or at a long distance.

So, in that case the separation of variable or eigen function expansion technique whatever we are discussing over here is not valid. So, we have to keep it in mind to that this technique that expansion technique is applicable only when we have a situation, that is a bounded domain. Another also this kind of things like we have discussed in the previous day also, that is the domain rather the boundary conditions in the finite domain should be homogeneous. If not homogeneous we have to do some manipulation some tricks to make it homogeneous.

Now, now we will talk about the next one, which is the commonly we come across equation commonly encountered equation is the wave equation or hyperbolic equation hyperbolic PDE.

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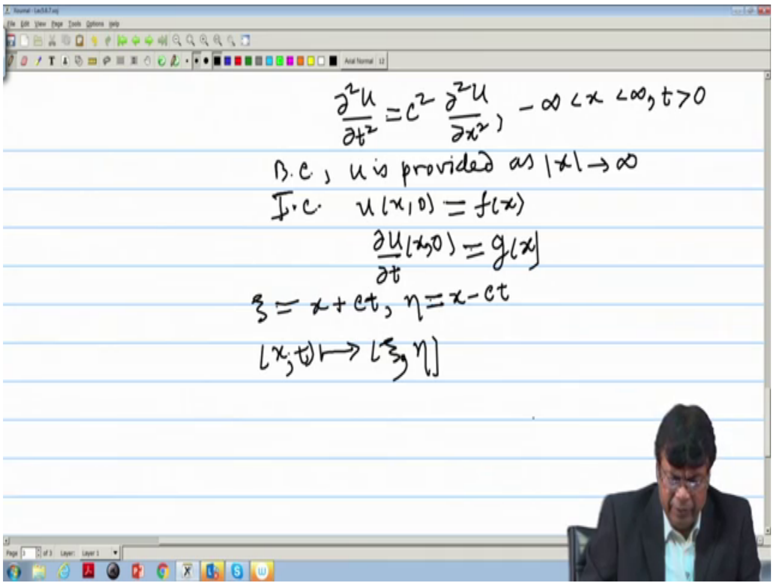

So, a simple wave equation is given by del 2 u del t 2 equal to C square del 2 u del x 2. Now let us first consider a domain of infinite domain say infinity t greater than 0 t is the time.

So, the domain is the entire real axis; that means, for any choice of x. Now this is a equation we come across in many many particularly if it is a say simple harmonic wave or anything a process which is repeating with the time without decay. Now this is termed as the wave equation now why? Why its what is the characteristics of this kind of equation or solutions specific nature of the solution? Now if I have the say initial conditions it should have a boundary condition.

So a u is some u is provided as x tends to mod x tends to infinity so; that means, x is going either minus infinity or plus infinity that is the boundary condition and this is the initial condition; that means, u. So, there should be two initial conditions because it is a second order in t so say fx and the other initial condition is given as del u del t at x equal to x at t equal to 0 is gx.

Now, a very simple way to solve this equation is as follows. Now if I make a transformation say xi equal to x plus ct and eta equal to x minus ct if I take a variable, we consider xi and eta two variable, now if I make a transformation from x t to xi eta. So, what we get is the equation you can do little manipulation and what will find that equation will become reduced equation will be like this del 2 u del xi del eta equal to 0.

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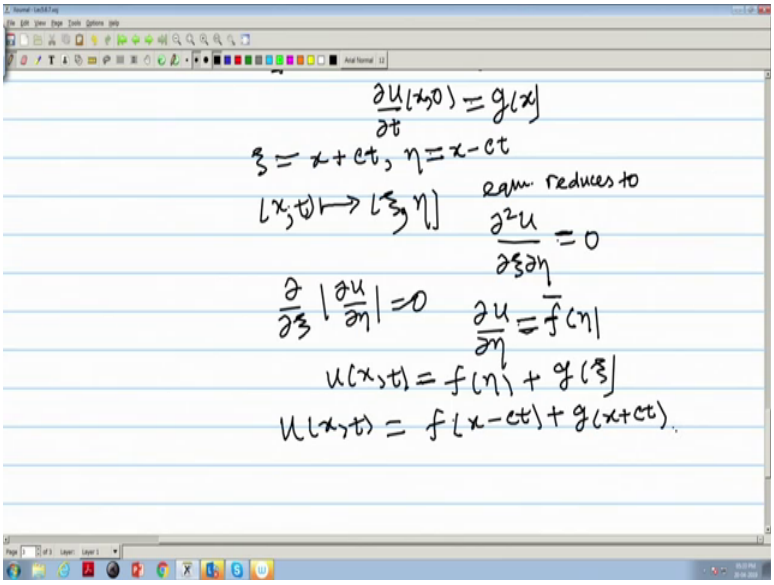

Equation reduces to this. Del 2 u del xi del eta equal to 0 homogeneous equation of course, homogeneous and linear that is the most important assumption if the homogeneous means this side has to be 0 only. Now integrating is very simple, if I now write this as del del xi of del u del eta equal to 0. So, what I if I integrate both sides with respect to xi. So, I can say that del u del eta is nothing, but a function of eta, is the constant which is independent of xi to the function of eta.

Now, if I again integrate. So, this can be written as u x t this can be written as f eta say f bar if I call. So, if I integrate, I get a some a f eta and the remaining constant I can call as g xi that will be a function of xi, because we are have a partial derivative adjuster to eta xi is kept to be constant. So, when I integrate the constant will be a function of xi.

So, what it says that if I now substitute back? So, u xt equal to f eta is x minus c t and g is x plus c t. So, left side unknown function is x and t depending on t, but here they are the single variable function x minus ct and x plus ct. So, though x t are varying, so, x minus ct also varies x plus ct also varies, but as a single variable ok. Now next thing we will talk in the next lecture.

Thank you.