Mathematical Methods For Boundary Value Problem
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Lecture - 03
Solution of BVPS by Eigen Function Expansion

Welcome back; now the same Eigen Function Expansion technique what we are discussing for ordinary differential equation or boundary value problem a single variable situation. Now, we will talk on the partial differential equations. Now partial differential equation and ordinary differential equation the main difference is that here the variable are more than one independent variables are involved say for example, u if u is a function of x and t u is a function of x and t.

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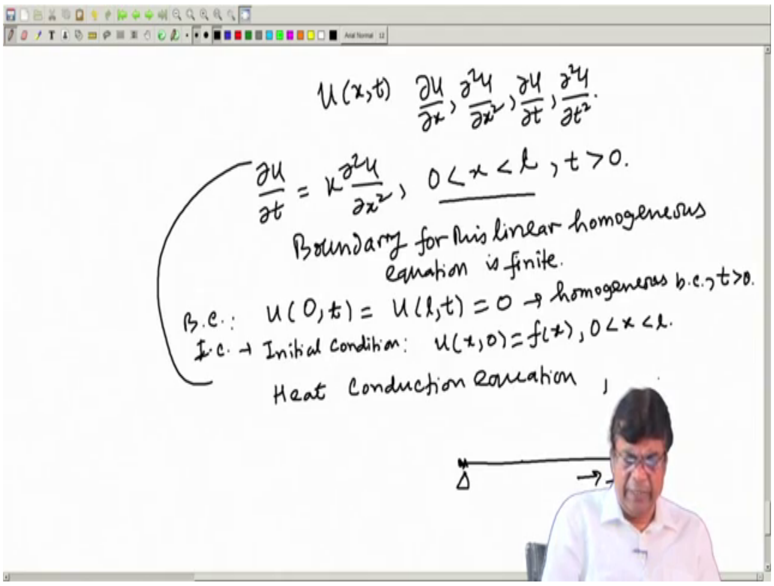

So, in that case it is derivatives will have the partial derivative del u del x del 2 u del x 2 del 2 u del t etcetera. So, this kind of things appears. So, in this situation then the partial differential equation involves so; that means, the differential equation which is a relation or equation which is involving the partial derivatives are referred as the partial differential equation. We have little bit discussion on these partial differential equations classification etcetera and the preambles already in the lecture 1.

Now, one of the important class of equation or important type of equation which we come across in heat transport equation or any kind of transport equation which is governed by the diffusion mechanism. So, this follows this kind of equation del u del t equal to k del 2 u del x 2. This is again a boundary value problem it is the initial boundary value problem because t is in involved.

So, you have a boundary say x can vary between 0 to say certain finite region 0 to l and t which is the time can be positive direction. Now, if we have a situation where l is say finite then the same expansion technique what we discussed in the previous lectures can be applicable here as well if the boundary is finite boundary for this linear equation linear another important things the homogeneous equation for equation is finite. For the time being we are not talking about what we will do with the if it is a infinite domain. So, that will come in due course of time.

So, for the time being we are considering a x is varying between 0 to l; l is a finite number and u is satisfying a partial differential equation which is also called the parabolic PDE and t is the time t equal to 0. So, what we have is the boundary condition; that means, this is the B C. We are writing say we impose again I made a mistake this is not x.

So, we impose homogeneous boundary condition so; that means, u 0 t u l t equal to 0 homogeneous. So, far we are restricting to homogeneous equation homogeneous boundary condition and initial condition I C. So, initial condition we are writing as u x 0 equal to say sum f x for 0 less than x less than l and this is happening for any t greater than 0 and this is happening for any x within the domain 0 to l.

So, this combined. So, you have a situation you have a differential equations with the conditions given by boundary condition u 0 t equal to u l t and initial condition u l 0 called fx. This combined situation is referred as the initial boundary value problem and this if we want to make a correspondence with the physical significance. So, it is a one can say that heat conduction equation.

So, this is a heat conduction equation. So, basically what it represents a thin rod say of length say this is x a thin rod of length l which was initially heated with some temperature effects a temperature distribution initially was f x and at the 2 end it has a fixed temperature it is maintained at 2 fixed temperature which are as a reference temperature.

So, let us call it 0 so, at this 2 end. So, then finding the heat distribution at any time at any point that satisfied the heat conduction equation and k is the conductivity of the material. So, if it is a iron or silver or copper so, k changes. So, basically this is our problem is to solve. So, this is heat conduction equation. Now, what we can start doing is that the equation is homogeneous and boundary conditions are homogeneous.

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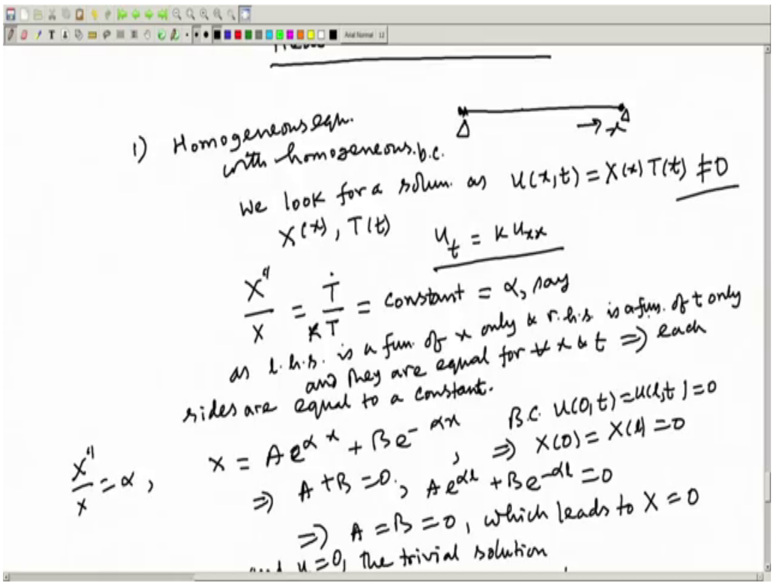

So, what we have a homogeneous equation with homogeneous boundary condition; homogeneous boundary condition. I am repeating the same thing several times this homogeneous thing. So, homogeneous bc; I think that is enough to say that it is a homogeneous bc.

So, we look for a solution as u x t equal to X x and T t. So, in other words X is the function of X is function of x only and T is a function of t only ok. So, we look for a solution as u x t equal to X x equal and T t. So, if I substitute. So, our equation is ut equal to k u xx. So, if I substitute so, what I get? X double dash by X equal to T dot by T this relation we get. Now, this l.h.s is a function of x and r.h.s is a function of t. So, this has to be a constant as l.h.s is a function of x only and r.h.s is a function of t only and this is happening this relation is happening for any x and t.

So, for that what we must have that these should be equal to a constant. Now, let us take this constant equal to some say alpha say. Now, this is a purely function of x it is purely function of t and they are equal all the time all the time all the x is a function of t and they are equal for all x and t. So, this implies a constant ratio is a each side is a each sides are equal to a constant. Now, if we take this a constant as alpha. So, what we can write is X double dash by X equal to alpha.

So, what I can write as X equal to Ae to the power i alpha x plus Be to the power minus i alpha x. Sorry why I am sorry why should not be there. So, this is nothing, but alpha x and Ae to the power alpha x plus Be to the plus Be to the power minus alpha x. Now, what we boundary condition is homogeneous boundary condition. So, u 0 t equal to ul t equal to 0. So, this implies X 0 equal to X l equal to 0. So, what do you find that this implies that A plus B equal to 0 and also Ae to the power alpha l plus Be to the power minus alpha l equal to 0.

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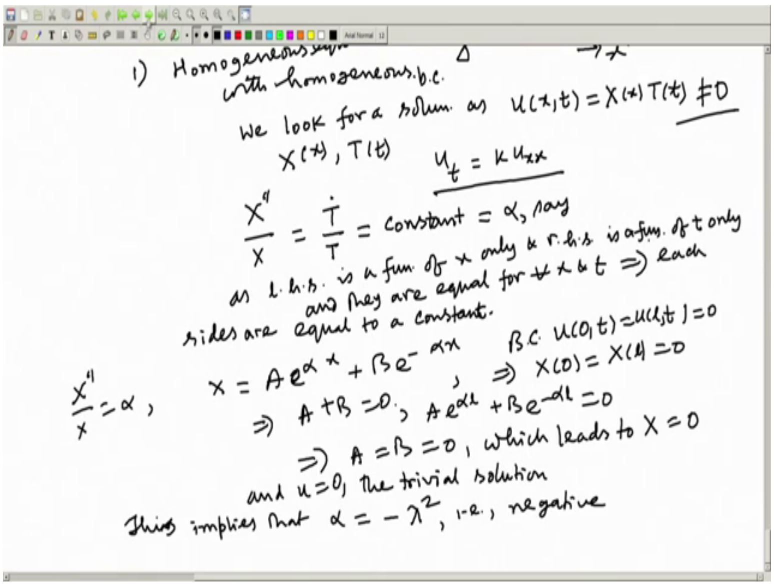

So, this implies A B both are 0 which leads to a trivial solution X equal to 0 and u equal to 0 the trivial solution, but basically we are interested to find out a non-trivial solution so; that means, what we are interested that this should not be equal to 0. If it is equal to 0 then it is a trivial solution and always since it is a homogeneous equation trivial solution is 0 solution is on always a possibility. So, we are looking for a solution which is not 0.

So, what it implies then that this implies that this constant should be equal to a this implies that alpha is a negative constant that is negative. It cannot be positive if it is positive then we have a difficulty we will have a trivial situation. So, alpha should be negative. So, if alpha equal to minus lambda square negative means I can take as minus lambda square.

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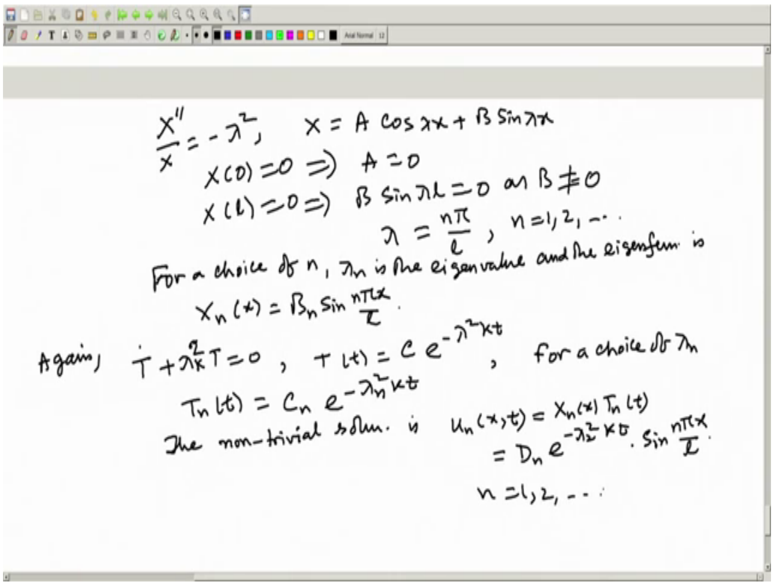

So, what I find is X double dash equal to X equal to minus lambda square. So, this gives you X equal to A cos lambda x plus B sin lambda x because now it is e to the power i to the power situation. So, now, if I apply the boundary condition; so, X 0 equal to 0 implies what this implies A equal to 0 because this is out and X l equal to 0 implies B sin lambda l equal to 0. Now B cannot be 0 as B is not equal to 0 because again it will lead to a trivial solution um. So, this means sin. So, lambda lambda l should be equal to n pi; so, n pi by l ok.

So, this is n equal to 1 2 3 etcetera; so, this lambda n for a value of for a choice of n. So, n the lambda n is the eigenvalue and the corresponding eigenfunction X n equal to x n x in this case X n X equal to one can write as B n sin n pi x by l. So, this is our eigenfunction. Now with the eigenfunction; now also we have to find out that T again T dot plus betas beta [FL]. Here it is a we have written lambda square. So, lambda square T equal to 0 and so what I get is T t equal to sum C e to the power minus beta square. There is one K was missing K was missing yeah this is will be K. So, this is a K.

So, we have to put K over here. So, please note the change. So, K should come over here. So, lambda square K will turn over here.; so, e to the power C e to the power minus lambda square K t or kappa t whatever way. Now for a for a given n for a choice of n some lambda n say eigenvalue. So, you can find out the write down the T n t is C n e to the power minus lambda n square K t. So, the non-trivial solution is u n xt equal to X n into C n. So, X n x T n t and we can write as some new constant say D n e to the power minus lambda n square K t into sin n pi X by l n equal to 1 2 etcetera. Now, each of these u n x t are satisfying the our homogeneous equation.

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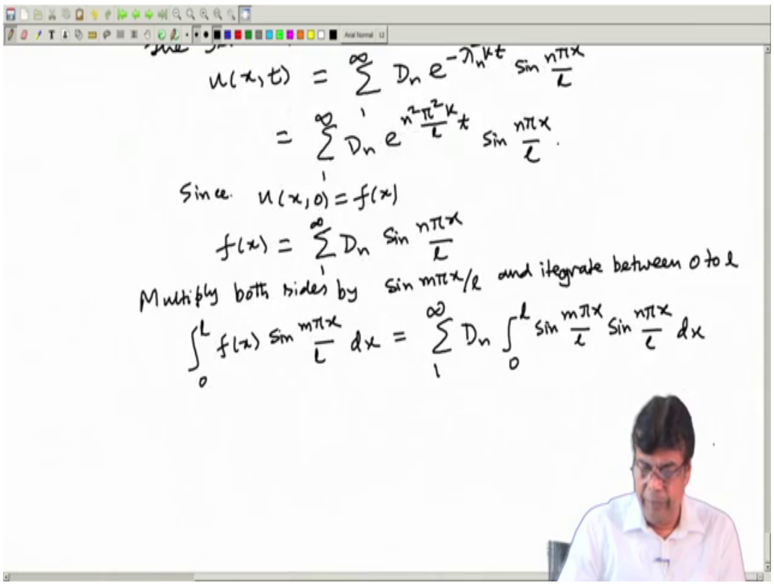

So, the general solution will be any combination of this; that means, u x t is nothing, but sigma sum D n e to the power minus lambda n square we should have written the lambda n square in the form n pi x by l. So, basically this is lambda n lambda n square. So, n equal to 1 to infinity; so, this is 1 to infinity D n e to the power lambda is pi n square pi square by K by l into t sin n pi x by l.

So, now, what I left is to find out the D n and we get the solution. Now D n can be obtained by using the condition another initial condition. Now since u x 0 what we have is u x 0 equal to f x. So, one can write as f x equal to x 0. So, one can write as D n sin n pi x by l 1 to infinity. So, again the orthogonality property of the eigenfunctions sin n pi x l by l are the eigenfunctions.

So, I can use the orthogonality property. So, what we have to do is multiply both sides by sin m pi x by l and integrate between integrate between 0 to l. So, what I get is 0 to l f x sin m pi x by l dx equal to sigma. We can put inside the integral sin integral inside the summation sign. So, 0 to l sin m pi x by l sin n pi x by l dx.

Now, these are the orthogonal functions because they are forming the they are forming the eigenfunctions they are also this is a sign property.

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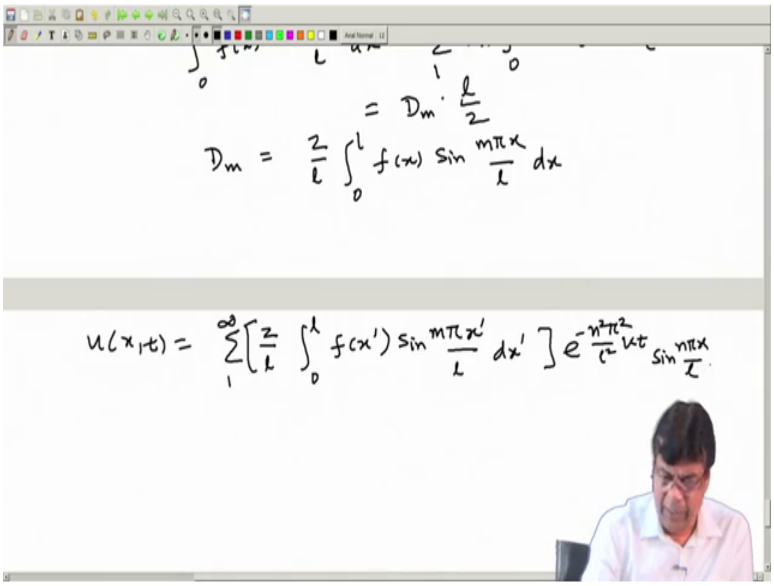

So, what you can do is what we find that if I when I sum up sum over n. So, when n equal to m then only this term will be non-zero and remaining terms all will be 0. So, what we get here is D m and this integral will be nothing, but l by 2. So, and. So, what we get is D m is 2 by l 0 to l f x sin m pi x by l dx. So, I find out the constant in terms of the initial condition. So, whatever was prescribed.

So, u x t we can now write as u x t which is the solution of the heat conduction equation sigma 1 to infinity 2 by l 0 to l f x dash because they are the integration variable instead of writing the variable x dash which is our function variable. So, e to the power minus n square pi square by l square K square t if a pi square K t sin n pi x by l yeah n square pi square K by l; n square should have been because lambda n square is l square n pi by l. So, this is l square.

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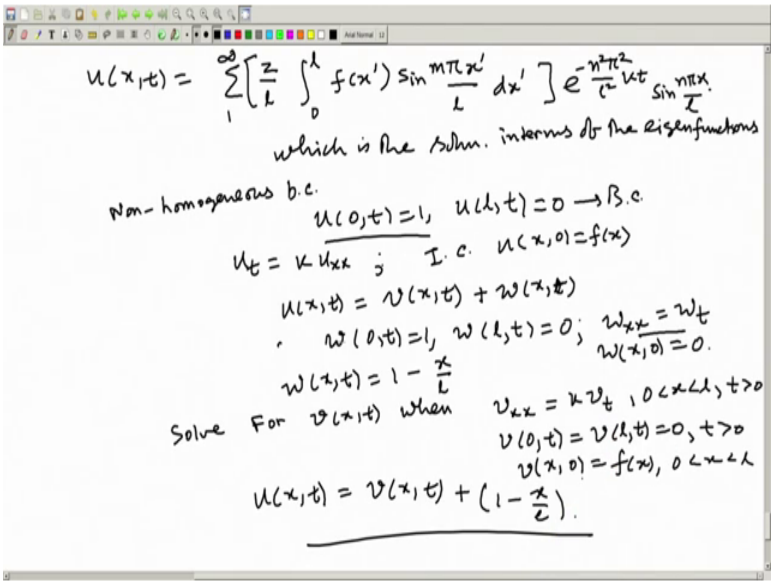

So, this is the solution of the which is the solution in terms of the eigenfunctions eigenfunctions. So, as a eigenfunction expansion we could obtain the solution for this kind of homogeneous PDE homogeneous partial differential equation with homogeneous boundary condition. Now here one point to note that we have taken the boundary condition to be homogeneous, but we may not be lucky all the time that we will have a; so, homogenous boundary condition.

Now that can be violated. So, say if may have a non-homogeneous boundary condition. So, in that case a non-homogeneous situation say you may have a situation like this u 0 t equal to 1 and say the other one is 0. So, now, this is a simple example discussing. So, what we do that u t is equal to K u x x and of course, this is the B c and I c is as usual say u x 0 equal to f x.

Now what you have to do is one trick is this that I substitute u x t equal to say v x t such that it is taking the homogeneous boundary condition with homogeneous equation plus say sum w x t such a way that w x t satisfy the given parabolic PDE and also it is taking care of the homogeneous boundary non-homogeneity so; that means, I need e z w x w x 0 w 0 t equal to 1 and w l t equal to 0.

And in addition w x x equal to w t that need to be satisfied. Now I can choose these and also these w because at t equal to 0. So, what we have is x 0. So, that has to be 0. So, one simple situation say w x t I can take as just thing like this one ok. So, 1 minus x by l 1 minus x by l so, if I take 1 minus x by l. So, first of all it is steady distribution. So, for any time 1 minus x by l; so, it satisfy these equation and also it is a 0 t and w l t is 0 w l t is 0 and w 0 t is 1. So finally, an one can solve for v x t with homogeneous equation when v xx equal to k v t for 0 less than x less than l t greater than 0 and since u x t equal u 0 t is taken care by w 0 t. So, what you have is 0 t equal to v l t equal to 0, but t greater than 0 and the initial condition is unperturbed because w is independent of t. So, initial condition was as usual x 0 equal to the same thing f x or 0 less than x less than l.

So, finally, we can write the solution as whatever you find out the solution in the series expansion form as we discussed before then substitute v x t plus 1 minus x by l which is the no. So, this is one way of taking care of the non-homogeneous situation non-homogeneous boundary condition.

So, so likewise this non homogeneity can be smartly avoided or can be taken into account whenever it appears in the boundary condition. So, for separation of variable technique or for that matter the Fourier series expansion or eigenfunction expansion is also referred as Fourier series expansion. I do not want to use the term because we are not using the Fourier series for the time being.

So, in the in that case what we need is a what I need is the domain to be finite equation to be homogeneous boundary condition to be homogeneous. If not some tricks to be adopted, but of course, domain has to be finite that is another.

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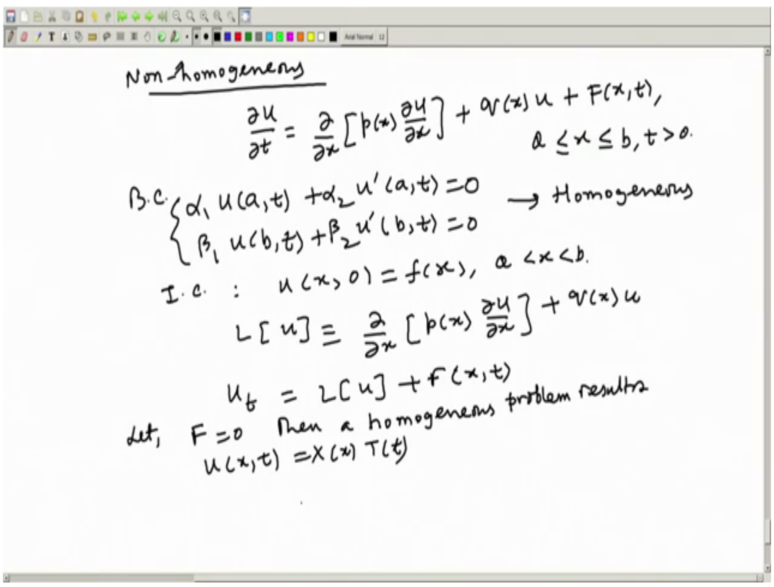

Another condition to be imposed; now if we have a non-homogeneous situation say we have a equation like this del u del t equal to say we have a situation like this say this is the l x p x del u del x in general I am writing any general form q x u plus f x t with the say finite boundary of course. So, in this case this equation has a non homogeneous term f x t and boundary conditions.

Let us assume Bc's are homogeneous bc's are u. So, let us take alpha 1 u a t plus alpha 2 u dash a t equal to 0 and beta 1 u b t plus beta 2 u dash b t equal to. This is a problem u b t equal to 0. So, this is a homogeneous Bc and the initial condition I have only one should have only one because first order derivative in x so, f x. Now here if we want to find out the solution in that expansion situations. So, I introduced this operator l u as del del x p x del u del x plus q x u.

So, what I can write this equation as u t equal to l u plus f x t. If f equal to 0 that is the previous the same way as we proceeded before can be obtained; so, let first assume that f equal to 0. So, in the same process if we obtain the solution so; that means, then it is the homogeneous problem results. So, we look for the solution u x t equal to X x T t separation of variable.

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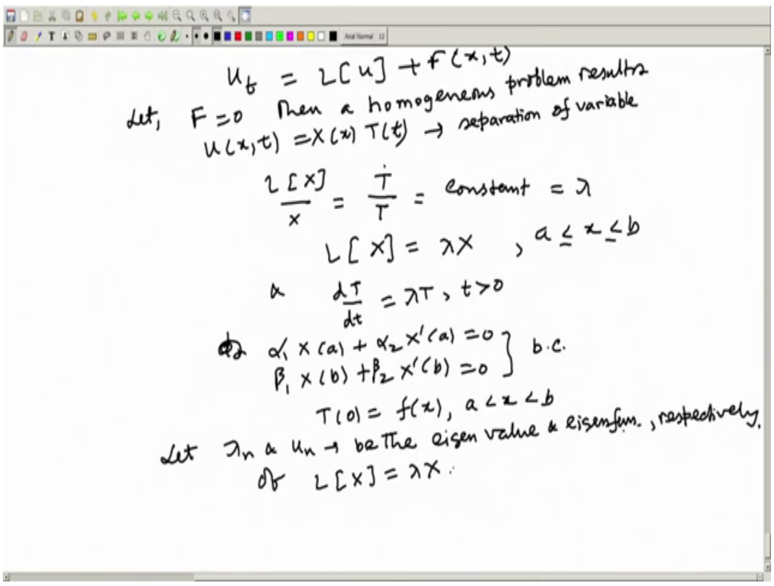

So, if I succeed there this is not this term is not there. So, what we have is l X by X equal to T dot by T equal to some constant and that constant say let us take some lambda. So, what I get is l X equal to lambda X so; that means, say eigenvalue situation for a less than x less than b and d T equal to lambda T for t greater than 0.

So, lambda is the separation constant and what we have here is a 1 alpha 1 right alpha 1 alpha one alpha 1 X a plus alpha 2 X dash a equal to 0 and beta 1 X b plus beta 2 x dash b equal to 0 and this is the bc and what we have is t 0 equal to f x this is our i c a less than x less than b. So, one can find out the eigenvalues. So, let us call let lambda n and u n be the eigenvalues and I am sorry eigenvalues eigenvalue and eigenfunction respectively. So, this can be obtained of say l x equal to lambda x.

So, this can be obtained and then we will apply the this eigenvalues expansion technique to find out the solution for the non-homogeneous case. So, this is I think we will continue in the next lecture. So, the same way we have to obtain that xx. So, say let us call this u x t. So, if I call lambda n n u n. So, we get a x n x and t n t. So, this is the thing we will continue in the next lecture.

Thank you.