Mathematical Methods For Boundary Value Problem
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Lecture - 20
Alternating - Direction Implicit Scheme and Successive - Over - Relaxation Technique for Poisson Equations

So, we have discussed about the stability and consistency of numerical scheme. Now, why these are important now? Basically any numerical procedure, what we need is the solution of the finite difference equation or the approximate equation should converge or should match with the exact solution of the differential equation of the PDE.

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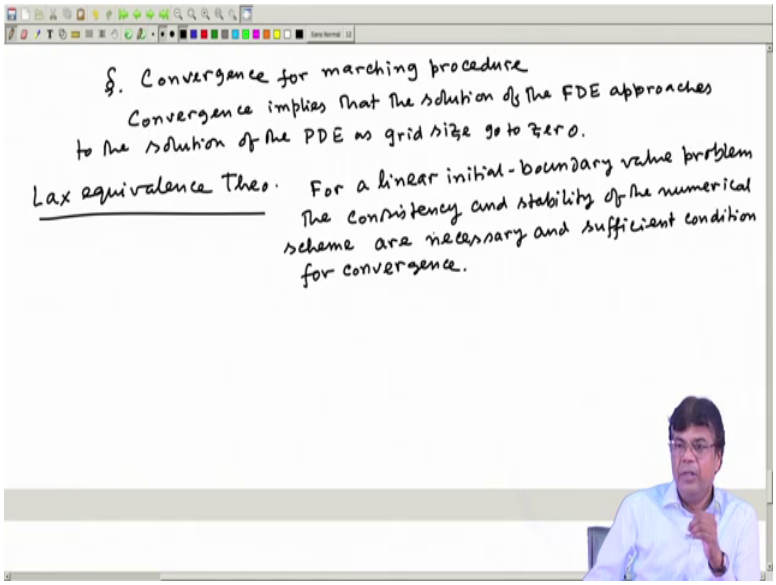

Now, for a marching procedure whatever we can say like this way; convergence for marching procedure whichever is adopted here. Now, converge implies that the solution of the numerical or FDE, we can call numerical scheme approaches to the solution of the PDE as grid size are as grid size go to 0. That is as I reduce the grid size smaller and smaller, the difference between the exact solution and the numerical solution should also reduce in the same order.

So; that means, at tends to 0 it will also should tends to 0. So, that is possible that is one was given by these theorem is referred as the Lax equivalence Theorem and this theorem is for this type of initial boundary value problem. So, for a of course linear, the stability

analysis what I did is for linear situation. For any a linear initial boundary value problem, the consistency and stability of the numerical scheme is of course or it will be are necessary and sufficient condition for convergence.

So, consistency we have already defined that means, what I require is that the truncation error should tends to 0 as the step size goes to 0 and the numerical scheme has to be stable. So, if this two happens, then we guarantee the convergence. So, this is for a linear situation. Of course, the same procedure or same outcome will handle or apply for the non-linear also because the non-linear PDE, what we do is we solve in a iterative fashion. So, iteratively that means, at any iteration a reduced linear form is considered.

So, the stability criteria is still holds good for those situation, but those method which are not adopted with the marching procedure like any boundary value problem or elliptic PDE and all that maybe we will discuss now. So, there is no marching procedure is adopted. So, this kind of stability does not come into the picture. So, there a consistency is good enough to have a convergence, but what we need to check is that the round up error should not go out of bound. So, that is possible if you have the coefficient matrix is diagonally dominant.

So, that is enough for to guarantee the stability. Because there it is not that from time level N we are going to N plus 1 the process is not started from a stage. So that means, a forward marching procedure is not adopted for that kind of boundary value problem situation. It is only when we have a initial boundary value problem.

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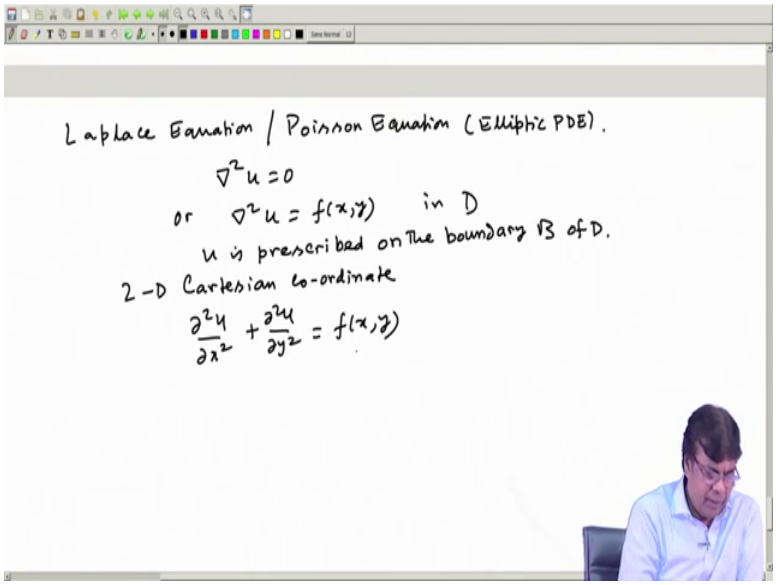

Now, we consider the pure boundary value problem; that means, say a Laplace equation or Poisson equation; this kind of. So, basically this is the elliptic PDE. So, what we have is say del 2 u equal to 0 or del 2 u equal to f x y. So, this is given in some D and u is prescribed on the boundary B of D. So, if it is a 2-D say 2-D Cartesian co-ordinate, we have del 2 u del x 2 plus del 2 u del y 2 equal to f x y.

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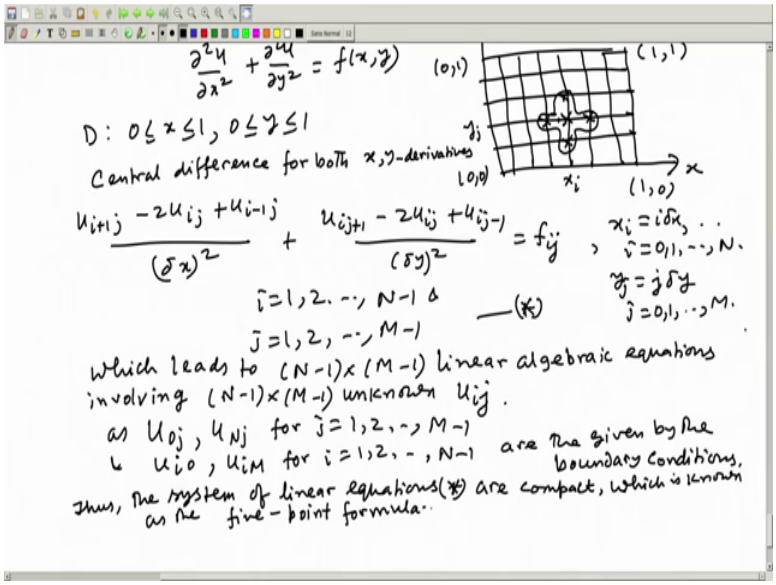

Now, so that means, say suppose D is a square domain and D say 0 less than x less than equal to 1, 0 less than y less than equal to 1. So that means, say 1 by 1 square

dimensional. Say suppose, we have a situation like this say this is x; this is y. So, 1 0, 1 1, 0 1, this is 0 0. So, the same manner, we considered the subdivided the domain into number of grids. So, let us call this is say this line this is say x i and this is y j. So that means, this is the point say x i grid size.

Now, if I apply the central difference scheme x and y derivatives, this gives u i plus 1 j minus 2 u i j plus u i minus 1 j by delta x square u ij plus 1 minus 2 u ij plus ui j minus 1 by delta y square. So, this is equal to f i j. So, we have already defined xi is i delta x; say i varying from 1, 2; i can be 0 also. i 0 means the first boundary 1 to say N. So, 0 N another boundary points and y j is j delta y. So, j is varying from say 0 1 to M.

So, these are the variation of i and j. Now, if i vary i from 1 2 N minus 1 and j from 1 2 M minus 1; so that is the interior point. So, which leads to N minus 1 cross M minus 1 linear algebraic equation involving N minus 1 cross M minus 1 unknowns. Unknown u i j because at as u 0 j u N j for j equal to say 1 to M minus 1 and u i 0 and u i M for i equal to 1 to N minus 1 are given at the given by the boundary condition.

So, let us call this system of equation star. Thus, the system of equation star are compact. Compact in the sense that same number of equation, same number of variables. So, if you solve we get a unique solution; same number of equation is some variable. So, basically what I did this is called the five-point formula. So, to find the solution at the grid point ij, we have used i. So, this is ij minus 1; ij plus 1; i j i plus 1 j; i minus i minus 1 j. So that means, this 5 five-points are clubbed together. So, this is also referred as the compact as the five-point formula.

Now, one thing is that is not forming a tri diagonal. So, if it is not forming a tri diagonal so that we cannot use the Thomas algorithm.

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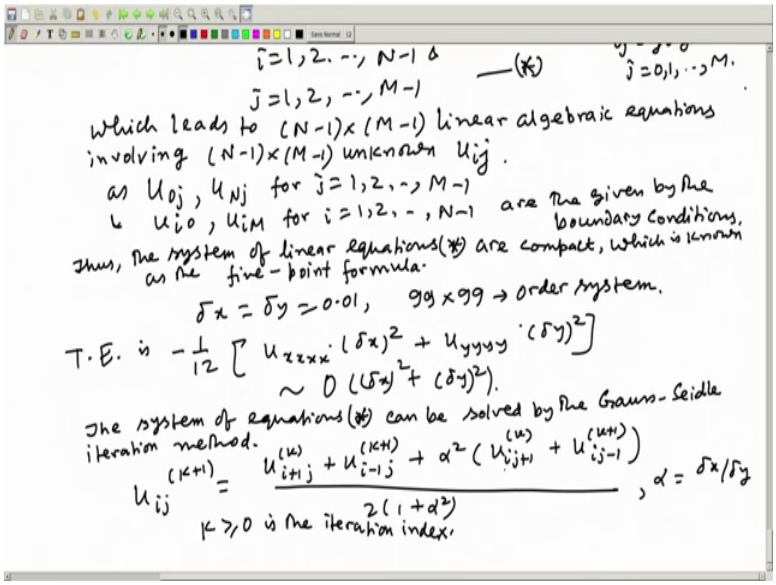

So, basically these delta x delta y are quite small. So, we have a quite a large system. Say for example, even if I have x y varying between 0 to 1 and delta x delta y say 0.01 showing that case we have a 99 cross 99 order system. So, this many number of equation and this many number of variables. So, 10 to the power 4 type; so, which is very difficult to obtain the solution by some elimination gauss elimination and other. Another is that if we do the same process as we described before the truncation error.

So, we can find out the truncation error because it is the local and global at the same because we have not taken any specific grid any general grid discretization was adopted to a general grid. So, the truncation error comes to be is truncation error is minus 1 by 12 u 4 x into delta x square plus u fourth derivative into delta y square. So, basically that is the order delta x square plus delta y square. Obviously, that has to be because we have used a central difference scheme.

So, now this system of equation can be solved this star by a iterative procedure, by the Gauss-Seidel iteration method. So, in the Gauss-Seidel iteration method, what I do is the ijth equation, we treat the variable u ij. So that means, we are going by this way. So, at the ijth many equation, when you come to this we are going by point by point. So that means, when I am here.

So, already this two are solved obtain the solution; so, at that iteration. So, the ijth iteration oops see we treat these two from the previous iterated value and these two from

the rather opposite way; these two from the current iterated value and the other two by the previous iterated value.

So, right this way u i j k plus 1 as u k i plus 1 j plus u k plus 1 plus 1 j plus say alpha square which is the ratio between these two grid size i j plus 1 at the kth iteration plus u k plus 1 i j minus 1 by 2 into 1 plus alpha square. So, where alpha is nothing but delta x by delta y. So, one can choose these to be 1 and k greater than equal to 0 is the iteration index.

So, that mean say ijth equation, we solved for the uij and the points which are prior to i j; that means, i minus 1 j and ij minus 1 are already obtained solution at the k plus 1 the iteration level. So, we have taken the upgraded value and the points which are ahead of this i j this is i plus 1 j and i j plus 1 are kept at the previous iterated value. So, this is the way the iteration process or recurrence relation is there iteration process performed.

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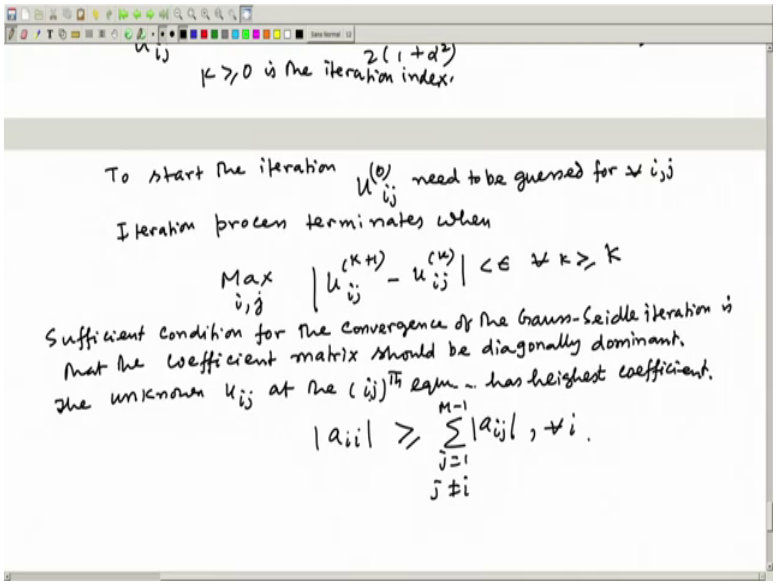

So, to start the iteration, u 0 ij need to be guessed for all ij; that means, at all the grid points u 0 ij is assumed and iteration process terminates if or when max over ij uk plus 1 ij minus u k i j is less than epsilon, for all k greater than equal to some capital K. So, this is a sufficient condition for if the this the or I mean if the sequence converge then this will be satisfied.

So, these guarantee that if convergence is achieved. Now the convergence of course, depends on how good the choice of u 0 ij is say u 0 ij is perfectly chose after few iterations after certain iterations the convergence is obtained. Now, the sufficient condition for Gauss-Seidel iteration or convergence of the Gauss-Seidel iteration is that the coefficient matrix should be diagonally dominant.

Now, the same what we did in the procedure, the same way we have adopted the procedure. So, where u i j at the ijth equation the ui j has the highest value. So, if we look into the numerical scheme the unknown u ij at the i rather ijth equation has highest coefficient. So, these guarantees the diagonal dominance; so that means, what you should have is that u i i should be greater than equal to sigma j equal to 1, 2 whatever M minus 1; j not equal to i mod of ai j. So, that should happen for all i. So, this is happens.

So, without failing to that also it can converge, but this is a that is why it is referred as the sufficient convergence, sufficient condition for convergence. Now one trick to accelerate the convergence for the Gauss-Seidel method, basically the Gauss-Seidel method is little above than linear order of convergence.

So that means, it convergence rate is very slow. The error at every iteration reduces in a very slow rate. It is a kind of little above than the linear order of convergence. To accelerate the convergence, there is a procedure that is called the Successive-Over-Relaxation technique.

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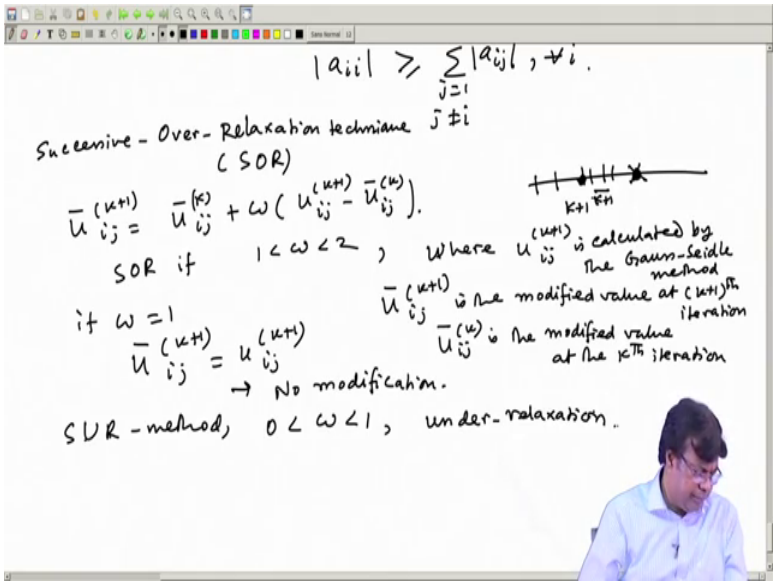

Successive-Over-Relaxation method technique or one can call as simply SOR. So, in SOR so that means, say suppose this is the basically what I know is that this is the solution and you are going step by step like this way and you are converging towards the solution. Now, to accelerate at this stage at this stage say this is k plus 1 to accelerate that the same direction i extrapolate the solution to some level at the same level k plus 1 bar if I call.

So that means, we did this combination u bar k plus 1 i j equal to u bar k ij plus omega into u k plus 1 ij minus u bar k ij. If over relaxation if omega rather 1 less than omega less than 2 and under relaxation so that means, basically what I did is the combination. We are at the k plus 1 is the Gauss-Seidel iterated value and we know that we are linearly converging to the solution.

This is the exact solution. This is the solution or converge solution. So, at this stage in the same direction, we are little advanced. So, omega is greater than 1. So, the direction is determined by this ui j k plus 1 minus u bar k ij.

So, in the same direction there is a line if I call this at the position vector. So, the equation of the line is given by this way. So, we are going little ahead of that is the extrapolated values and where, u where uk plus 1 is calculated by the by the Gauss-Seidel method and u bar k plus 1 ij is the modified value at k plus 1 iteration same way u bar k ij is the modified value at the Kth iteration.

So obviously, if omega is equal to 1 so, what I find that no modification; so, u k plus 1 ij no modification that is the no modification. Now, we will apply the SOR provided what do you find that error is gradually reducing, where to find that values are gradually reducing and we could see that it is going to terminate after a large number of stages.

But in some cases what will happen is that the value is undershooting overshooting. So that means, there is an oscillatory situation is occurring. So, in that case then we choose this omega the parameter less than 1. So, that is called the successive SUR method. So, SUR method that is sometime is useful.

In that case, we take omega in within these less than 1 that is called the under-relaxation. So, if we find that the solution, the every introduction the value is sometime bigger

sometimes smaller than the previous one. So, these are the some modifications are suggested to perform the convergence acceleration of these Gauss-Seidel method.

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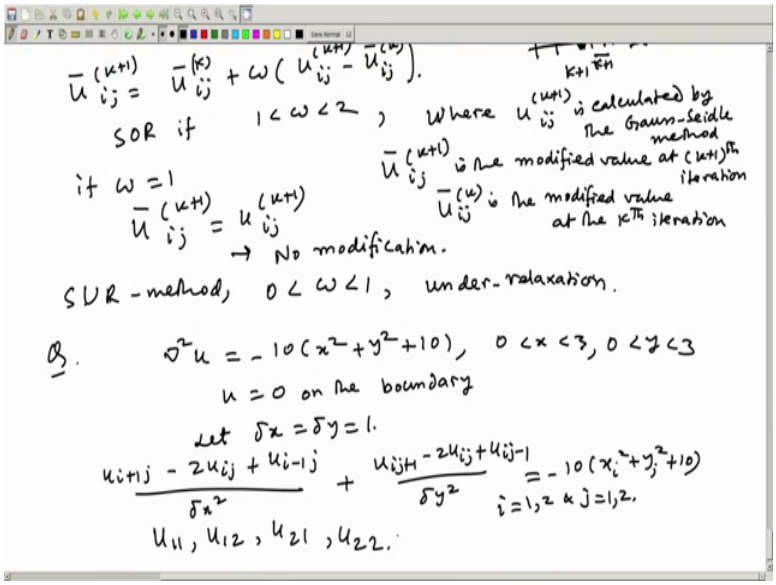

Now, let us solve one problem say del 2 u equal to minus 10 x square plus y square plus 10 say 0 less than x less than 3 0 less than y less than 3. So, there is a 2-D Poisson equation and u equal to 0 on the boundary. So, boundary is a square and let us take delta x equal to delta y equal to 1.

So, if I choose that. So, I can now write the discretized equation as u i plus 1 j minus 2 u ij plus ui minus 1 j by delta x square which is 1 and ui j plus 1; 2 u ij plus ui j minus 1 by delta y square equal to minus 10 into x i square plus y j square plus 10. So, i is 1, 2 and j is also 1, 2. So that means, basically we have 4 unknowns and 4 equations.

We need to find out u 1 1, u 1 2, u 1 3 is the boundary and u 2 1 u 2 2. So, if we apply the Gauss-Seidel procedure and obtained the solution so, that can be computed by the Gauss-Seidel steps. Now, there are several modifications for this kind of because in many situations, we come across the elliptic PDE or Poisson equation or Laplace equation, particularly when you have a force potential electric potential or velocity potential kind of equations.

Now, these are sometime can be treated in a form because of this restriction because of this drawback the convergence of the solution very slow process. So, one of the remedy

is to treat it as a unsteady way and then, that unsteady one can be solved in a manner which can have a rapid rate of convergence.

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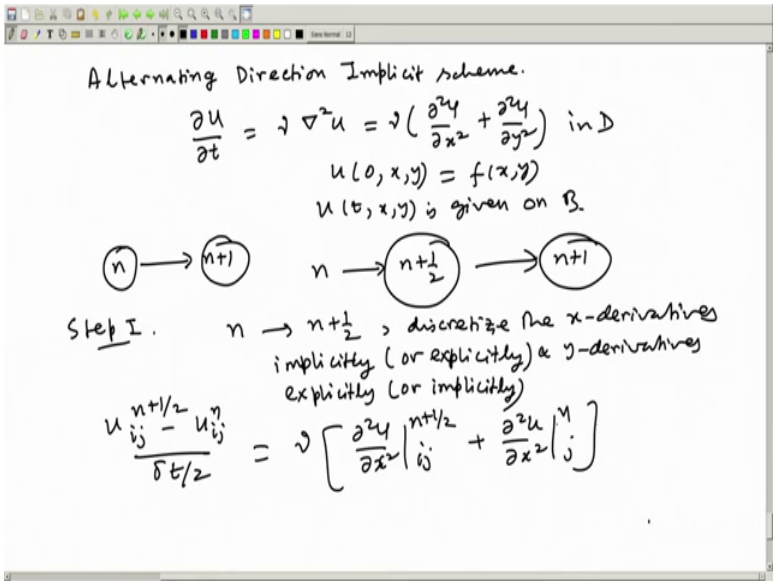

So, I will before concluding I will just introduce one more method this is called the Alternating Direction Implicit scheme. So, I can take a situation like this either this t can be fictitious or maybe a diffusion process with multi dimensional. So, even for a steady one I can take this as a pseudo unsteady and continue with the time till another steady state is reached.

So, you have a initial condition to be prescribed and u is it is in D is a and u t, x, y is given on B. So, to solve this, what I do that the same time marching procedure. But instead of going n to n plus 1 directly from one time step to another what we do is we go via a intermediate time step. n plus half and then, we go to the next time step n plus 1.

So, that is why this is termed as the implicit alternating direction implicit scheme. So, in the step 1, there what I do is we go from n to n plus half, discretise the x derivatives implicitly and y derivatives explicitly can be interchanged. So that means, what we are doing is u n plus half ij minus half ij minus u n ij by delta t by 2 is nu del 2 u del x 2. This I am discretizing in the current time step that is the implicit and this one at the previous time step; known time step. You know why I am doing this?

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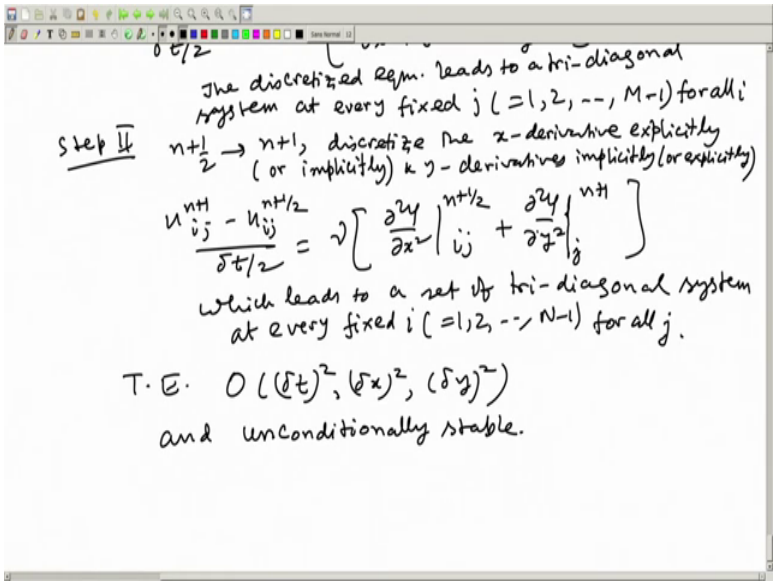

Because, the discretized equation leads to a tri-diagonal system at every fixed j either say 1, 2 etcetera M minus 1 for all i. In the step 2 we just reward the situation that means, now we are going from n plus half to n plus 1; n plus 1 discretize the x derivatives explicitly, if we have taken implicitly.

So, it is explicitly or implicitly, if we have not done that before and y derivative implicitly. So, the discretized equation looks like u n plus 1 ij minus u n plus half ij delta t by 2 equal to nu del 2 u del x 2 n plus half i j plus del 2 u del x 2 in sorry here, it will be y del y 2 n plus 1; n plus 1 j ok.

So, what is the advantage? The same advantage as I said. So, which leads to a set of; here it should be written set of set of tri-diagonal system, tri-diagonal matrix, tri-diagonal system at every fixed i which can be 1, 2 up to N minus 1 for all j. So that means, now situation is reversed and this procedure is much faster compared to thee because tri-diagonal means we will have a direct method that is the Thomas algorithm can be used and this gives a much faster way to get the convergence by adopting a pseudo time dependency.

So, only thing is that we have to change the direction that is one direction if in the first time step that is n plus n to n plus half, if x is the explicitly discretized; then y should be implicit should be in the implicit manner it should be discretized and it will be reverse the direction will be reversed.

So, that is how so, order of accuracy and the stability are remain the same. Here of course, the a time marching procedure is adopted and it is a implicit scheme. So, the truncation error can be is can be shown to be order delta t square, delta x square, delta y square and unconditionally stable. Because it is a kind of all the good properties of the Crank Nicolson scheme is present.

But, Crank Nicolson scheme cannot be reduced to a tri-diagonal system because they are all at a time at the same time level both x derivative and y derivative are treated to be unknown. So, we do not have a tri-diagonal pattern for the Crank Nicolson scheme. So, that is why ads scheme is more popular.

So, with that I stop here and the assignment is all followed which will illustrate the methods and some hints will be also given how to how to use this method to solve the problems for both numerical as well as the previous one whatever we have done for eigen function expansion and a little bit of greens function etcetera. Thank you.

Thank you all.