Mathematical Methods For Boundary Value Problem

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Lecture - 02
Strum-Liouville Problems, Linear BVP (Contd.)

Welcome back. So, we were talking about the eigen value and eigen function of a linear operator.

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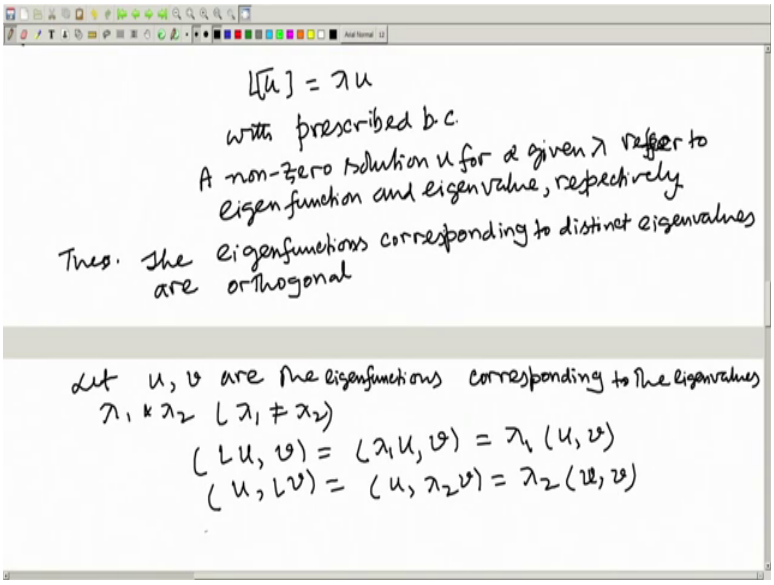

So, we have defined the linear operator as L u equal to lambda u. So, any for a given u with subject to boundary condition with boundary conditions with prescribed; let us call this as prescribed boundary condition; prescribed boundary conditions. So, a non zero solution u with for a given lambda nonzero solution u for a given lambda refers to eigen function and eigenvalue respectively. Now, we will show some important property; one of the important property is that the eigen functions are corresponding to distinct eigen values are orthogonal.

So, one of the important theorem one can say is that the eigen functions; the eigen functions corresponding to distinct eigen values are orthogonal; orthogonal means the inner product is 0; if they are different. So, let u and v are the eigen values corresponding to eigen sorry; this is not eigen values they are eigen functions eigen functions there. So, u and v are the eigen functions corresponding to the eigen values say let us call lambda

1, lambda 2; lambda 1 and lambda 2 and they are distinct; that means, lambda 1 not equal to lambda 2.

Then what we have to prove is that this inner product of u v is 0. Now what we know is Lu v is equal to what? We can write as Lu is lambda u v and equal to one can take out as lambda 1; of course, lambda this is the eigen value lambda 1 corresponding to this lambda 1; so, lambda 1 u, v and if we take these inner product v; u L v. So, this is u lambda 2 v and this is can be taken as lambda 2 u v; now what we know is that they are self ad joint.

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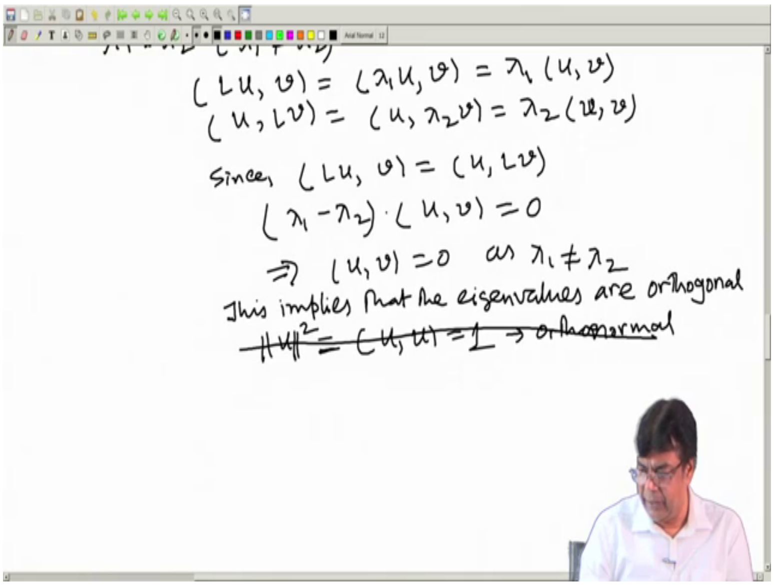

So, since L u v equal to u Lv; we have already shown that so; that means, lambda 1 minus lambda 2 into this is the problem here. So, lambda 1 minus lambda 2 into this inner product u, v equal to 0; this since implies u, v equal to 0 as lambda 1 not equal to lambda 2. So, this is happening for any pair of u, v.

So, this implies they are orthogonal implies that the eigen values are orthogonal. And orthonormal if we will call if the inner product of with itself is 1; so this may not be. So, if this is the one then orthonormal, but orthonormal means we have to this is nothing, but the length of or norm of the vector u or the length of the function; so, that has to be 1.

So, we can scale that way; so that it becomes 1 itself. So, this is may not be the situation over here; so, this is may not be applicable all the cases. So, let us take an example for the eigenvalues and eigen function; now why we need these is a very important thing because this theorem is an important one. Because this in the domain a B or 0 a over which we are finding the solutions or where our attention is focused for the boundary value problem, the eigen values are forming a orthogonal set. Now, if we have an orthogonal set of vector in a certain domain.

So, any other vector can be expressed as a combination of these eigen values; this orthogonal vectors; so that is called the eigen function expansion. So, this eigen function expansion enable us to construct solution by using the eigen functions of the operator for non homogeneous cases, we will come to that. So, prior to that let us understand what eigen values; eigen functions are and what at their role.

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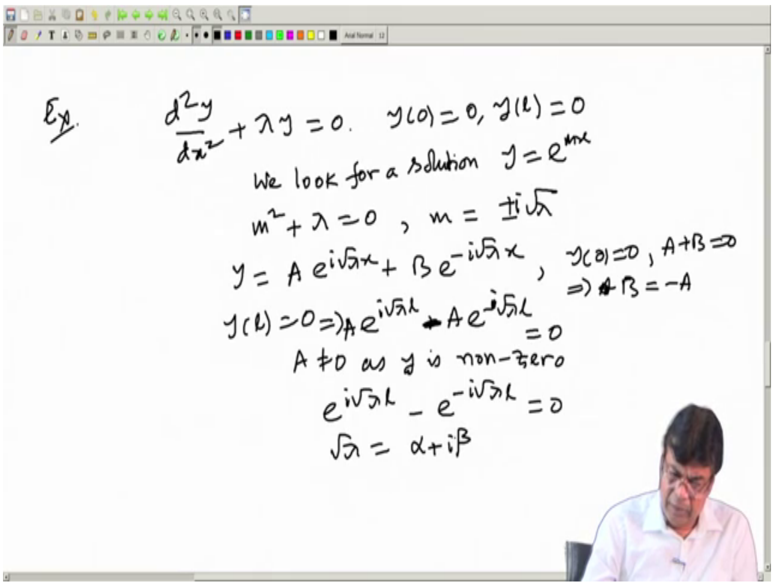

So, example one of the examples say d 2 y; d x 2 plus lambda y equal to 0 and let us call y 0 is 0 and y l equal to 0; o less than x less than l. So, one can find out; so let us we look for a solution; y equal to say e to the power mx.

So, what I get is m square if we now substitute here; m square plus lambda equal to 0. So, m equal to plus minus root I root lambda; now so; that means, y is becoming A e to the power i root over lambda x plus B e to the power minus i root lambda x; now what we have is y 0 is 0.

So, y 0 is 0 implies A plus rather we can say that B equal to minus A a plus B equal to 0. So, B equal to minus A another also y l equal to 0 implies e to the power i root lambda l this is A. So, one can write as this we are writing in terms of B. So, this is A and this is B minus A e to the power minus i root lambda L equal to 0.

So, since A cannot be 0; A is not equal to 0 as both A B 0 means as y is nonzero; we are looking for a nonzero solution non trivial solution. So, both A and B cannot be 0 because if A equal to 0 B is also 0; so we cannot have that. So, what we can have is i root lambda l minus e to the power; not minus minus e to the power minus i root lambda l equal to 0. So, in other words what we are going to have is. So, let root lambda if I substitute as alpha plus I beta. So, lambda can be a real or imaginary.

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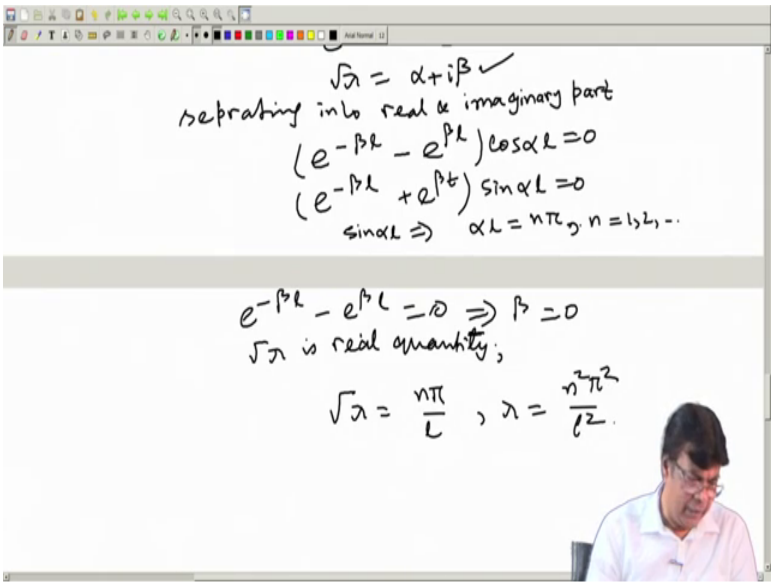

So, now if we separate into real and imaginary parts; into real and imaginary part we get e to the power minus beta l minus minus e to the power beta l cos alpha l equal to 0 and e to the power minus beta l plus e to the power beta l sin alpha l equal to 0. So, this gives sin alpha l equal to 0. So, this implies that sin alpha l equal to 0 implies alpha l equal to n pi; n is an integer 1, 2 etcetera again n cannot be 0.

Because n equal to 0 means it will be a trivial solution and cos alpha l cannot be 0 in that case, so what we have is e to the power minus beta l minus e to the power beta l equal to 0; this can be happening only if beta equal to 0. So; that means, root lambda is real quantity and it has to be because lambda is the ratio between two real quantity both y and y double dash are taken to be real.

So, lambda is real; so root lambda has to be real quantity only it is just proof that even if we take root over lambda is alpha plus i beta; what it comes out to be that root over lambda is nothing, but a real quantity. So, what we have is root lambda equal to n pi by l.

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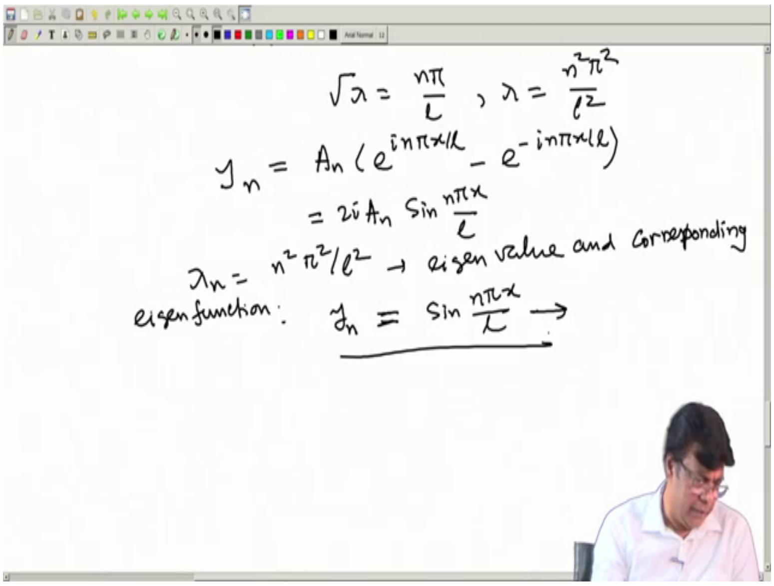

So, that means, lambda is n square pi square by l square this is the eigen value of the n equal to 0 is ruled out because it cannot be 0; because it cannot have a non trivial solution and the corresponding y we call as; so y n corresponding to this root lambda we can say as A n e to the power i n pi x by l minus e to the power minus i n pi x by l. So, this is equal to say 2 i into A n.

So, sin n pi x by l; I can take this as a constant. So, this eigen functions because we are talking about a non homogeneous; homogeneous equation, so any multiplication. So, this is the homogeneous equation. So, any multiplication is also a solution any constant multiplication. So, we call this lambda n as n square pi square by l square; the eigen value eigen value and corresponding eigen function is eigen function is given by y n we can say; y n is sin n pi x by l.

So, this is a non-zero solution because there is a nonzero y. So, that why you can call this as the eigen values. So, let us write in a better way. So, these are the eigen values and eigen functions for the this simple operator.

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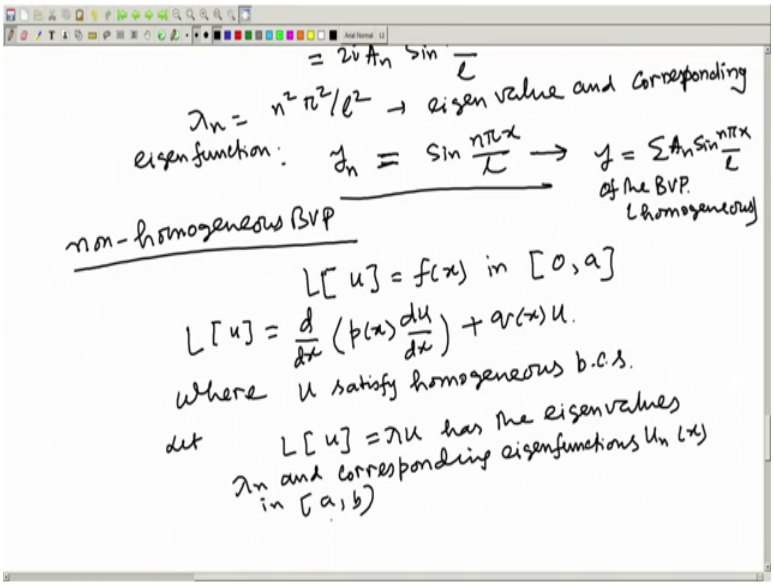

Now, we talked about the non homogeneous boundary value problem. So, what we found that a homogenous boundary value problem by determining the eigen values we can find out the eigen functions and a any combination. So; that means, the solution y can be expressed as any combination in sin n pi x by l; this is the general solution of the of the BVP; the linear homogeneous BVP; homogeneous what discuss sorry I am writing non homogeneous, but it is basically a homogeneous boundary value problem.

So, any combination will be a general solution; now we will apply this trick to find the solution for a non homogeneous situation. So, if we have a situation like this L u equal to f x in say let us call the interval as 0 to L and L operator is that simple operator linear operator. So, right in a form as any second order linear operator can be written in this way p x d u; d x plus q x u. So, now our intention is to solve this and subject to where u satisfy homogeneous boundary conditions as indicated in two prior to this.

So, this is our next task now; so we know this that we know that the L u let L u equal to lambda u has the eigen values lambda n. And corresponding eigen functions corresponding eigen functions we call as u n; u n x u n x. Now as we have checked before that this u n x are inside the domain a, b. Now, or rather instead of writing u n x because that may create confusion because u is generally we are writing.

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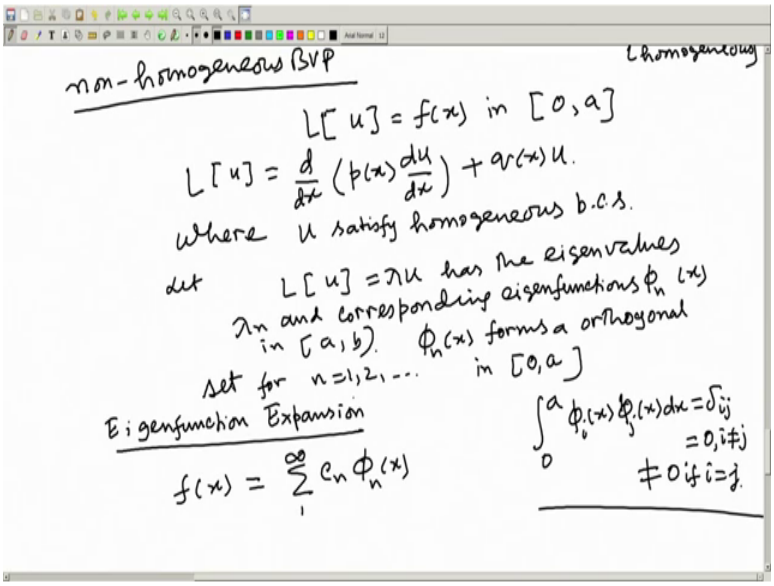

So, let us write this is as phi n x; phi n x ok. Now this phi n x; phi n x forms a orthogonal orthogonal set for n equal to 1, 2 etcetera in a. b. Now what we will do is eigen function expansion; now this f x in sorry not a, b; it is 0 a we are talking about.

So, a, b can be of course, shift it to the interval 0, a now what it says that we know that 0 to a phi i; x phi j x we have denoted this way d x equal to delta ij. So; that means, this is equal to 0 if i not equal to j is nonzero if i equal to j. I am not telling that it will be 1, but non zero; if i not equal to j then i equal to j then only it is nonzero and if i; so i not equal to j; it is 0 and it is nonzero provided if i equal to j; so this is forming an orthogonal set.

Now what we are doing is; want to do is f x we want to write; f x as an expansion 1 to infinity some C n; phi n x, expansion of this form. Now, if f x is continuous and second order contour C to a b; I mean continuously differentiable for second derivative; then we can have a expansion of this form. So, this infinite series converge and it converts to f x. Now the question is how to find out the C n?

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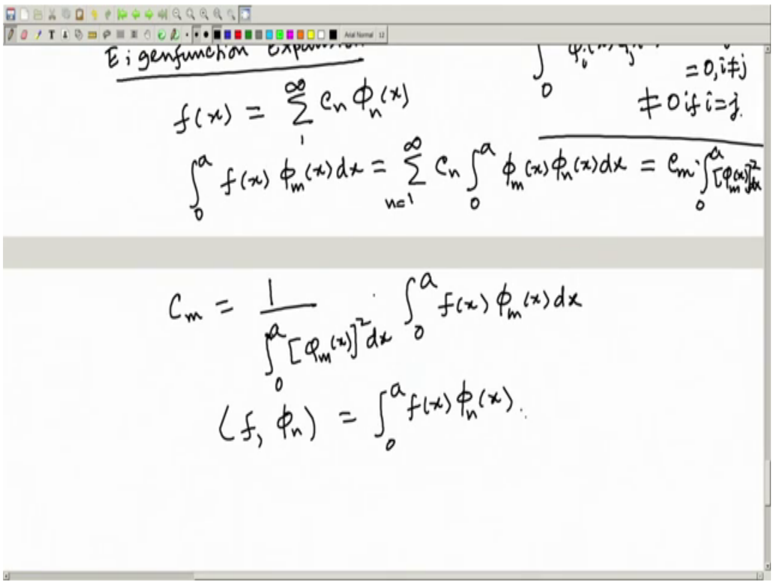

Now this C n by using this orthogonality property; so, if I now multiply with some phi m x; say we multiply both side by f x into say phi m x; d x and this one; 1 to infinity. So, C n is a constant can be taken out 0 to a; phi m x phi n x; d x. Now what will happen is that these are all 0 except when n equal to m. So, we because this is the summation over n; so; we are summing from 1 to infinity; so for all the cases it will be 0; except when these n become m, so in that case we will have this one C m into some constant. So, C m into 0 to a phi m x whole square d x.

So, now we can write from here that C m is C subscript m is equal to 0 to a phi m x whole square; this is also called the norm of this or length of the function; phi m x eigen functions 0 to a f x phi m x; dx. So, in that process we get a x function so this is nothing, but f n, phi n this is nothing, but the inner product between these f and the eigen function phi n. So, these f x is expressed as a series of series function of the eigen functions phi n x.

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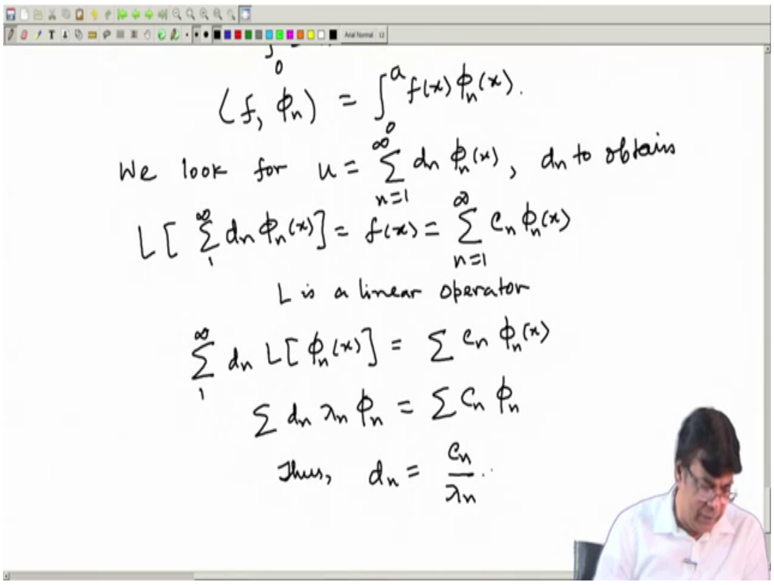

Now, what we will do is we will look for a; we will look for a solution, u of the non homogeneous equation in this form; d n phi n x; d x n equal to 1 to infinity; d n yet to be find out, d n to obtain we look for this solution.

So, if I now substitute; so L of; L of u so; that means, this is d n, phi n x equal to f x this side. So, f x we are writing as sigma C n; phi n x is also 1 to infinity; now L is a linear operator. So, L can be taken inside the summation. So, what I get is if I take L inside, so this is sigma 1 to infinity d n; L of phi n x equal to sigma C n; phi n x. Now L of phi n x is nothing, but lambda n phi n. So, what I get is d n; lambda n, phi n equal to sigma C n; phi n.

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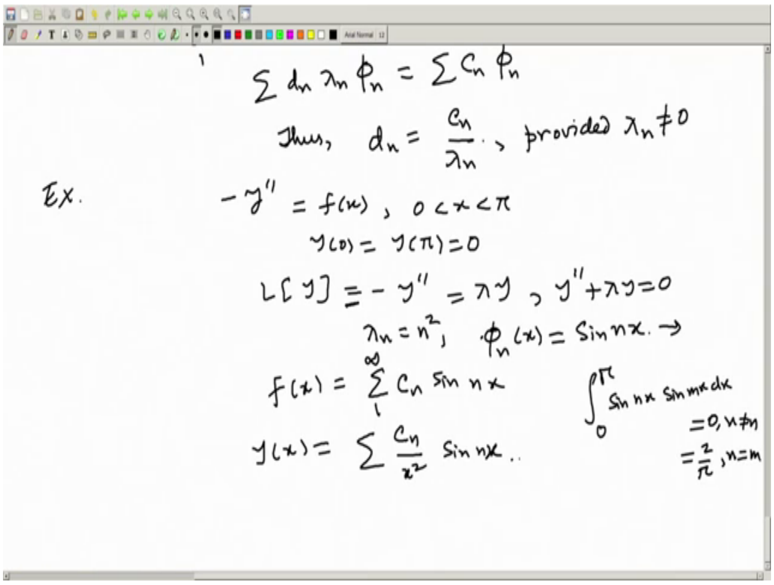

So, from here what I can find that d n; these coefficients are nothing, but C n by lambda n, but this is provided lambda n is not equal to 0; if lambda n equal to 0 of course, this term will not appear. So, so this is one way of solving the non homogeneous boundary value problem in terms of the eigen functions. So, let us take an example.

So, basically what property we have utilized is that the orthogonality of the eigen functions that orthogonal eigen functions are forming an orthogonal set. And, in that authority leads us to consideration of the non homogeneous function; f x to expand in terms of the eigen functions. And, in the same way the solutions also can be expressed as an expansion of the eigen functions.

So, consider this example minus y double dash equal to f x for 0 less than x less than pi; very simple problem, y is 0 equal to y pi equal to 0; we have to choose homogeneous boundary condition. So, if we take L y; L y equal to minus y double dash and that is equal to; so this is 3, this operator. So, lambda y; so what we can get is y double dash plus lambda y equal to 0. So, lambda n is nothing, but n square and y phi n x can be shown to be sin n x ok. So, this is the eigen values and eigen functions.

So, f x one can write as sigma 1 to infinity, C n sin n x because they are forming the eigen values; sine nx are the eigen functions. So, this one can find out the c x; C n by multiplying sin m x. So, as you know that 0 to pi; so sin n x and sin m x; d x will be 0 if n not equal to m and it will be 2 by pi I guess if n equal to m; so we apply those property. So, one can find out the y x is given by C n by x square sin n x.

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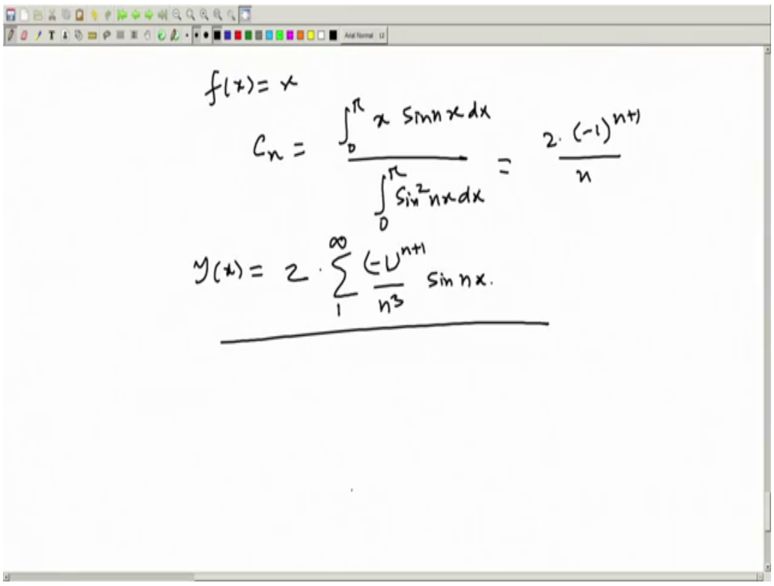

Now, suppose if I take if I take f x equal to x x a. So, in that case C n can be written as 0 to pi; x sin n x; d x by sin square x n x; d x, 0 to pi and that comes out to be 2 into minus 1 to the power n plus 1 by n. So, y x then we can write as 2 into sigma; 1 to infinity minus 1 to the power n plus 1 by n cube; sin n x; so this is a simple way of expressing the solution. Now f x in this case is very simple function, but we may not have a simple function all the time.

So, in that case these eigen function expansion is handy and we will show that in particularly in the and this kind of; one can find out also there are several examples like Bessel equations or Legendre equations and all. So, and can express the there where that; if it is a non-homogeneous term we cannot obtain the solution very straightforward way. So, in that case we have to go by a expansion in terms of the eigen functions because they are forming the orthogonal basis in the domain.

Thank you, we will talk in the next class.