Mathematical Methods For Boundary Value Problem
Prof. Somnath Bhattacharyya
Department of Mathematics
Indian Institute of Technology Kharagpur

Lecture - 19
Stability Analysis of Numerical Schemes

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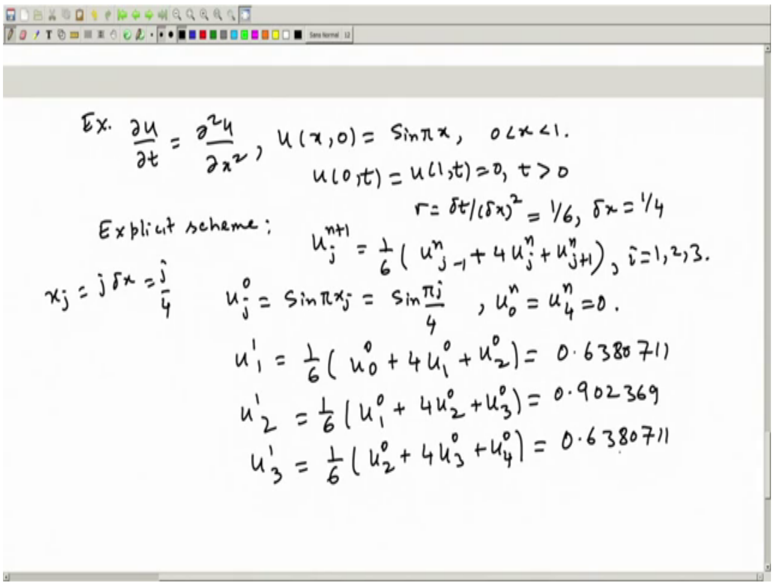

Now, let us take an example to illustrate the scheme we have discussed. Consider this problem del u del t equal to del 2 u del x 2, u x 0 is the which is the initial condition sin pi x and u 0, t equal to u 1, t equal to 0. So, this is 0 less than x less than 1 and this is t greater than 0. So, that means, the boundary is 0 and 1, these are the two boundary.

Now, if I first explicit scheme. So, what I get the explicit scheme is u n plus 1 j can be expressed if I consider let us define this parameter r as here nu is 1. So, delta t by delta x square and let us choose this as 1 by 6 and delta x equal to 1 by 4. So, if I choose this. So, explicit scheme comes to be 1 by 6 u n j minus 1 u n j minus 1 minus or for here 4 u n j plus u n j plus 1; j is varying from 1, 2, 3 because 4 here is the last boundary.

So, and what we have given is u 0 j equal to sin pi x j; that means, sin pi, x j is what? x j equal to j by 4 j delta x; so, that means, j by 4; so, sin pi j by 4. So, and what we have is u n u n 0 equal to u n 4 equal to 0, these are the boundary conditions are provided. So, with that one can obtain the solution and if I do the calculations I get the solution like this u 1

1 equal to 1 by 6 u 0 0 plus 4 u 0 j I am putting as 1. So, this is u 0 2 and that comes out to be 0.6380711; that means, using all the initial conditions

Similarly, u 1 2 equal to 1 by 6 u 0 1 plus 4 u 0 2 plus u 0 3 comes out to be 0.902369. Similarly, u 1 3 is 1 by 6 u 0 2 4 u 0 3 plus u 0 4; u 0 4 is the boundary condition. So, this is 0.638071. So, this is the next time level solution and in the same way we can proceed for the higher time level; that means, subsequent time level solution can be obtained.

Now, if I do the same problem by explicit scheme, if I do the same problem by implicit scheme, so, in that case what it will look like? That means, if I discretize by say implicit scheme. So, first order implicit scheme itself.

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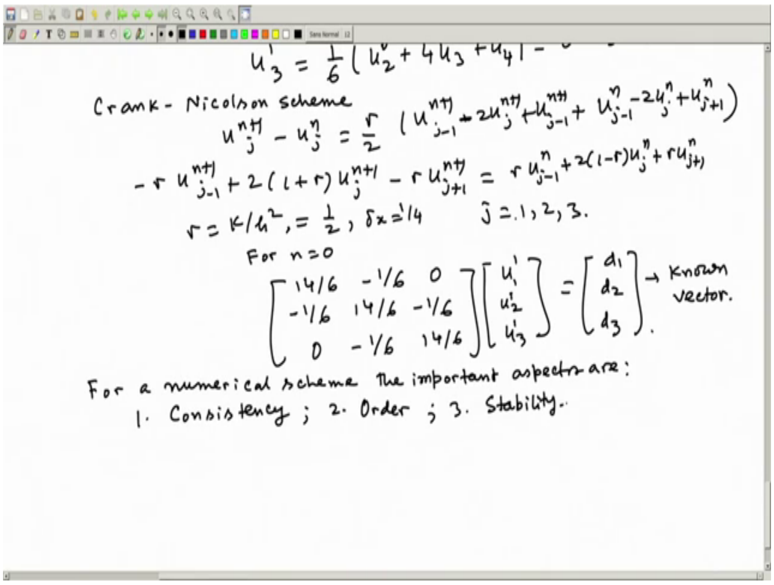

So, in that order if I go by Crank Nicolson scheme; so, if I go by Crank Nicolson scheme so, what we have is u n plus 1 j minus u n j equal to r by 2. So, r was the nu delta t by delta x square c.

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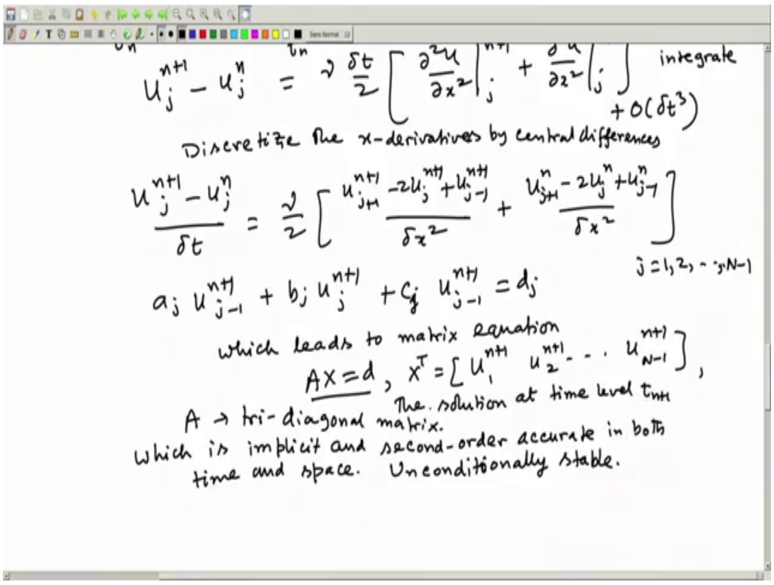

This is the thing we are writing here. So, u n plus 1 j minus 1 plus minus 2 u n plus 1 j plus u n plus 1 j minus 1 and u n j minus 1 minus 2 u n j plus u n j plus 1. So, if I now bring to this side, so, what we get is r by 2. So, you get minus r u n 1 plus 1 j plus 2 into 1 plus r u n plus 1 j this is j minus 1 and so, if I multiply and this one is minus r u n plus 1 j plus 1 that becomes r u n j minus 1 these are all known quantities because this is with superscript n or equal to if I choose r is k by delta h square.

So, if I choose r equal to say half and delta x 1 by 4 so, one can find out the solutions and we get a metric system because in this case j will be from 1, 2, 3. So, at any time level say if I start with n equal to 0 for n equal to 0, so, we get a tridiagonal system like this 14 by 6 minus 1 by 6 using the initial conditions and the coefficients minus 1 by 6, 14 by 6, 14 by 6; u 1 1, u 1 2, u 1 3 this is equal to some known values d 1, d 2 and d 3. So, once these are known vector so, once we invert this one we get the solution for these Crank Nicolson scheme and subsequently once u 1 j are known then we go to the u 2 j and so on. So, that is how the method proceeds.

Now, our next task is to consider; so, we know how to implement the numerical scheme for parabolic PDE or there is a conduction equation. Now, our next task will be to check what is the order of accuracy and what do you mean by what is the stability and under what condition the solution because whatever the solution numerical solution we are finding out it needs to finally converge to the solution which are bound to obtain by the differential equation. So, that is the exact solution of the differential equation ODE. To do that we define so, three factors which need to be important for a numerical scheme.

The important aspects are 1 is consistency; 2 is the order; and 3 is stability. Now, what is consistency? Now the consistency; so, whenever we are applying a numerical scheme what we are doing is we are retaining only up to the second order terms, third order terms, first order terms and truncating a infinite series to a finite number of terms. So, in that process we are committing a error. So, that error since it is arised due to the truncation is referred as the truncation error.

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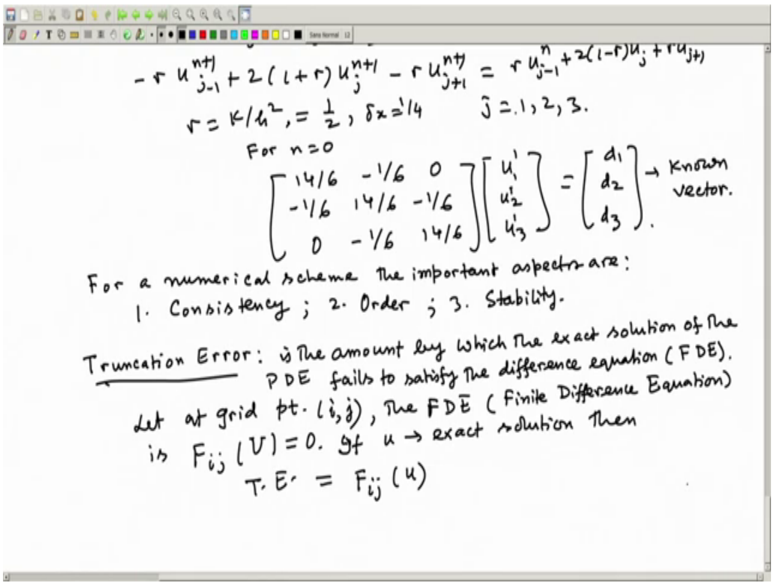

So, truncation error now, first we define what is truncation error. Now, truncation error we define in the way is the amount by which the exact solution of the PDE fails to satisfy the difference equation or finite difference equation. Let us call this as a FD FDE, Finite Difference Equation. So, let at a grid point i, j this is our the FDE the finite difference equation; finite difference equation is let us call as say F i, j and is a function of say small u ok.

Let us write this as the capital U because F i, j. Now, this is the finite difference equation equal to 0 say. Now, if small u is the exact solution, then truncation error this is the TE is nothing, but F i, j small u so; that means, I replace the solution I replace the u by the solution of the finite difference equation. Sorry, the exact solution of the differential equation we now substitute into the finite difference equation. So, whatever the residue whatever the leftover that is we are calling as the truncation error.

Now, this truncation error of course, it is does not convey anything unless we know the order of the truncation error. Now, to get an expression of the truncation error, so, what we do is we expand by Taylor series and get an estimate of the truncation error. Now, one thing is that there is a local truncation error let us say local and global will come into picture if we have a different way of discretization depending on the grid. So, that means, if I vary the grid and the discretization procedure changes then the truncation error will change vary with a grid. So, then in that cases we can call as a local truncation error.

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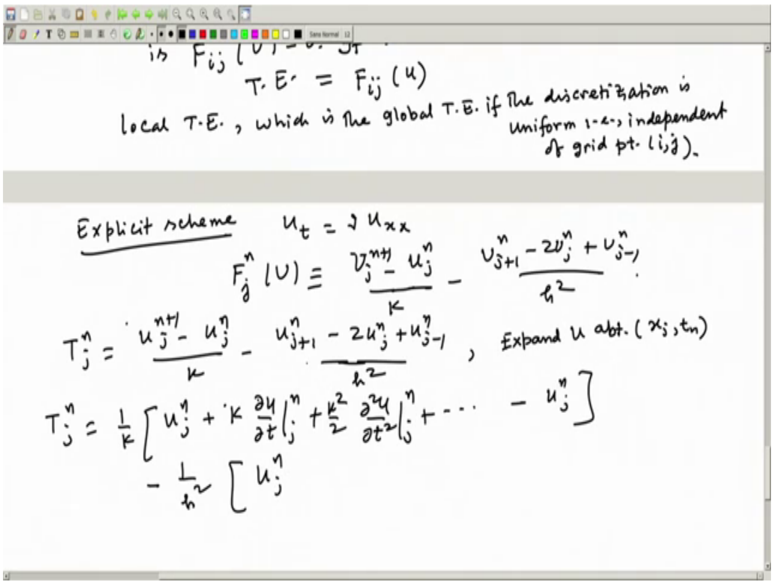

So, this is the local T.E., which is same as a global T.E. which can be treated as a which is the global T.E. if the discretization is uniform or that is independent of grid size independent of grid it independent of the grid point i, j. So, for all the grid points I have the similar or a uniform procedure if I have adopted then the local truncation error and global truncation error is same.

So, for the explicit scheme whatever we have just described what should be the truncation error. So, what they have explicit scheme of this equation the T ij whatever the first of all our difference equation F ij is u equal to nothing, but u we are writing in a capital version U n plus 1 j minus or rather ij means here it is n j in n j let us call. So, this is u n j by delta t minus u n j plus 1 minus 2 u n j plus u n j minus 1 by h square. So, let us call this is K this is called this is a K. So, this is K. So, this is the finite difference equation for the explicit scheme.

So, the truncation error at the grid point n j is or since our notation is like this, so, superscript n time and subscript j is the space. So, T n j is nothing, but if I replace this u by the exact solution; so, T u n plus 1 j minus u n j by k minus u n j plus 1 minus 2 u n j plus u n j minus 1 by K square.

Now, to know about the form of the truncation error expand u about x j, t n if when I expand all the variable about x j, t n. So, I get T n j is equal to 1 by K u n plus 1 j. So, u n j plus delta t or rather K del u del t evaluated at n j plus K square by 2 del 2 u del t 2 evaluated at n j and so on; the last one is u n j that we have need not have to expand sorry this is h square this is h square. So, this is 1 h square, so, u n j. So, here it is x.

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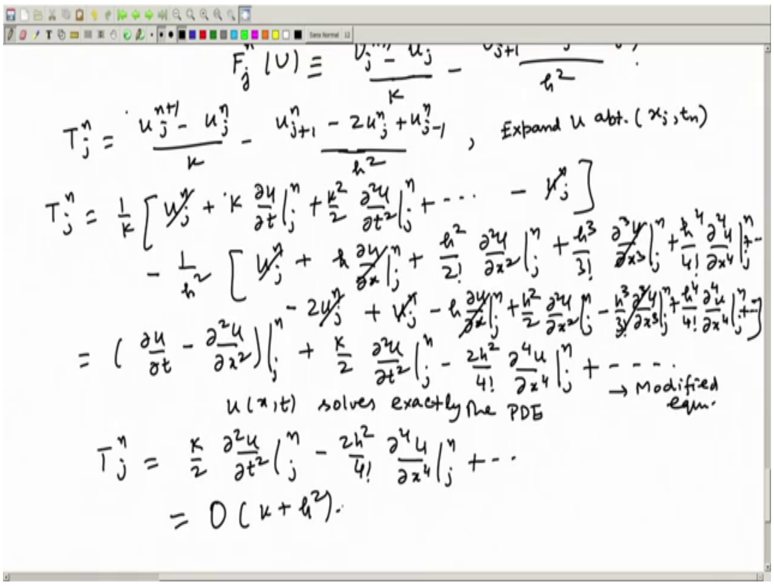

So, plus delta x del u del x n j plus del x again instead of del x I can write as h, so that square we can just write like this way. del 2 u del x 2 n j plus h cube by 3 fact del 3 u del x 3 n j plus x to the power 4 by 4 factorial del 4 u del x 4 n j plus etcetera terms minus 2 u n j nothing to be done for this the next one is a reduction del u del x n j plus h s square by 2 del 2 u del x 2 n j minus h cube by 3 factorial del 3 u del x cube n j plus this will be plus h to the power 4 all the even term will be plus because h is negative n j etcetera.

So, this is the truncation error expression is a infinite series. Obviously, this can be little mix little simplified. So, this even j 2 u n j and there is also u n j are get cancelled. So, our first let us collect the terms which is independent of K and h. So, we get del u del t and this is here this is also get cancelled. So, this third order also get cancelled. So, here you get h square h square and outside is 1 by h square. So, we get del 2 u del x 2 which has no multiplication with h or K. So, this is free of h and K. So, this is the term.

And, then next what we I get is K square by 2 del 2 u del t 2 n j and K cube by sorry not K square because K is outside. So, K by 2 K by 2 K square by K by 2 and for h h cube is out. So, the least term the next term will be of lowest degree h is h square. So, this is minus 2 h square by 4 factorial del 4 u del x 4 n j plus all these terms which are higher order in K and h square.

Now, u is the exact solution of the differential equation. u x, t solves exactly the PDE basically this is the equation is called a modified equation by this is a infinite number of terms partial differential equation by this numerical scheme basically we are solving this differential equation these infinitely long differential equation which is referred as the modified equation.

So, we are solving exactly this differential equation of this form. Now, if I consider u is the exact solution we have considered u as the exact solution of the PDE. So, the first term is satisfied. So, T n j comes out to be del K by 2 n j plus or minus 2 h square by 4; this is the least order term del 4 u del x 4 n j plus etcetera. So, that means, it is of order K square not K square K plus h square.

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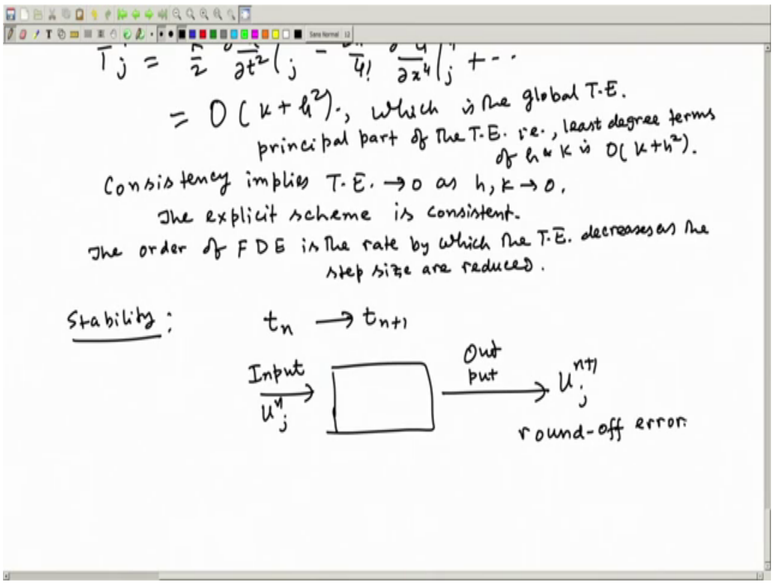

So; that means, it is a first order in time first order accurate in time and second order and this is the global truncation error which is the also the global truncation error there is no ambiguity because. So, always what you have to see is the principal part principal part of the T.E. that is least degree of least degree terms of h and K is order K and h square. So, that means, first order in time and second order in space. So, this determines the order.

Another term we talked about consistency. So, consistency implies where T.E. should tends to 0 as h, K tends to 0. So, that means, h as the step size goes to 0 the truncation error should approach to 0. So, this is happening here; so, that means, all the term in the truncation error should be multiplied with h or K or their power. So, there should not be any term which is independent of h or K. So, consistency implies here.

So, the explicit scheme what you can say is consistent undoubtedly and the. So, we can say that the order of a FDE is the rate by which the T.E., T.E. decreases as the step size are reduced. So, this is the consistency this is the order. So, order gives the measure that how accurate the solution will be and the consistency is very important.

Consistency is to know that if I make the step size smaller and smaller whether the numericals the truncation error should tends to 0. So, it may be first order second order, but what we need is the truncation error should tends to 0.

Now, another important aspect is the stability. Now, stability what it happening is that every time we are solving so, we are kind of say from the time level t n we are going to t n plus 1. So, that means, you can say there is a black box; so, your input is u n j and your output is u n rather u n plus 1 j. So, some operation is taking place in between. Now, in this operation there are several approximation or a similar process goes on one of the one is the round-off error. So, that means, what we have to do that we have to have an infinite decimal places to be truncated or chopped out to a finite number of decimal places.

So, in that process a error is committed even other errors if I do not allow to creep into the situation. So, now, stability means what we have to check is that the error which is somehow kept into the system at time level say n; so, whether that error grows in the subsequent time or it remains bounded or decay. So, if it is remain the same or it decades then we call the scheme is stable, but if the method is such that the error which is inherited at a time level say n-th time level and what I find that it is keep on growing. So, then in that case it will be unstable.

So, one simple way to check that whether the error the stability is the von Neumann analysis that I will quick just introduce now; so, basically in a.

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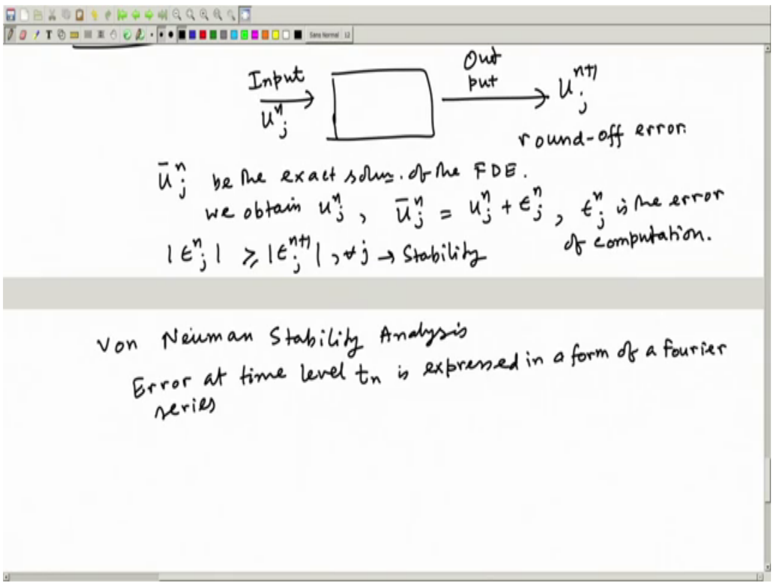

[FL] So, we were talking about how to check the stability of a numerical scheme. So, even if I take all the precautions, but we may come across the error like round-off error; that means, a reducing a infinite decimal number to a finite number of decimal places. So, so that means, if the say u bar n j be the exact solution of the numerical scheme of the numerical method or of the FDE say which will never get and we are getting we obtained u n j instinct. So, that means, there is an error we obtain u n j.

So, there is an error; so, you can write u n j equal to u n j plus some error epsilon n j where epsilon n j is the error of computation error of computation and we assume that this is quite small because other errors are not considered in the in this case in this situation. So, now this u bar n j satisfy the numerical scheme and this is a linear. So, what I do this epsilon n j is a discrete distribution. So, the error at time level say t n is approximated is represented ok; now, before that this is the von Neumann stability analysis, but before that stability.

So, what you need for stability is that this epsilon n j should be less than rather should be epsilon n plus 1 j for all j. So, this is the thing is required. Now, now we talked about how to do that one how to check that for stability. So, this is the way I can define the stability. Now, the von Neumann stability analysis there what I do the error at time level say t n is expressed in a form of Fourier series Fourier series because there is a discrete distribution and it is a finite.

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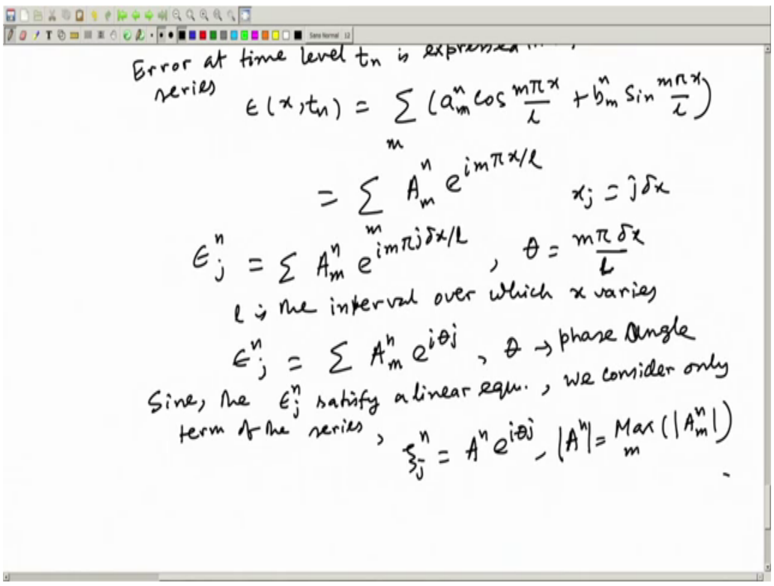

So, integrable; so, I can write the Fourier series distribution say f x n, t n is equal to say sigma a n cos m pi x by m pi x by l say and b n b m sin m pi x by l; so, m summation. So, this can be written as a m A. So, this is of course, because of the time level; so, these coefficients should A m e to the power i m pi x by l. So, epsilon n j; so, this can be written as A n m e to the power i m pi x j.

So, x j is nothing, but j delta x; so, j delta x by l. So, if I say this theta which is the constant m pi delta x by l this is l; l is the interval over which l is the interval over which x varies. So, in that case I can write epsilon n j equal to sigma A n m e to the power i theta j; i is the imaginary number theta is called the phase angle.

Now, since the FDE since the epsilon n j satisfy a linear equation. So, we can consider only one term consider only one term of this Fourier series of the series. The summation can be taken out and let we call this term as zeta n j n A n e to the power i theta j. Now, we taken A n which is the maximum over m of all this amplitude whoever has the maximum amplitude I call that is as that is denoted as A n; so, for some value of m, if it is happening.

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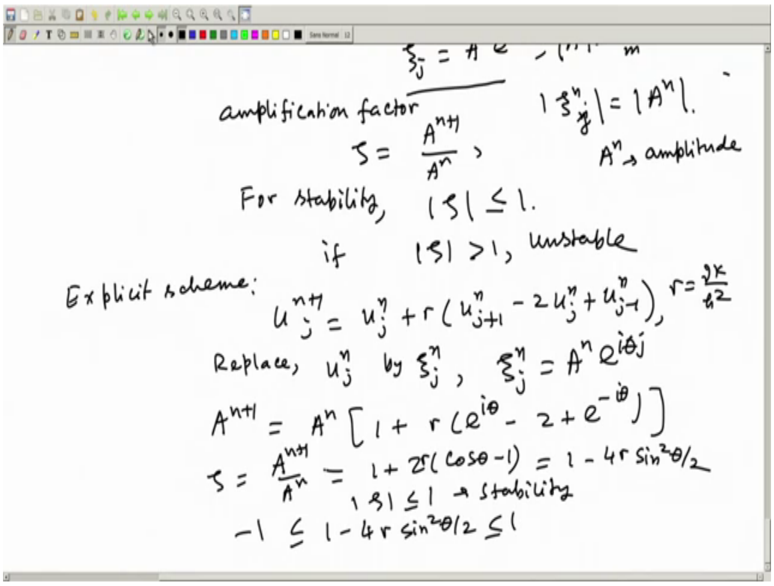

So, for stability we define another amplification factor that is I call as zeta is A n plus 1 by A n. Now, for stability what we require mod zeta should be less than equal to 1. So, if mod zeta is greater than 1 unstable. Now, obviously, the mod of xi n j is nothing, but xi n j is nothing, but mod of A n. So, this is A n is the amplitude. So, amplitude and theta is the phase angle. So, this xi n j represents a wave, n is the amplitude; it is a complex number, it can be complex number.

So, what we require that the amplitude should reduce the magnitude or modulus of the amplitude should reduce as time progresses. So, this is the procedure. So, that means, to check the stability we expand by error satisfy the same equation as the FDE and the component of the error one component we expressed by Fourier series. So, one component we are considering is zeta xi n j and we have to find out the zeta whether zeta is reducing mod of zeta less than 1.

So, for the explicit scheme or I can write as this was the explicit scheme even j plus r u n j plus 1 minus 2 u n j plus u n j minus 1. So, r was nu K by del h square. So, epsilon j also so, we replace because the same equation is satisfied by the error by xi n j and xi n j we are writing as A n e to the power i theta j. So, what I get is A n plus 1 equal to A n and into 1 plus r e to the power i theta minus 2 plus e to the power minus i theta.

Now, what we need to find out? So, zeta equal to A n plus 1 by A n and that is given by this is equal to 1 plus 2 into cos theta minus 1 2 into r r is multiplied. So, this is equal to cos theta minus 1 is equal to so, 1 minus 4 r sin square theta by 2. Now, mod zeta less than 1 for stability; so, that means, what we require is minus 1 less than equal to 4 r sin square theta by 2 less than equal to 1 whether that is happening.

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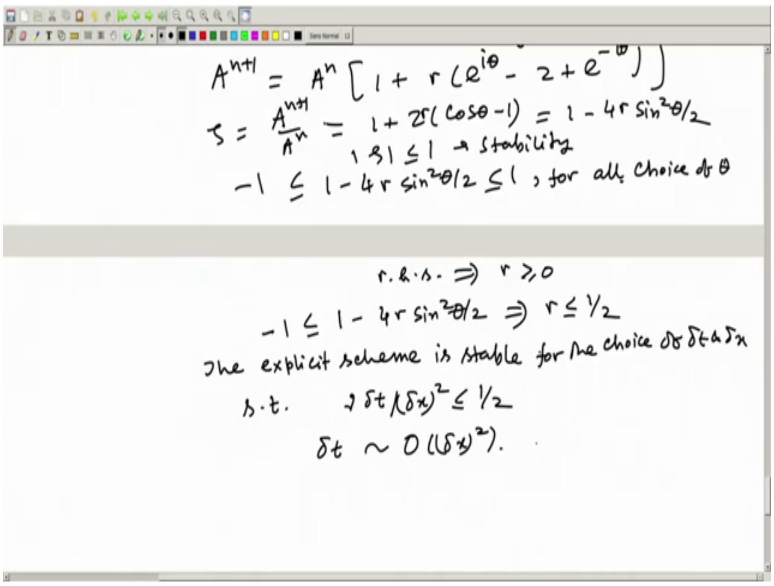

So, for any choice of, for any real theta any choice of theta or for all choice of theta; now, what I find these the this inequality the RHS implies r should be greater than 0, that is enough. r is greater than 0 and the lhs this gives 1 less than equal to 1 minus 4 r sin square theta by 2, this implies that 4 r sin square theta by 2 should be less than equal to half; so, that means, r should be less than equal to half.

So, this implies that the explicit scheme is stable for the choice of delta t and delta x such that such that what is nu delta t by delta x square should be less than equal to half; so, that means, delta t should be of order delta x square. So, this is a that means, delta t is has to be choose very very small if it is a explicit scheme.

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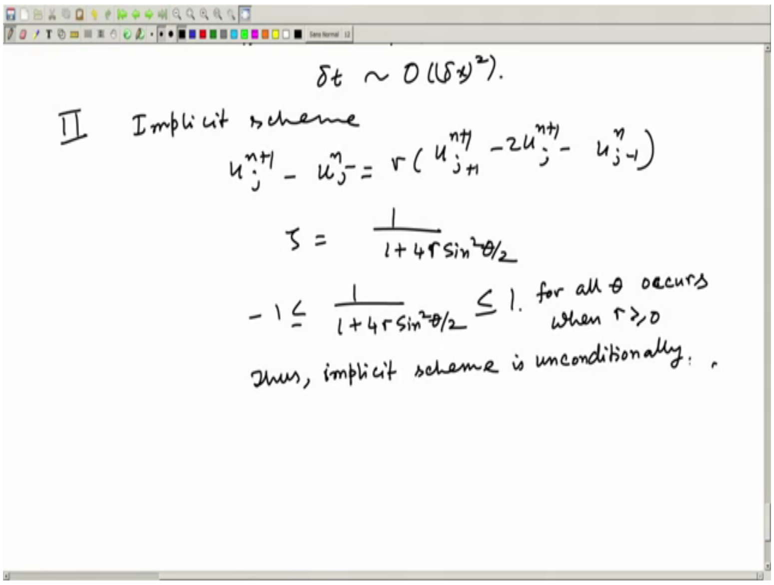

Now, either implicit scheme, the same procedure if I apply for the implicit scheme what I get this is the u n plus 1 j minus u n j the same way r into u n plus 1 j plus 1 u n plus 1 j minus plus 1 j minus 1. So, zeta comes out to be 1 plus 4 r sin square theta by 2. And, so, this is should be 1 plus 4 r sin square theta by 2 less than equal to 1 and this is happening for all theta occurs or any choice when r greater than equal to 0.

Thus, implicit scheme is because this is always a positive and this is a 1 plus a positive quantity it is less than 1. So, thus the implicit scheme is unconditionally stable. So, this gives an advantage that any choice of delta t delta x can be possible. Now, for the explicit scheme that condition for stability impose the restriction that the time step has to be very

very small very refined; so, that means, the normally this diffusion process is a long time process. So, you have to have a large computation.

So, we stop this lecture now.