Mathematical Methods For Boundary Value Problem

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Lecture - 18
Crank- Nicolson Scheme Implicit Scheme

So, we will start today for a computing partial differential equations that is boundary value problem, which is provided or described in terms of partial differential equation. Now partial differential equation means here the unknown variable is function of more than one independent variables.

So, we start with the most simple one that is the parabolic PDE which is the most simplest of the type, which is we can call as an initial boundary value problem to describe the parabolic PDE.

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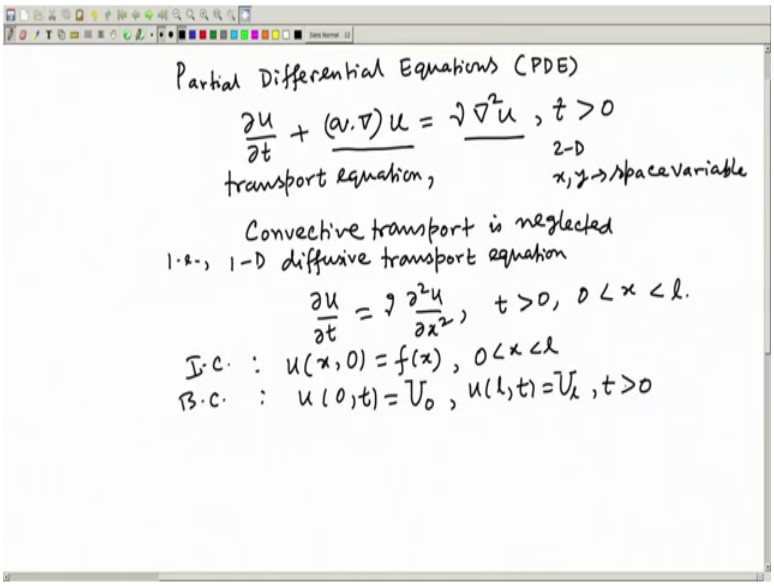

We need to prescribe an initial condition and two boundary condition with respect to the special variable. So, what we have is, so, if we have the. So this is the partial differential equation PDE. So, any kind of transport equation which is governed by say del u del t plus some q dot grad of say u equal to some nu del 2 u. So, this is a transport equation. So, transport equation which is governed by both the convection as well as diffusion. Now to this is a second order in the terms of differential equation is a second order initial boundary value problem. So, we have t greater than equal to 0 and suppose coordinate say, if it is a 2 dimension if we call 2D.

So, we have the xy or xy or the space variable. So, the conditions are given on the boundary of the special domain. Now simple one so this is the general transport equation. So, we have the 2 mechanisms diffusion and convection. Now if I want to go for a simple situation so; that means, we are considering only the space variable a single 1 dimensional transport and if we assume that for the time being that no convection convective transported absent convective transport is neglected.

So; that means, whatever the transport is occurring there is only were diffusion that is the nu del 2 u term and if it is a that is a 1 d diffusion or diffusive transport if we call. So, in that case we can write this equation as del u del t we are neglecting the second term. So, this is the one equal to del 2 u del x 2 say t greater than 0 and 0 less than x less than l is the domain; that means, x is varying between 0 to l and so u is the variable which is a function of x and t.

Now, we have the initial condition I c; that means, u is given for t equal to 0. So, u is given for t equal to 0 so; that means, u 0 is given to be some f x for 0 less than x less than l and B c so; that means, at the 2 boundary that is x equal to 0 and x equal to l this value of u is prescribed so; that means, what we have given is u prescribed at the two end values of x so; that means, you have been provided with 0 t equal to some say U 0 value and u l t which is given to be some U l for any time t greater than 0.

So, this is the a pure transport which is governed purely by the diffusion mechanism. Now this can be also represents as we discussed before a heat conduction equation also; that means, you have a rod of length l whose to end temperatures are provided and initially it was heated with a certain temperature f x then the temperature distribution that is temperature at any point at any time is given by followed this equation.

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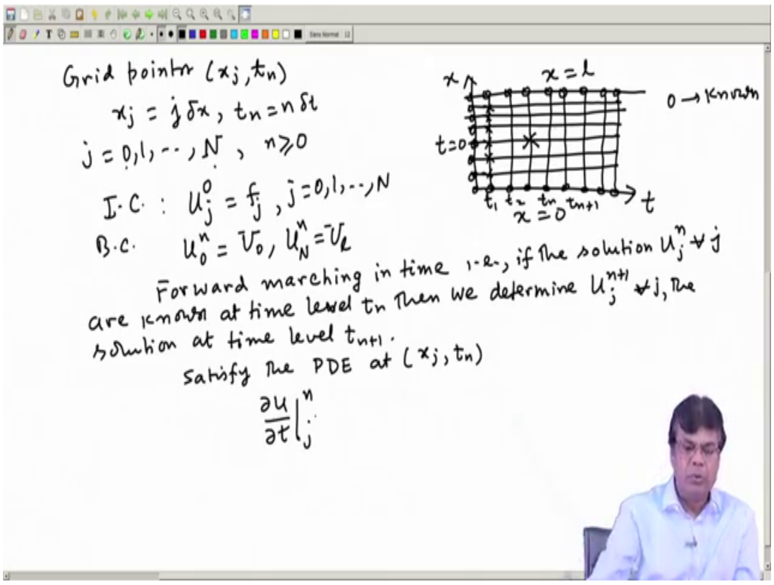

So, what we have now in the coordinate system, if I consider this is the say this coordinate is x and this is t? So, what we have given is this is x equal to 0 line and this is x equal to say l line? So, over x equal to 0 and x equal to l u is prescribed and this is the t equal to 0 line. So, along this line this value of u is prescribed. So, we need to find out the value at in between points in this semi infinite region because we do not have the other end for t. So, this is a semi infinite domain in that sense.

So, as usual what we do is first we choose the grid point that is at those points where we want to compute the solution should I define certain grid points. So, grid points are defined like this way say we call as x j t n are the grid points, grid points x j t n. So, x j we call we can provide by this j delta x and t n equal to n delta t a step size delta x and delta t are choosed and so, j will be varying from 0, 1, to say N. These are zero and n at the two boundaries and small n is any number any integer n greater than 0.

So; that means, basically what we did is we are considering the this is the time levels at t equal to 0, this is t 1, this is t 2 like this way it is going. So, at certain stage t n this is t n plus 1 etcetera. This is the way it is proceeding and x wise here horizontal line we are taking like this way. So, we have this grid point distribution.

So, basically what we are looking for the solution. So, what we have been provided? Now initial conditions now can be translated is this way that u 0 j is given to be f j for j equal to 0, 1 up to N and B c can be translated this manner u at any time n 0 is given to be u 0 and u n capital N is given to be U l.

So; that means, this points this encircle points are all given these values are given. All these things all the points which are lying on the boundary the solutions is known. So, at all these points we know the solution. So, you have to find out an intermediate point. Any intermediate points we need to find out the solution. So, in circle this points known. So, wherever the solution is known we are designated by the circle symbol.

Now, what we do? We adopt a strategy a forward marching in time strategy; that means, what we do say to start with we know the solution at t 0 so; that means, along this line use that we were first find out the solution at t 1. Next step then once I obtain the solution at all this points t 1 for all j then we use this t 1 go to t 2 and then t 3 like that way a forward marching in time procedure is adopted, forward marching in time in time that is if time level t n well I think I should write in this way. If the solution u n j for all j are known at time level t n.

Then we determine u n plus 1 j for all j the solution at time level t n plus 1. So, to do that what we do because this differential equation the PDE is satisfied at every point or within this domain. So, satisfy the PDE the PDE at the grid point x j, t n. So; that means, what I did is del u del t were writing at n j we put the superscript as n superscript as n to designate as time and this is del 2 u del x 2 n j and nu is the.

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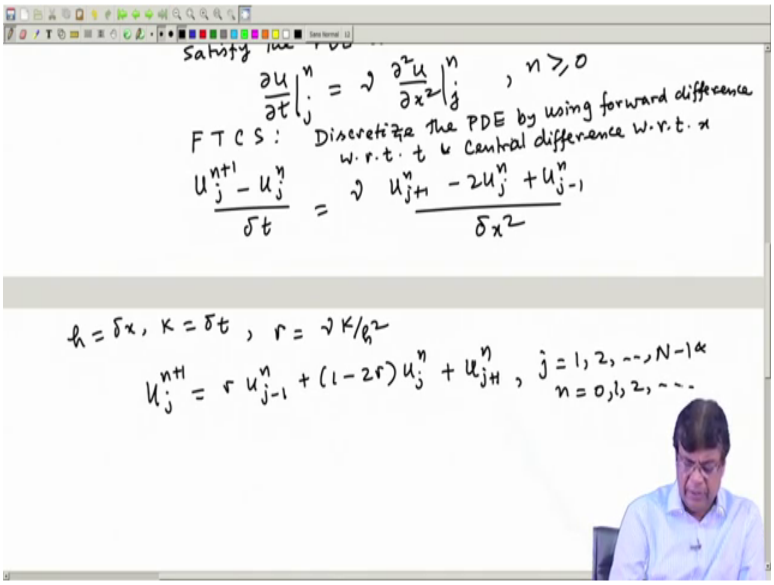

If we talk about the temperature, heat transport then nu is the conductivity of the metal and this kind of physical quantities can be specified or can be defined for this nu.

So, PDE is satisfied at the time level n and j. Now n can be any number, n is n in an integer; that means, n can be greater than 0, it can be if we start we can take n equal to 0. Now what we apply is forward time central space; that means, this is the one discretize the PDE by using forward difference with respect to t and central difference with respect to x.

Now, this t may not be all the time as type, it can be any other variable also, it can be any other time like variable, but thing is that we should have a first order derivative in one variable that is t and one single condition is given. So, if I now apply FTCS. So, forward difference so; that means, u 1 plus 1 j minus u n j by delta t equal to nu u n j plus 1 minus 2 u n j plus u n j minus 1 by delta x square. So, this is the central difference we have used for the x coordinate and first order for our difference in time coordinate.

Now, what we have assumed that we know the solution t n time level n; that means, u n is known for all j we need to find out the even plus 1 j. So, this equation which I can write if I define a so let us assume instead of do not instead of writing as a big letter delta x we write as h and K for delta t and this r let us call this is r equal to nu k by h square. If I denote by a parameter nu r given by this way. So, then the unknown term u n plus 1 j unknown variable can be directly written in terms of all the known quantities.

So, this is r u n j plus 1 or let us write in this way customary is u n j minus 1 and this is u n j is going to other side; so, 1 minus 2 r u n j plus u n j plus 1. So, j is varying from 1 2 up to N minus 1 and n equal to 0, 1 2, etcetera. So, this is the a numerical scheme which is referred as explicit scheme, explicit scheme because as the unknown u n plus 1 j is expressed explicitly in terms of known quantities, known quantities u n j u n j plus 1 u n j plus u n j minus 1 etcetera all with superscript n.

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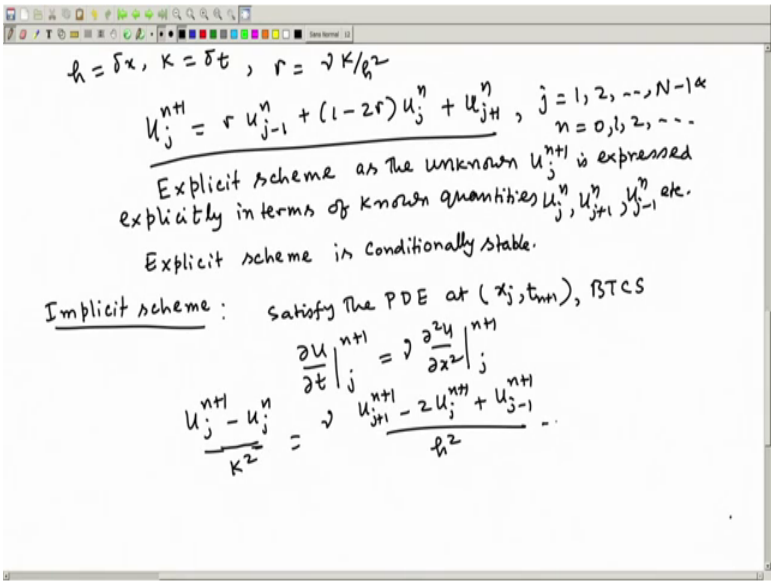

So, this is our known quantity. So, he could express the unknown even plus 1 j in terms of all the known quantities. So, that is why this is called the explicit discretization. Now this explicit discretization or explicit scheme and; obviously, the way we have discretized it shows that it will be a because we have taken the first order approximation for time and second order for a central difference for x. So, it will be a first order in a time and second order in x. Now this a explicit scheme has drawbacks.

So, that we will discuss what for the timing let us keep it in mind that explicit scheme though it is very simple to use is conditionally stable, but this it can become unstable as well. So, that is why we have to go for little a better version or better approximation. So, another way also what we can do is that we satisfy another way is that is called an implicit scheme, implicit scheme.

So, if we satisfy the satisfy the PDE at x j t n plus 1 rather x j, t n plus 1 and what we do is and apply BTCS, Backward Time Borrower Central Space. So; that means, what we did del u del t we are giving as n plus 1 j equal to nu del 2 u del x 2 n plus 1 j. So, if I now give the backward difference in n in time. So, this gives u n plus 1 j minus u n j by delta t equal to nu u n plus 1 j minus 2 n plus 1 j.

Sorry n plus 1 j plus one j n plus 1 j minus 1 by h square let us not write. So, this is K square and that is it.

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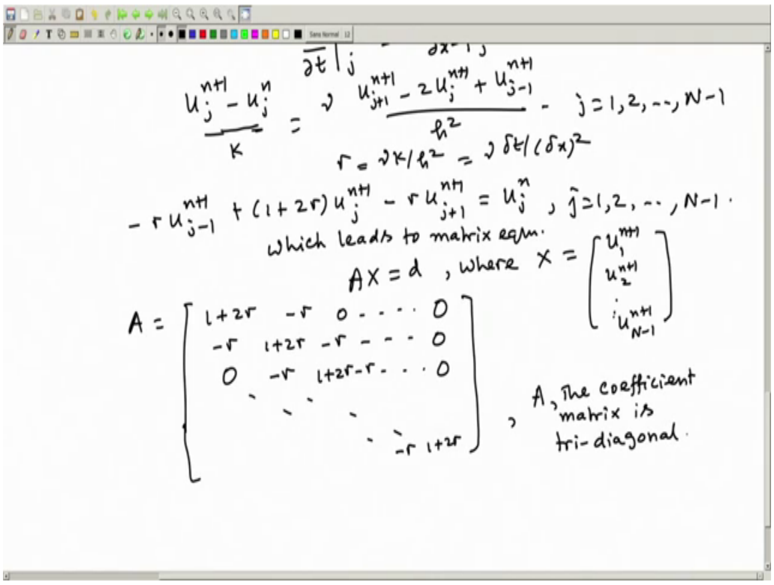

So, when j is from 1, 2, n minus 1 because our j equal to 0 and n is given. So, with that what we are finding is how many equations we have here n minus 1 equations and one difficulty here that u n plus 1 is appearing in both side. So, this is not a one where we have we can express the unknown directly in terms or explicitly in terms of all the known quantities.

So, if I introduce that r equal to nu sorry this is not k square this is k only. So, nu k by h square basically there is a nu delta t by delta x whole square if we recall that one. So, if I introduce that r so, what we get is and r u n plus 1 j minus 1 and then next term u n j u n correspond j. So, that will be minus 2 r and so plus 1 minus 2 r un plus 1 j plus r u n plus 1 j plus 1 equal to u n plus equal to oh I am sorry this is if I write in this way. So, this is u n j hm. So, this is the one.

So, plus and outside is minus. So, now, if I take this minus so; that means, this is plus and this is minus. So, this is forming a matrix equation. Now so, if I vary j from 1, 2 up to N minus 1, so this is resulting to a tri diagonal matrix system. So, so if I put if I put our j equal to 1. So, I know this j minus 1 this solution is known. So, that can be taken to the other side. So, we have u n plus 1 1 and u n plus 1 2, similarly the second and third etcetera.

So, minus 2 r minus r now which leads to a matrix equation A X equal to say d where X is the all the solutions to be obtained u n plus 1 u n plus 1 2 etcetera, u n plus 1 n minus 1 which need to be obtained and A is a tri diagonal matrix.

So, A is first element is 1 plus 2 r minus r 0 0 0 0 all are 0, next is minus r 1 plus 2 r minus r remaining entries are all 0. So, minus r 1 plus 2 r minus r etcetera. So, I get a tri diagonal system as we have seen for the central difference approximation. So, the last row will be minus r 1 plus 2 r. So, this is the formation it is creating. So that means, A is the coefficient matrix A the coefficient matrix is tri diagonal, tri diagonal.

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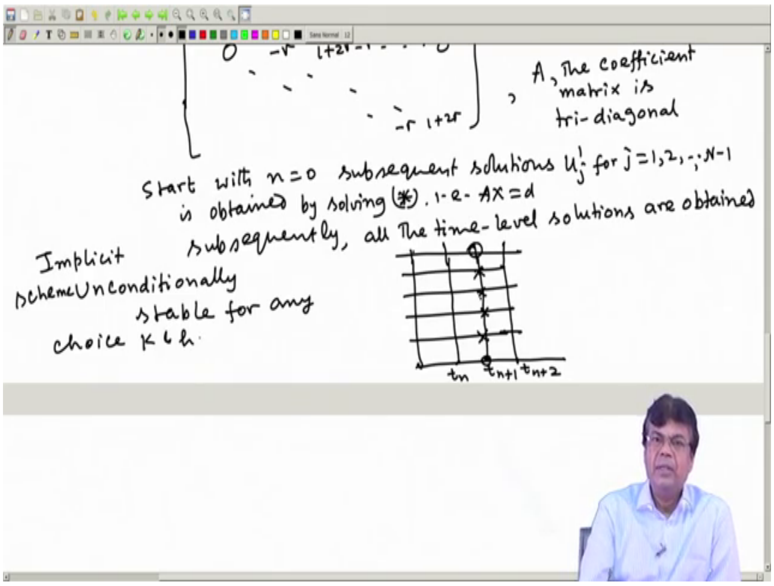

So; that means, what you have to do? So, at every time, so start with n equal to 0 and subsequent time solution subsequent solutions u 1 j, u 2 j, u 1 j for all for j equal to 1, 2, N minus 1 and then is obtained by solving the system A X equal to d by solving this system of equations. So, let us call this system of equation as star this one we call as a by solving star that is A X equal to d, A X equal to d and subsequently all the time level solutions are obtained whatever the desired solutions is required.

So, in other words what we did is here see if I draw it here. So, say this is at t n which the solution was known and we want to find out that t n plus 1. So, this t n can miss the starting one that is t equal to 0 and all these j level these are the j level. So, what we did is using this t n we are finding. So, this two these are the boundary and remaining one forming all this j equal to 1, 2, 3 n minus 1. They are solving these tri diagonal system X equal to d.

So, we invert this A X equal to d either by the Thomas algorithm or whatever. So, we get the solution as the time level n plus 1. Once I obtained the solution then my next task will be to get the solution at t n plus 2. So; that means, a forward marching step by step forward marching in time procedure is adopted. Now so far it looks like this is implicit scheme which will show that it is unconditionally stable so; that means, you have the freedom to choose any implicit scheme implicit scheme unconditionally stable.

So; that means, I can choose any for any choice of delta h for any choice of K and h, K and h. But so far what we did is first order in time. We could not come out of the first order discretization in time.

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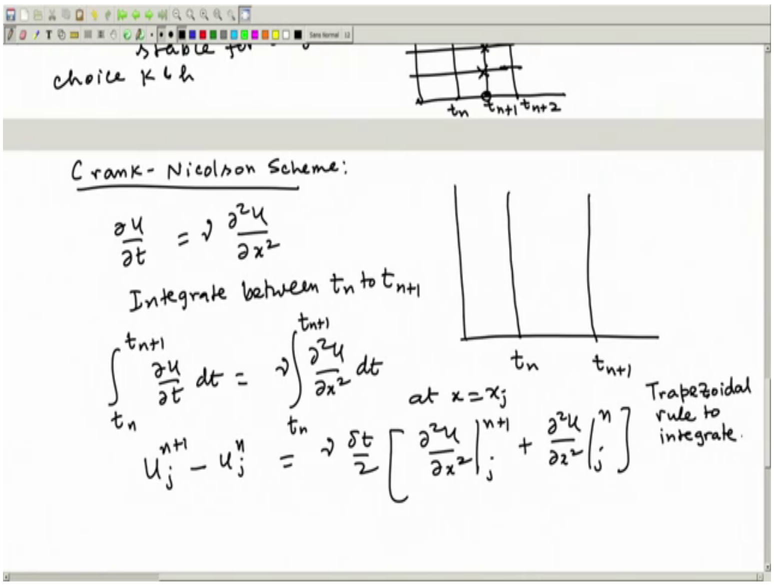

Now, the most popular one is the one which is the Crank-Nicolson Scheme. So, in the Crank-Nicolson Scheme what is done is again it is also a forward marching time so; that means, you are at time level t n say and you would like to find out the solution at t n plus 1. So, what they do you have the equation del u del t equal to nu del 2 u del x 2 integrate between t n to t n plus 1 with respect to t.

So, this is t n plus 1 del u del t d t equal to nu del 2 u del x 2 d t t n plus t n 2 t n plus 1 ok. So, this one I can write very easily u n plus 1. So, at some say at x equal to x j at some point x equal to x j. So, n plus 1 j minus u n j and here what we do is we apply trapezoidal formula. So that means, nu so the step size is delta t by 2 so; that means, this is del 2 u del x 2 this is at n plus 1 j plus del 2 u del x 2 at n j. This is the trapezoidal rule is applied, trapezoidal rule to integrate.

So, now what we do all this space derivative.

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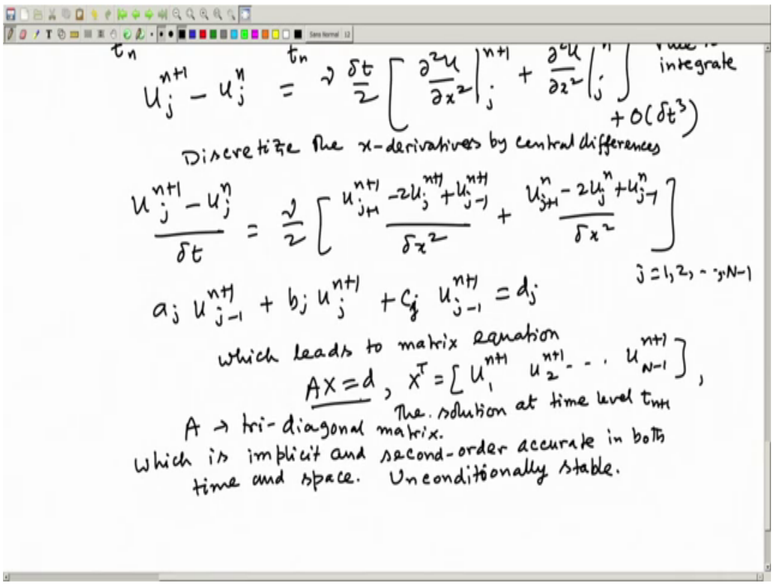

Now, since we are applying the trapezoidal rule, so it will be of order delta t cube with respect to t. We have not done anything with respect to x right. Now up to that now we will do is discretize centrally with respect to x. So, discretize the x derivatives by central difference scheme. So, what I get is u n plus 1 j minus u n j by delta t, if i divide it this way the nu by 2 u n plus 1 j plus 1 minus 2 u n plus 1 j plus u n plus 1 j minus 1 by delta x square plus this is in the, this is the implicit term because this is at the current time level and this is the explicit terms that is the known time level.

So, what appears is it will be a second order in both time and space. So, at the fix j a fixed in j is varying from 1 to N minus 1. So, which can be expressed in a form as a i not a I, a j u n plus 1 j minus 1 plus b j u n plus 1 j plus c j u n plus 1 j minus 1 equal to d j. So, again it will lead to a tri-diagonal system because all the known are superscript with n are known quantities, which is transferred to the other side so which leads to a matrix equation. Again I can write as a x equal to d x is the x transpose I can write as instead of writing column wise I can write this u 1 u 1, 2 etcetera.

U n plus 1 n minus 1 solution at time level t n plus 1. So, this is the one is the A is the tri-diagonal matrix. So, this is the most popular one because first of all it is implicit. So, which is an implicit and order delta t square and second order in both time and space. So, for this the Crank-Nicolson Scheme is most popular one to handle because implicit. So it is unconditionally stable; that means, any choice of step size is permissible. No restriction for the choice of h or k delta t or delta x.

. So, again the same procedure; that means, you start with n equal to 0 and the initial condition then you solve this by solving the by inverting the tri-diagonal system AX equal to d. You get the e 1 j for all j j equal to 1 to n minus 1. Once I got u 1 j then I repeat the same process; that means, u n j 2.

Now, you go to second time level u 2 j and subsequently all the time level the required one you need to find out. So, now in the next class we will talk about the stability. What do you mean by stability and the error? How to compare the what is the advantage of Crank-Nicolson Scheme? We have described which is little complicated compared to the first order implicit scheme. So, we know make a characterization of this scheme ok.

Thanks.