Mathematical Methods For Boundary Value Problem
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Lecture - 17
Control Volume Formulation

So, in continuation with the non-linear boundary value problem; now in several situations we come across of non-linear boundary value problem. For example, say boundary layer theory; similarity transformation of boundary layer theory. Say if we consider the development of boundary layer over a flat plate.

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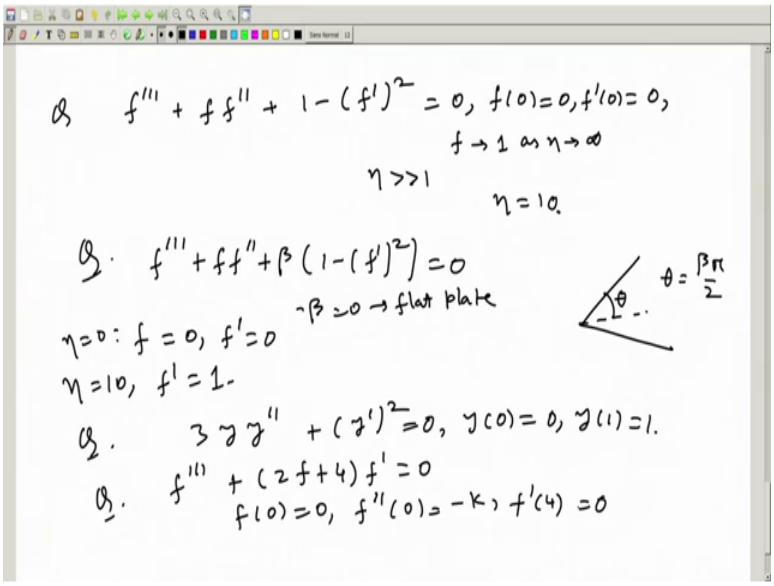

So, in that case the similarity transformation gives the a boundary layer boundary value problem like this. F f double dash sorry this is f dash ok, f double dash plus 1 minus f dash is squared equal to 0; f 0 equal to 0 and f dash 0; basically f dash 0 is a stream wise velocity component. So, that is 0 only say a flat plate and it is tending to 1 as say eta is the variable tends to infinity. So that means, at the edge of the boundary layer.

Now of course, we cannot handle a infinity in a numerical situation, numerical scheme. So, what you have to do is we have to assume that infinity means it is the large, so that means, eta is quite large. So, there are some a few run we have to execution one has to make and show that it is approached in an asymptotic fashion. So that means, the approximate value for f approach 1 in an asymptotic fashion.

So, that is the idea to handle the infinity boundary condition. So, if you feel that it is complicated to think about that manner. So, let us take eta equal to 10, a numerical value. It is a no harm in taking this. Now, or also say we can have a situation more generalized say flow past a wage. So, if we have a wage of same wage angle as theta. So, in that case also this can also be written and what is termed as the (Refer Time: 02:45) skun equation.

So, that can be expressed in this form, it will be in terms of a parameter beta. So, this is f triple dash plus f f double dash plus beta into 1 minus f dash is square equal to 0. So, this beta is nothing but or rather this theta. Theta is the wedge angle beta pi by 2, so, half wedge angle. So, if beta equal to 0, this is your the one is the flat plate situation. The same boundary condition if at eta equal to 0. So, independent variable I call as eta. Eta equal to 0 f equal to 0, f dash equal to 0; eta equal to say 10, f dash equal to 1 ok.

So, this kind of boundary value problem, we always come across in many situations say for example, to analyze the as I said fluid flow problems or also the in heat transfer or in bending moment of a bar or something. So, this kind of problems we the non-linear boundary value problem is a common occurrence. So, y is 0 equal to 0; y 1 equal to 1. Say another problem, boundary value problem which is non-linear say flow.

So, if you have a orifice kind of thing. So, the jet the strength of the jet which is coming out of the orifice can be governed by this f dash which solved this equation. So, f 0 equal to 0 and f double dash 0 equal to minus k some flux and f dash eta equal to 0 or rather not eta say some value f dash 4 equal to 0. So, this kind of non-linear problems.

So, as we talked about the Newton Linearization Technique. So, there what we do is first we discretise the equation and then, we start solving the set of non-linear algebraic equation by the iteratively by the Newton's linearization technique. So, this is one way of handling the Newton's linearization technique or non-linear BVP. Say, if I illustrate this with an example say a simpler example that is take this one.

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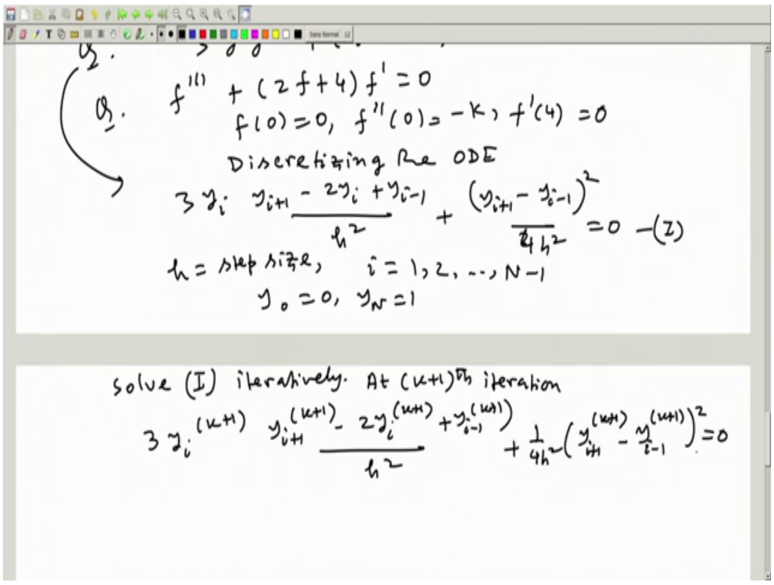

So, this one. So, what I do is first at the discretization. Discretizing the ODE, we get 3 y i y i plus 1 minus 2 y i plus y i minus 1 by h square plus y i plus 1 minus y i minus 1 whole square by 4 h square equal to 0 and i is i varies from great point if I choose h say some value step size is chosed. So, accordingly i will be from 1, 2 etc N minus 1 and what we have given his y 0 is 0 and y n is 1. So, I solve this. This is a non-linear equation say 1. So, solve I system of non-linear equation iteratively.

So, at the at k plus 1 iteration, what we have is 3 y i k plus 1 y i plus 1 k plus 1 minus 2 y i k plus 1 plus y i minus 1 k plus 1 by h square by 4 plus 4 h square y i plus 1 k plus 1 minus y i minus 1 y i minus 1 k plus 1 whole square equal to 0.

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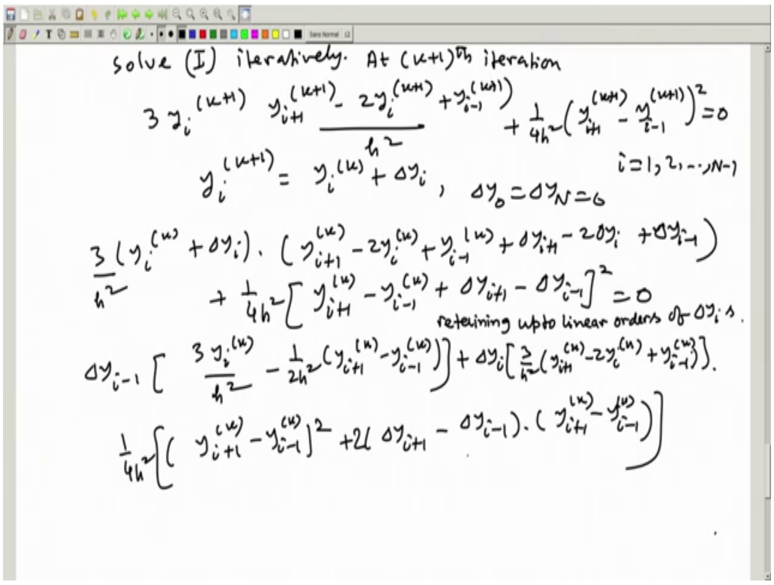

Now, what we will do is will approximate y i k plus 1 with the this is the recurrence relation delta y i; i is from 1, 2, N minus 1 and delta y is 0 equal to delta y n equal to 0 if I now substitute this. So, and retaining only the linear terms. So, what I get is 3 y i k plus delta y i in to 3 by h square into y i plus 1 k minus 2 y i k plus y minus 1 k plus delta y plus 1 turns as 2 delta y plus a delta y i minus 1.

Now, 1 by 4 h square. Now, if I write this 1 y i plus 1 k minus y i minus 1 k plus delta y i plus 1 y i minus 1. This is whole square equal to 0. Now, so only unknowns are delta y i's. All these superscript with k are known. So, all the known terms can be sent to the other side. So, now, if I want to collect the coefficient of delta y i minus 1, so what will be the coefficient from here? Delta y minus 1 will be 3; 3 y i k by h square.

So, this is the only one term and this is neglected. Delta y i into delta y i minus 1 is neglected. So, dropping only retaining only up to up retaining up to linear orders linear orders of delta y i. So, this is the one and from here. Now, here if I do this a plus b whole square business. So, what will be I can write this 1 y i plus 1. So, this will be ok. If I write it this way y i plus 1 minus y i minus 1 whole square and these b square is out. So, what we have is delta y i plus 1 minus delta y i minus 1 into y i plus 1 minus y i minus 1. This is k k k k. This is no roll over here.

So, all divided by 4 h square. So, and the other term that is a square plus 2 a b of course, this is 2 a b plus b square; b square is out. So, from here if I do this take the coefficient of delta y i minus 1 that will be minus 2 by 4 h square into this. So, minus 1 by 2 h square y i plus 1 k minus y i minus 1 k, so, this is delta y i. Then, delta y i into delta y i. So, delta y i is a big term is coming from here. This is 3 by h square y i plus 1 k minus 2 y i k plus y i minus 1 k. No extra term or no other terms; y i minus 1 k, no other terms are coming.

But for this term delta y i plus 1, this will be same as the delta y i 1; only thing is that it will be plus. So, we can now remove these algebra. So, what I get here is the. So, if I now remove this one.

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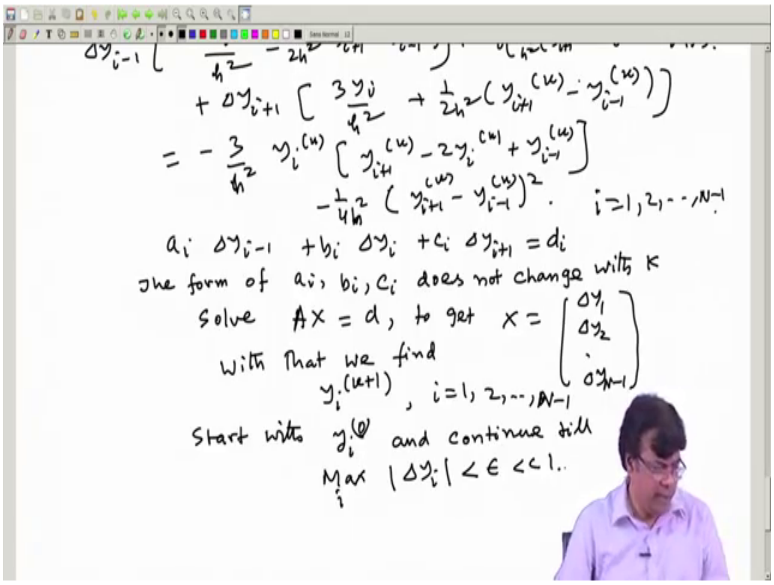

And then, if I write the only the term delta y i plus delta y i plus 1. So, delta y plus 1, one term is coming from here that is 3 yi by h square, 3 y i by h square because this 2 is out and plus 1 by 2 h square y i plus 1 k minus y i minus 1 k and all this thing this is the unknown terms and remaining all are taken out to the right side.

So, this is y i k this can be all are superscript k. So, and this can be a non-linear it does not count because this is all known quantity. So, this is 1 way 4 h square; 4 h square what we have is y i plus 1 minus y i minus 1 k k this is whole square. So, this is i equal to 1 2 n minus 1 this. Set of equations, I can write as a i delta y i minus 1 plus bi delta y plus c i delta y i plus 1 equal to d i; i equal to 1 to minus 1.

Now, one thing to note that the coefficient the function form of a i b i the form of a i, b i, c i does not change with k. So, that means, if i vary when I vary the iteration index. So, what is varying is the y i k values, but the function form remains same. So, once I evaluate these coefficients a i, b i, c i, d i. So, it is once for all for this problem. So, what you have to do is every time you upgrade the value, this y i k and evaluate these coefficients this a i, b i, c i and d i and solve this tridiagonal system.

So, which takes the form. So, solve a x equal to say d to get x equal to delta y 1 delta y 2 delta y m minus 1. So, once I get, so x to get x. Then, with that we find, we find y i k plus 1 for all for i equal to 1 to k minus 1 N minus 1 and so, we continue the iteration start with y i 0 and continue till we get that delta y i for maximum over i is sufficiently small less than 1 quantity. So, now these boundary condition depending on the boundary condition, what the boundary condition was taken is what does a boundary condition was 0 and 1.

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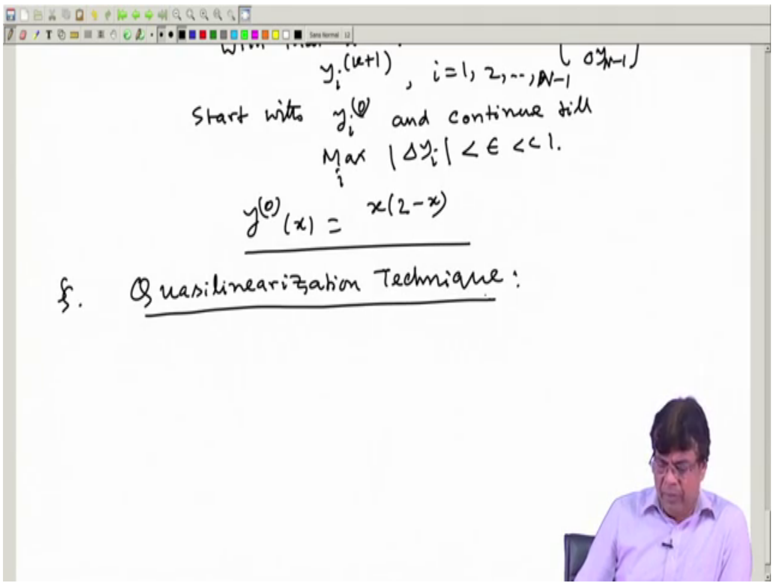

So, I can take this y 0 x equal to say x into 1 minus x. No, no this is not d 1 because x into 2 minus x can be a possibility x into 2 minus x. So, if I put x equal to 0, so it is 0. If I put x equal to 1, so it is 1. This was the boundary condition 2 minus x. So, in that process we need not have to feed in the computer one by one. The for all the grid points, we need not have to go that y 1 y 2 like this. So, if I give this function value y 0 x. So, that is good enough to compute the y i 0 for all the grid points i. So, that is how the Newtonian linearization technique goes.

So, as we talked about the boundary condition should be satisfied the initial approximation. So, then another very powerful method is the quasi linearization technique; quasi linearization technique.

So, in this technique it is also the same variation of the Newton linearization technique, Newton's method again, based on the Taylor series expansion. So, what we do we expect the equation. We do not discretized the non-linear equation rather what we do is at every iteration, we iteratively linearize the non-linear to a approximately no linear form and then solve that linear boundary value problem.

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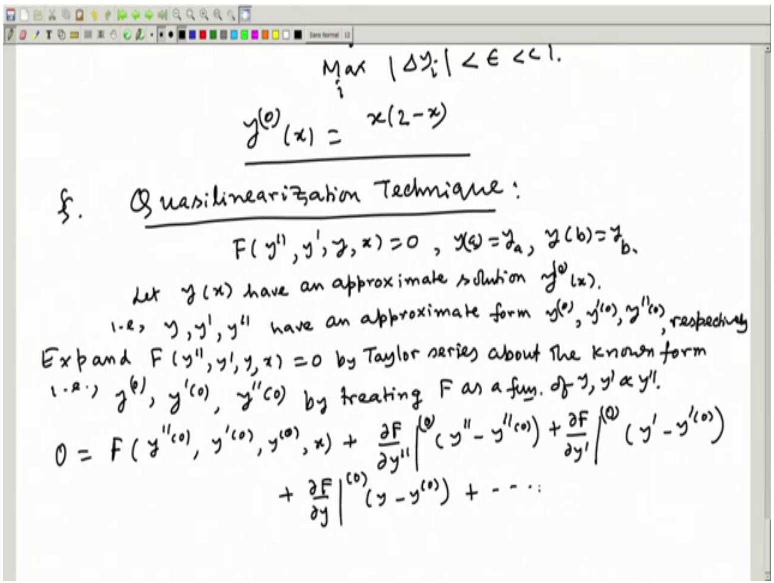

So, our problem is say this is the 1; x equal to 0 with certain boundary conditions say y a is y a y b is y b. Now, let y x have an approximate solution y 0 x y x which is unknown to us which has to be find out. So, let it has an approximate form as y 0 x here x is a variable.

So, that means, we are talking about the function not a discrete value. So, y 0 y x is a function which is unknown to us. So, let a approximate form of y 0 x is y x is y 0 x. So, that is to say that since y 0 x is known. So, y dash, y double dash have an approximate form y 0, y 0, y dash 0, y double dash 0 respectively.

So, what you do? Expand F y double dash y dash y x equal to 0 by Taylor series about the known form, about the known form that is y 0, y dash 0, y double dash 0 about the known form. So, if I do that, what I get is by treating now here by treating F as a function of y, y dash and y double dash; one thing is that to remember that though y, y dash and y double dash are related, but when we are doing the Taylor series expansion, we are treating this F as a function of functions. Function of y, y dash and y double dash. So, that say the partial derivative when you take in the Taylor series expression 3 variables, so that means wherever the these variables or these quantities y, y dash, y double dash appearing explicitly we take a partial derivative of that.

So, we are not considering a relation between y, y dash and y double dash for the Taylor series expansion. So, if we do that. So, what I get is 0. Of course, it is always it remains 0 this side and then, Taylor series expansion we are doing about these 0. Well, I am denoting this way y doubled dash 0 approximate form y 0 x, remain x then. Del F del y double dash, this is evaluated at 0.

An increment is y double dash minus y double dash 0. The next one is del F del y dash at 0; increment is y dash y dash 0 plus del F del y at 0 y y minus y 0 plus all these higher order. Next is second order derivative; third order derivatives and a infinite series will form.

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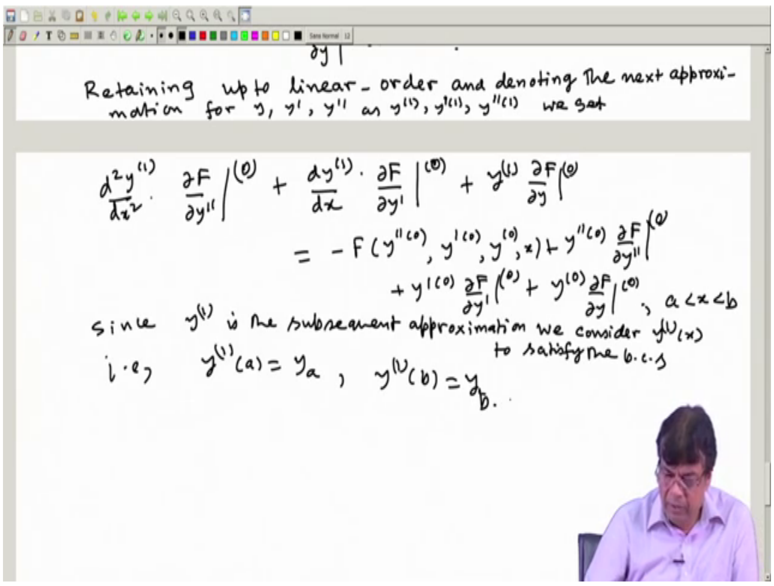

So, we if we drop all the non-linear terms, so retaining up to linear order; so, we always take the most convenient route or most convenient is up to linear; unknown which appears in a linear fashion. So, we can directly write unknown in terms of all the known quantities. So, we have written only the linear order. So obviously, the y, y dash and y double dash which is the solution of the boundary value problem cannot be said that the same will be satisfying the truncated one.

So that means, the solution of the truncated form if I written only up to the linear order. So, that and solve those quantities why this functions y, y dash and y double dash that is not going to be exactly the same as the boundary value problem solution. So, to distinguish it is that we call those are the best approximation, that is the better approximation; only the linear order and denoting the next approximation, denoting the approximation for y, y dash, y double dash as y 1, y y dash 1, y double dash 1 we get.

So, we get now d 2 y 1 d x 2 into del F del y double dash 0 plus so all with 0 super sub superscript 0 are known quantity. So, we transfer to the right hand side. So, the next one is d y 1 d x into del F del y dash 0, then the next one is y 1 del F del y 0. So, this is becoming minus F y double dash 0, y dash 0, y 0 x minus or plus y double dash 0 del F del y double dash. This is 0 plus y dash 0 del F del y dash 0 plus y 0 del F del y 0.

So, this is happening and this is happening for a less than x less than b and since y 1 is the solution of the BVP, is an approximate solution of the BVP is the subsequent approximation of y 0 x. So, we impose this boundary condition. Y 1 a is y 1 is y 1 x satisfy the boundary condition for satisfy the bc is considered to satisfy the bc. We consider y 1 x 2 satisfy the bc's that is y 1 0 a is y a and y 1 b is y b.

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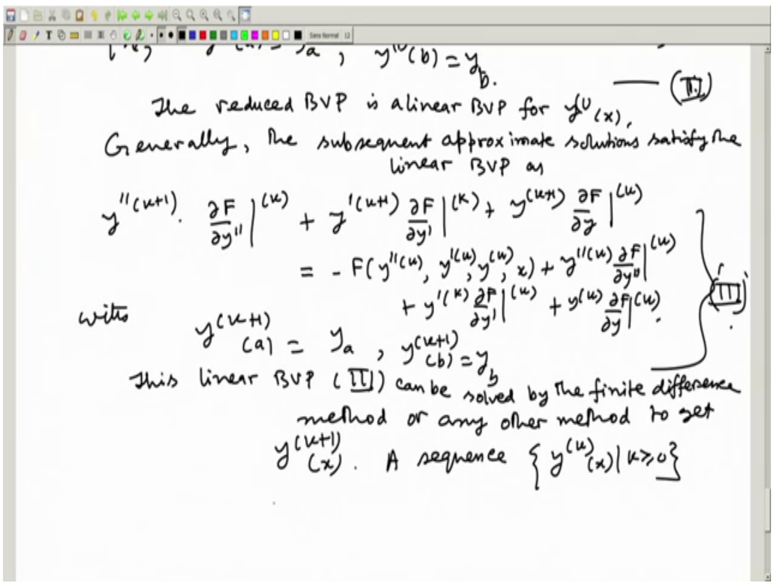

So, what I find that these together or let us call instead of star; let us call this as the reduced form as 2. So, the ODE or I can call the BVP, the reduced BVP is a linear BVP for y 1 x. So, now, by now we are master of solving this how to handle a linear two point boundary value problem. So, if I solve this I get that y 1 x and that is obviously not the exactly the solution for the boundary value problem.

We are looking for. So, we have to go with the subsequent approximation; so that means, in general generally I can say that the general generally the subsequent approximation approximate solution are up satisfy or ok, satisfy the linear BVP as say if I was at the kth iteration and if I am solving for the k plus 1 iteration.

So, y double dash k plus 1 del f del y double dash k plus y dash k plus 1 del F del y dash k plus y k plus 1; this is k del F del y equal to all these things, equal to some known quantity minus F y double dash k, y dash k, y dash k, x plus y double dash k del F del y double dash k plus y dash k del F del y dash k y k del F del y k and with b c y k plus 1 a is y a y k plus 1 b equal to y b.

So, this linear BVP, so let us call this is 3. So, let us call this as the equation 3. As there maybe, 3 can be solved by any method say finite difference method or any other method or any other method to get y i y k blush 1 x.

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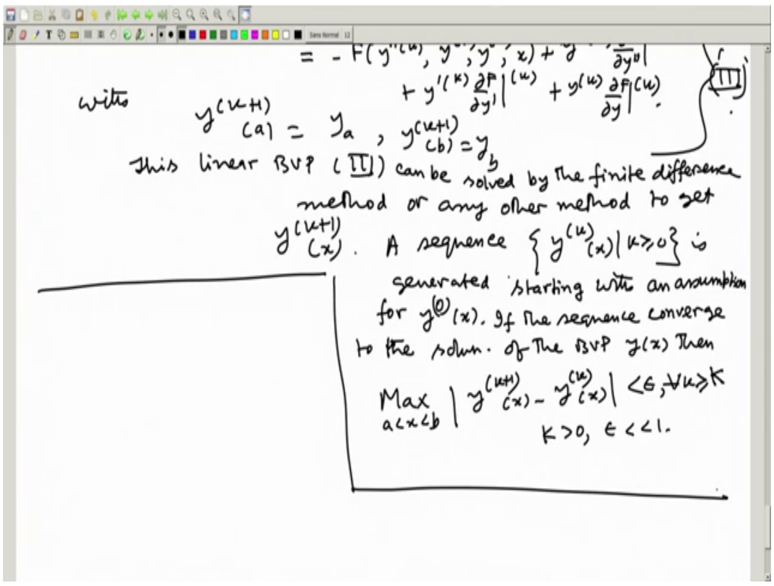

So, now, in this process a sequence y k x k greater than equal to 0 develops a sequence develops is generated starting with, starting with an initial approximation with an assumption for y 0 x this need to be assumed to start the method. So, now the if these sequence converge, if the sequence converge to the solution of the BVP of the BVP y x; then, we must have then we must have this max of a less than x less than b y k plus 1 x minus y k x should be less than epsilon for some for all k greater than equal to some capital K; capital K is greater than 0 and epsilon is a very very small positive number.

So, this is the criteria for convergence. So, this must be satisfied, if the iteration converges. If iteration converts there only this satisfaction appears. So, basically what we did is this stage. So, other things we can just forget about this. So, there is we have a BVP which we have written as F function of y double dash, y dash y and x and then, we have expanded in terms of treating this F as a function of y double dash, y dash and y and then, using a initial approximation y 0 x, we are obtaining the subsequent iterates which solves the BVP 3.

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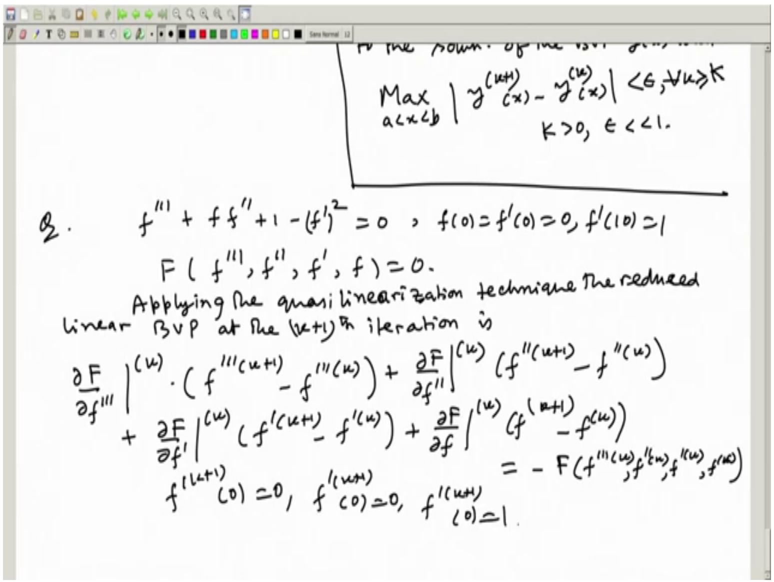

So, let me illustrate with an example. So, if we have a situation the same as flow past a flat plate. So, f triple dash plus f double dash plus 1 minus f dash square equal to 0. So, f 0 let us take f at 0, f dash at 0 is 0 and f dash at say 10 is 1. So, the BVP now it is third order; so, we write in this way. All this argument is 0; eta is does not appear explicitly, no need to write that.

So, now, what we do is we expand by Taylor series apply the quasi linearization technique, some technique the reduced linear BVP at the k plus 1th iteration is given by. So, what you have to do is del f by del f triple dash. This is at k into what you have is f triple dash k plus 1, then del F at del f double dash at k f double dash or this is maybe I was trying to do the next step. So, this may be a little step jump.

So, instead of doing this, if we go in a systematic manner; so this will be del F del f triple dash k into f triple dash k plus 1 minus f triple dash k plus del F del a double dash k f double dash k plus 1 minus f double dash k plus del f del F dash k f dash k plus 1 minus f dash k plus del F del f k f k plus 1 minus f k plus 1 minus f k and this side is minus of the same f a triple dash k f double dash k, f double dash k f k and what we have here the boundary conditions f k plus 1 0 is 0; f dash k plus 1 0 is 0; f dash k plus 1 0 equal to 1. What you get here a; what I get here a linear third order boundary value problem.

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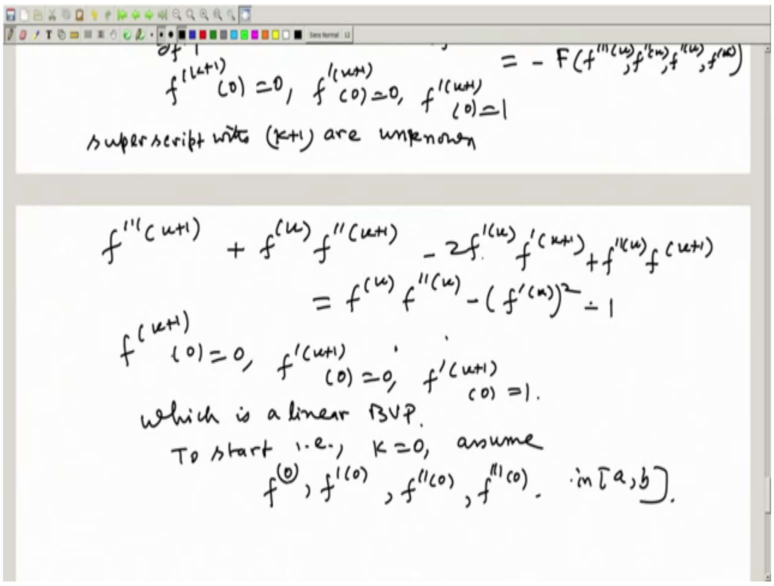

So, if I now write in a only the superscript with k plus 1 are unknown. See if I write only the unknown quantities to this sides. So, what I get is f triple dash k plus 1 plus f k f double dash k plus 1. The simple algebra if we do we get a form like this way f dash; sorry this is f dash k plus 1 and this is f double dash k into f k plus 1. So, obviously, its coefficients are all linear because k. So, k is a known quantity. Always superscript k is a known quantity.

So, either from the initial approximation and so, k greater than equal to 0 and you have this boundary condition as usual f k plus 1 equal 0 equal to 0 and. So, and which is a linear and BVP let us put to start with as k equal to 0; assume f 0, f dash 0, f double dash 0; there is a third derivative as well for all in a b which must satisfy the boundary condition.

So, this is how the quasi linearization technique develops that means, iteratively at every iteration, we are quasi linear the situation because basically they are not linear coefficients are locally frozen to the previous iterated values. So, that is coefficients are considered to be known. So, that is why it is termed as a quasi linear method.

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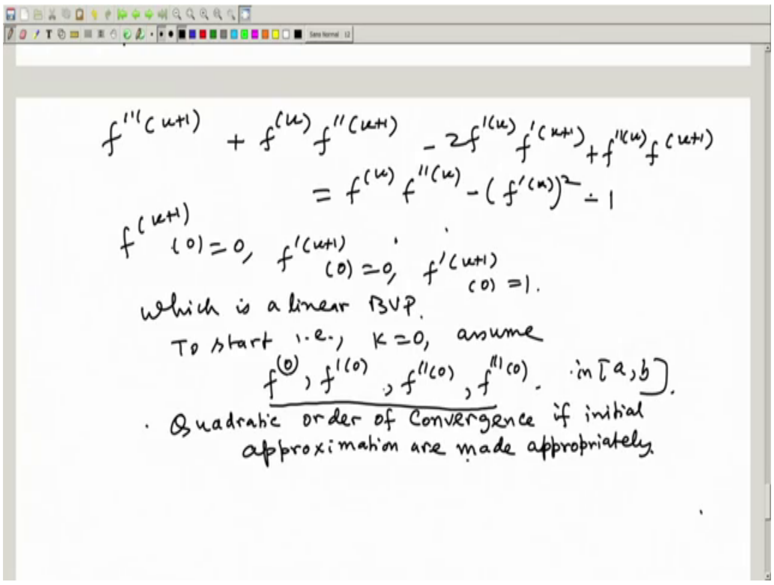

And this also have the same advantage. Quadratic order of convergence; so that means, convergence order is if a initial approximation are made appropriately. Now, this may looks little tough in the very beginning, but in many situations what we may have is a approximate linear form solutions or maybe some experimental data from there we can determine the initial approximation.

So that means, this set of data points this set of values can be determined for all a b and once I know this I first find out the k equal to 0; so that means, if k f 1 f dash 1 exetera then we go so that means once I know the f 1, then all these derivatives and all is obtained. Then we go to the f 2 if I calculate the calculate this by finite difference method. So, I need not have to consider this as a individual derivative also need not have to be evaluated.

We just approximate by finite difference formula. So, that is how the quasi linearization method proceeds. So, in the next we stop here. Next, we will talk about on the boundary value problem with involving partial differential equation linear and non-linear both.

Thank you.