Mathematical Methods For Boundary Value Problem
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Lecture - 16
Iterative Methods for Non - linear BVP (Contd.)

So we will start with the Non-linear Boundary Value Problem. So, far what we consider is a linear one. So, when you have a linear situation which when discretized by finite difference method we get a set of linear algebraic equations. So, that linear algebraic equation leads to a tridiagonal system.

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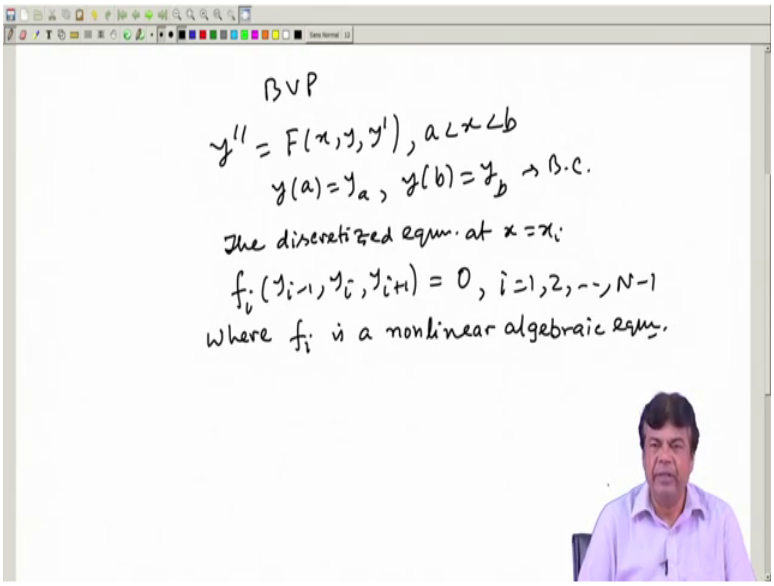

Now, when we do not have such linearity say if we have the BVP is given by a situation this is a general combination of say this BVP is say F y double dash equal to say F x y y dash say a less than x less than b and y a is y a and y b is y b. So, this is the 2 B C and F is any arbitrary function; F is any not exactly arbitrary because if we know the boundary value problem, so F is, so F is any non-linear function.

Now if I discretize, so definitely by say central difference scheme, so; that means, y double dash y dash are all replaced by 3 point formula i minus 1 and all. So, at the discretized equation; discretized equations at a say grid point x i will be something like f i y i minus 1 y i y i plus 1 equal to sum I can call a 0 I can call this set of equation i equal to 1, 2 N minus 1 where f i is the non-linear say non-linear algebraic equation algebraic equation.

So, problem remains the same; that means, we have N minus 1 set of unknown given by y 1, y 2, y N minus 1 and N minus number of equation. So, equation and are unknown are matching, but thing is how to solve this because we now have a set of non-linear algebraic equation. So, our task is to solve these set of non-linear algebraic equation, now to do that what we do is we apply the Newton's linearization technique, so; that means, we have to go by a iterative method.

So, iterative method means what we have to do is we have to assume a solution of the a initial solution some initial approximation and at every iteration we upgrade the solution and finally, we come to a convergence. So, you have to adopt a iterative procedure; iterative method.

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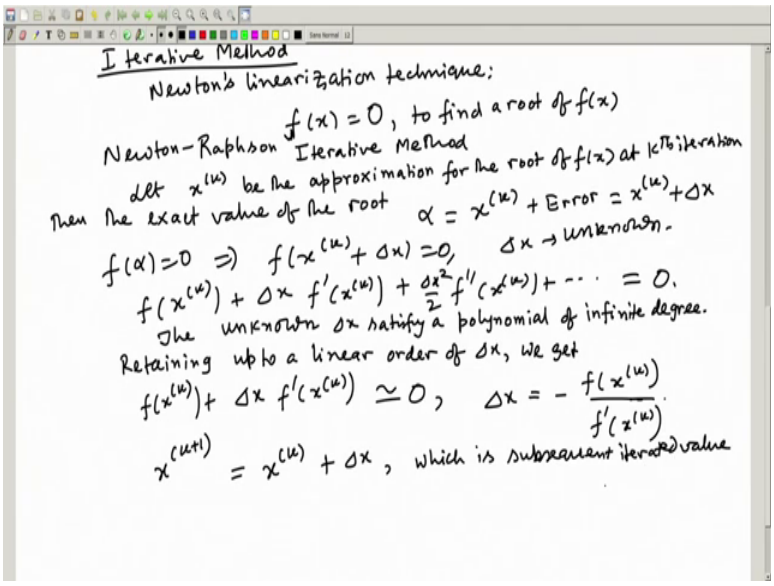

So, first we talk about Newton's linearization technique. Now, before I describe the Newton linearization technique to solve the boundary value problem first, let us recall what is the Newton linear Newton's method to solve a algebraic equation. So, before that a little preliminary so, suppose you have a equation f x equal to 0 and linear non-linear whatever, so this is an algebraic equation. So, you would like to find out to find a root find a root of f x; that means, what we would like to find a value of x say alpha or something which makes the right side as 0 f x equal to, so this side is 0.

So, to do that in Newton's (Refer Time: 05:12) Newton Raphson method. So, that is basically the Newton Raphson method that let I would say iterative method that will be more precise way to iterative method. So, what I do in this Newton Raphson iterative method that let x k be the approximation for the root; approximation for the root of f x at kth iteration. Then x, then the root the exact root exact value of the root say alpha equal to x k plus error this error which is unknown this error is unknown. So, let us called x k plus some delta x ok.

So, if I substitute this now if alpha equal to 0 as alpha is a root. So, this implies f of x k plus delta x equal to 0; delta x is unknown. So, if I know the delta x at this stage at the kth stage i get the root exact value of the root. So, trick is how to get the delta x. Now, to do that what I do first we expand by Taylor series, so what you get is f x k plus delta x f dash x k plus delta x square by 2 f double dash x k and so on. So, this is equal to 0.

Now, x k is known, so we can assume that all this f f dash f double dash etcetera are all known these unknown values. So, task is to find out the delta x. So now, the unknown delta x; the unknown delta x satisfy a polynomial of infinite degree, so; obviously, this is not possible to solve. So, what I do we can solve in finite degree. So, you can solve very easily up to the linear order. So, retaining up to a linear order of delta x we get this is will be the very easy way to find out delta x we get f x k plus delta x f dash x k is approximately equal to 0.

So, this delta x whatever we obtain from here, so we get we can write now delta x if I equal it to 0 we get the delta xs as you get the delta x as f x k minus f x k by f dash x k, we get the value of delta x is given by this way. Now; obviously, this delta x is not the one what we are looking for from this. So, if I now substitute, so x k plus this delta x this we call the next refinement. So, that is x k plus 1 the next iterated value. So, which is the which is subsequent iterates the iterated value well ok.

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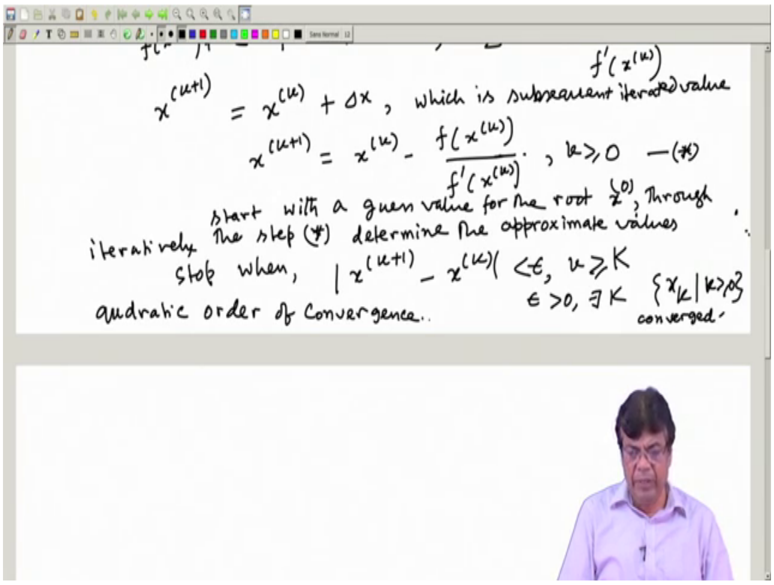

So; that means, next iteration we get as x k plus 1 equal to x k minus f x k by f dash x k k greater than equal to 0. So, this is the way the Newton Raphson method proceed. So; that means, to start with a guess; start with the guess value for the root x 0 for the root x 0 and through the steps let us call this is star upgrade or 2 star determine the approximate values iteratively approximate values iteratively.

So, this iteration iterative procedure converge when I say stop when x k plus 1 minus x k is less than epsilon for some k greater than equal to some given value of some choice of capital K there exists or for any choice of epsilon greater than 0 there exists a capital K when this happens. Then; that means, after a few iteration level if we get the two successive approximate solution for the root is very close differed by a small margin, then we can call that sequence of iterates; that means, this x k k greater than equal to 0 converges converged.

Now, of course, this is a if sequence converge, then these criteria satisfied; obviously, we cannot say that the other way round that is if the if we have a iteration convergence is guaranteed, then at certain stage this is going to happen. So, this is the Newton Raphson method; now Newton Raphson method the iteration converge provided the initial approximation is chosed appropriately. So; that means, it should be a close approximation; that means, x 0 cannot be very a wild case cannot be considered and one important thing is that or one advantageous thing is that the Newton Raphson method converges very fast quadratic rate of convergence, so quadratic order of convergence.

So, you get a very faster rate of convergence what the Newton Raphson method once it start converging also if it start diverging it will diverge in vary faster manner. So, within a few iterations say normally what will happen is that if we see that I choose the initial approximation and what I find that it is reducing or the value is next iterated value and then subsequent one is error is reducing. So, then we can call that the iteration is converging and maybe within four five iterations we get the converge solution.

But if we find that either is subsequent values are becoming larger and larger, so; that means, it is divergence it is going away. So, we can stop that iteration and start with a new approximation a much refined approximation. So, this is all about the Newton Raphson method.

So, this is for a single non-linear algebraic equation, so the convergence of that. So, the basically what we will be doing here in solving the non-linear finite boundary value problem or non-linear finite difference equation by non-linear ODE or any PDE also that is boundary value problem will be solving by the same principle the neutrons linearization technique.

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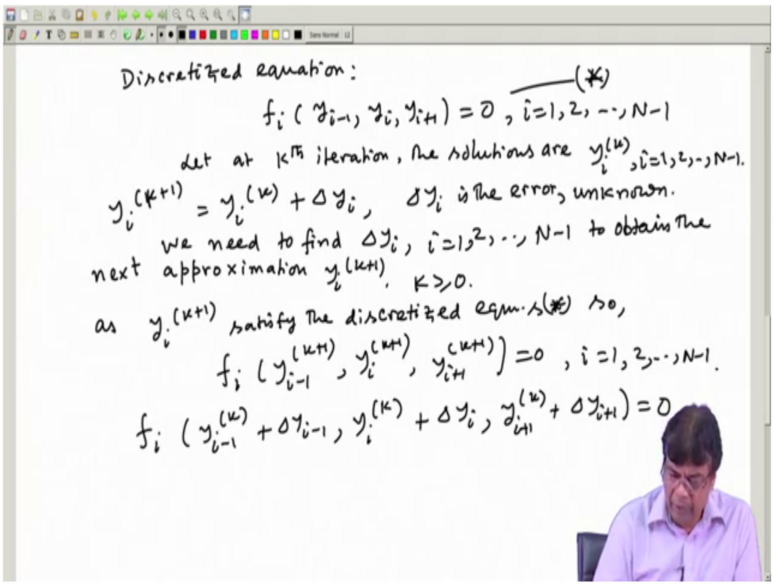

So, what we will be doing is here; that means, we have a algebraic equation, so discretized equation is we have expressed as f i y i minus 1 y i, y i plus 1 equal to 0 where i is riding from 1 to N minus 1 which is a non-linear set of algebraic equation which we are going to solve in a iterative fashion. So, let at kth iteration, the solutions are y i k i equal to for all i rather for all i is not a good appropriate here for i equal to 1 to N minus 1 this is the kth iteration level.

So, y i k plus 1 at the next iteration that is the better approximate for the y i is y i k plus delta y i; delta y i is the error unknown. So, we need to find delta y i, for i equal to 1, 2, N minus 1 to find to obtain the next approximation; approximation y i k plus 1. Now, delta y i is the unknown because y i k is known y i k, so k can be greater than equal to 0. So, I can say if I start from k equal to 0, so; that means, at the beginning, so; that means, you are assuming some y i 0 in the to start the method.

So, if I substitute now in the discretized equation, so f i y i because it will be this if y i k plus 1. So, we can write this y i k plus 1 satisfy the y i k plus 1 as y i k plus 1 satisfy the discretized equation; discretized equations let us call this as equation as star. So, we must have f i y i minus 1 k plus 1, y i k plus 1, y i plus 1 k plus 1, now the y i plus 1 this k plus 1 are all unknown.

So, what I did is we have approximated these by y i k plus delta y i minus 1, then y i k plus 1 plus sorry y i k here y i k; y i k plus delta y i, y i k plus y i plus 1 k plus delta y i plus 1 this is equal to 0, so i equal to 1, 2, N minus 1. So, the same way we expand by apply the way expand by Taylor series.

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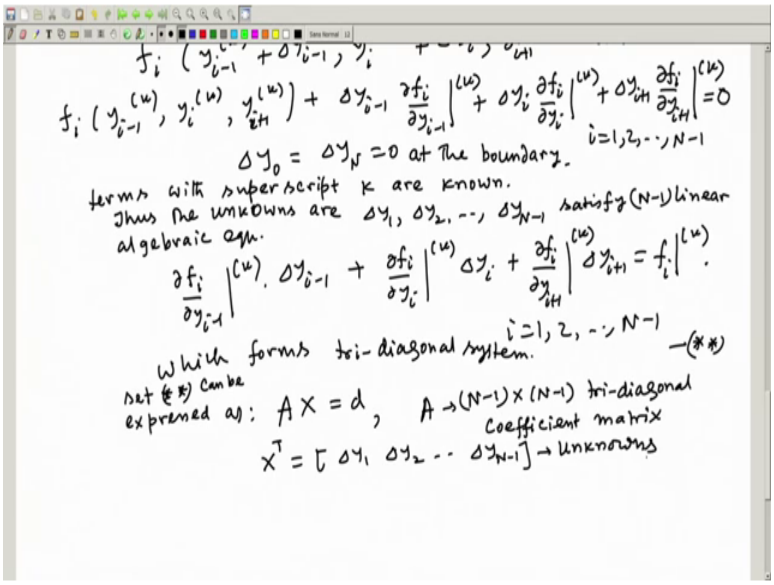

So, what I get is first one is f i y i minus 1 k y i k, y i plus 1 k all the known things, then delta y i minus 1 because they are the unknown which we are bringing out now by Taylor series expansion. So, del f i del y i minus 1 this is kept at k we are treating this as a variable y i minus 1 y i and y i plus 1 these are the three variables for the ith equation. So, the next one is delta y i del f i del y i this is k delta y i plus 1 del f i delta y i plus one is again at k and we written only up to linear order.

So, that is why we are not going beyond that because if I go beyond that this will be a quadratic and also again the solving the quadratic set of equation will become a problem. So, to get rid off all these kind of difficulties we written only up to the linear order. Now, one thing you have to we have to note here that always we should have delta y 0 equal to delta y N equal to 0 at the boundary because at the boundary y is given y or its derivative is given.

So, somehow the values are prescribed at the two boundaries. So, there is no iteration or no unknown situation arise. So, the solutions are prescribed at the two boundaries, so we need not have to do any tricks at the two end. So, that is why we can safely take delta y 0 and delta y N as 0. So, now, we have the unknowns are, so and all this with superscript k turns with superscript k are known.

Thus the unknowns our unknowns are del y 1, del y 2, del y N minus 1 satisfy a a linear a set of satisfy N minus 1 linear algebraic equation which a same thing we can write as del f i del y i i minus 1 z k into del y i minus 1 plus del f i del y i z k del y i plus del fi del y i plus 1 z k delta y i plus 1 and this is equal to your f i evaluated at k all these things and i equal to 1, 2, N minus 1. So, which forms a tridiagonal system.

So, always we should feel lucky if we have a tridiagonal system because tridiagonal system means the solving the algebraic equation a linear algebraic equation are much simpler because we have to we can use a direct method no iterative procedure or know any other tricks is required. So, if we have a tridiagonal, so discretization and all are made in such a way that it reduces to a tridiagonal system. So, we get in this case a tridiagonal set of equations. So, if I solve; so at every iteration. So, so let us call this is as set of equation double star.

So, which can be written as A x equal to say d the set of equation double star the set double star can be expressed as where A is a N minus 1 cross N minus dash 1 tridiagonal tridiagonal matrix tridiagonal matrix or we can say it is a coefficient matrix and x is at the x transpose I can write it as delta x 1 delta y 1, delta y 2 delta y N minus 1 the vector of unknowns.

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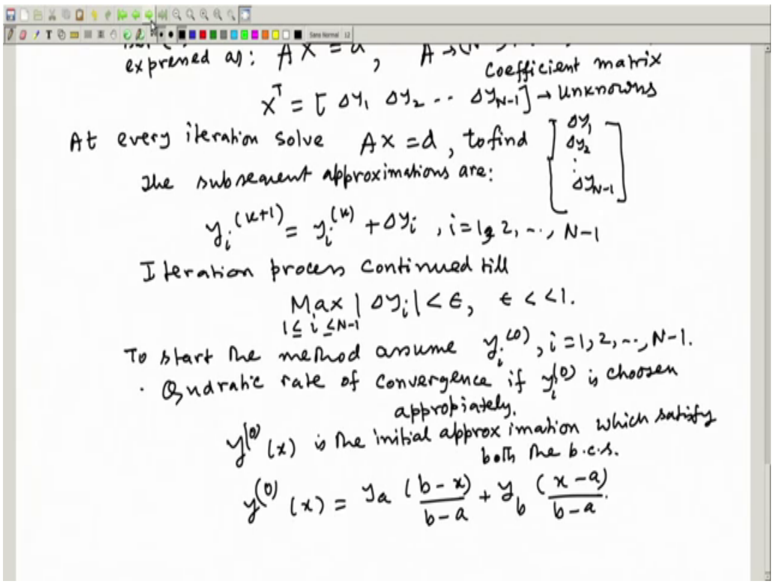

So, at every iteration we solve this tridiagonal system; so at every iteration; at every iteration solve A x equal to d to find to find the error to find this delta y 1, delta y 2, delta y N minus 2. So, using this; so find them, then next approximation the next the subsequent iterates; subsequent approximation are y i k plus 1 equal to y i k plus delta y i, i equal to 1, 2 N minus 1.

So, I continue the iteration procedure the iteration process continued till we get the maximum over i delta y i is less than epsilon 1 less than i less than equal to N minus 1 where epsilon greater is many many times is greater than 0 is many many times less than 1 quantity. So, once this step is achieved, so; that means, we get a convergence. So, this is how the Newton linearization technique process procedures are followed.

So, this is a; so to start the method the method assume y i 0 for i equal to 1, 2 etcetera and obtain the subsequent iterates by this manner and this is a quadratic convergence; quadratic rate of convergence, if y i 0 is choose appropriately normal for complicated problem.

Normally what is choose is that y i 0 say we can choose as y 0 x which satisfy is choose chosen to is considered to is the initial approximation instead of making a big data file as y i 0 in a form as a function which satisfied the boundary condition which satisfy the satisfy both the b c s. Say for example, if I choose y is 0 x equal to say what was given y a. So, if I choose y a into x minus or if I say b minus x by b minus a plus plus y b into x minus a by b minus a.

So, what will happen; what will happen is that if I put x equal to a, so this part is 0; so this is becoming y a and if I put x equal to b. So, this part is 0 and this will be coming y b. So, both the boundary conditions are satisfied this is one simple way to choose the initial approximation. So, once I have these initial approximation, then the procedure is, so if I give a little sketch of the procedure.

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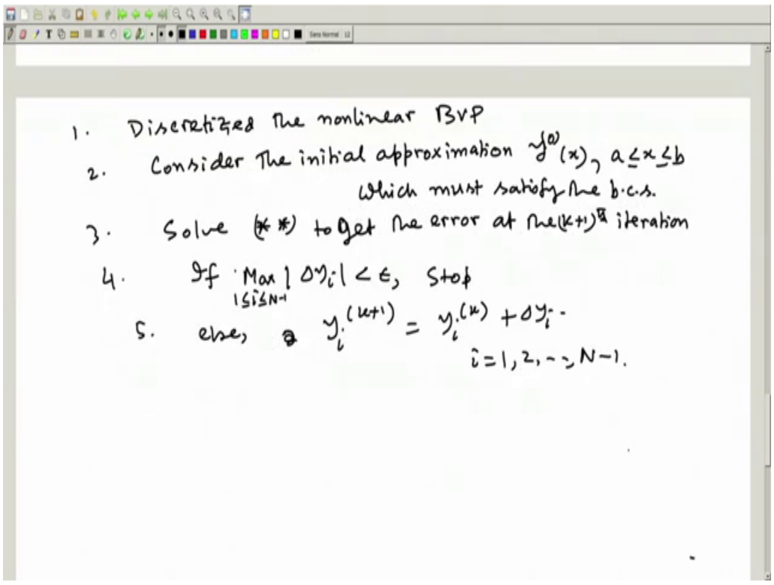

So, first discretize the non-linear BVP the usual way as we do for BVP, then consider a consider the initial approximation; initial approximation y 0 x for a less than x less than b which satisfy which precondition is which must satisfy the boundary condition, where the boundary conditions.

So, double star to get the error at each iteration at let us call at the k plus 1 iteration; at the k plus 1 iteration. Then determine y i k plus 1 or before that what you have to do is we were determining y i k plus 1 what you have to check is that, if delta max of 1 less than equal to i less than equal to N minus 1 delta y is less than epsilon, then if stop else y i; y i k plus 1 determine this y i k plus 1 equal to y i k plus delta y i. So, i equal to 1, 2, x N minus 1. So, this is how the iterative procedure goes, so let us illustrate with an example.

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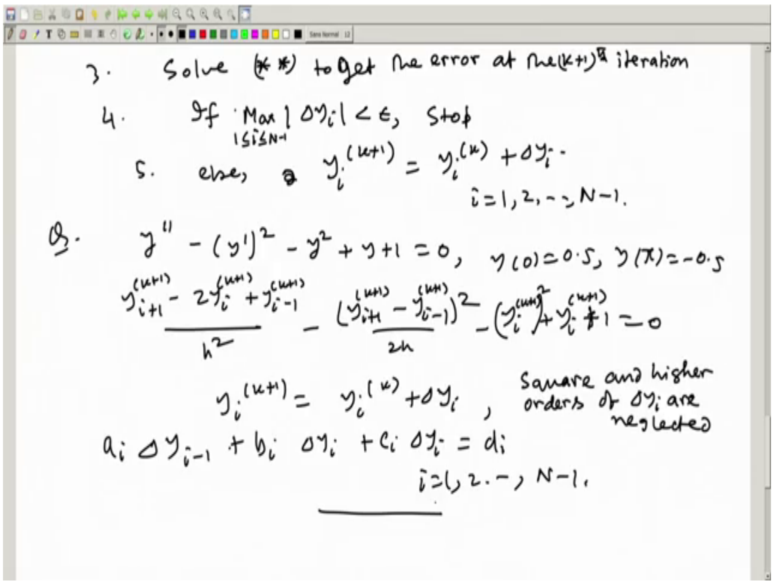

So, I have done with a derivative we have considered that derivatives Taylor series expansion and all. So, which is also equal equivalent can be very simply one can take an algebraic expansion only and retaining only up to the quadratic term. So, this is say y 0 is 0.5 and y pi is say minus 0.5, so this is the 0 to 5. Now, if I discretize this; so this is y I have plus 1 minus twice y i; y i plus y i minus 1 by h square minus y i plus 1 minus y i minus 1 by 2 h this is whole square minus y i square plus y i plus 1 equal to 0.

So, what I do this is satisfied at the k plus 1 iteration. So, at the k plus 1 iteration we approximate this y i k plus 1 equal to y i k and plus 2 plus delta y. So, this is at the k plus 1 because this are also solved at the k plus 1 iteration level. So, this is k plus 1 and this is k plus 1; this is k plus 1; so this is k plus 1 whole square and this is k plus 1 plus 1. So, if I now substitute and we retain only up to the delta y, so I can write these into delta y minus 1.

So, for example, this will be 1 minus h square and here there will be a square term. So, square term we square and product, so; that means, all these square and higher orders; square and higher order; square and higher orders of of delta y i s are neglected. So, if I neglect the square and higher orders of this delta y, so I will get a situation like delta y plus c i delta y equal to some d i is a i equal to 1, 2 etcetera in N minus 1.

So, this forms a tri diagonal set of equation system, so at every kth iteration. So, this is about the non-linear how to handle the non-linear boundary value problem by new translation technique. So, procedure is that we first discretized the non-linear BVP and get a non-linear set of algebraic equations. So, that non-linear set of algebraic equation we solved by a iteratively by this Newton linearization technique.

So, this is the one; so we have to choose these to start the iterative procedure we have to choose the appropriate initial condition initial guess and that initial guess through that every iteration we modify the solution and when we find that two successive approximate solutions are almost equivalent equal, so we stop the iteration. So, in the next we now stop here, the next lecture we will talk another iterative procedure to solve the boundary value problem non-linear cases and that is found to be more useful for PDs.

Thank you.