Mathematical Methods For Boundary Value Problem
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Lecture - 15
Iterative Methods for Nonlinear BVP

So, we started with the higher order boundary value problem and what we have shown in the previous one is that, there is a difficulty arise, if we straightaway discretize this third order boundary value problem by a finite difference method. So, what strategy we adopt now is that we will reduce to a second order one.

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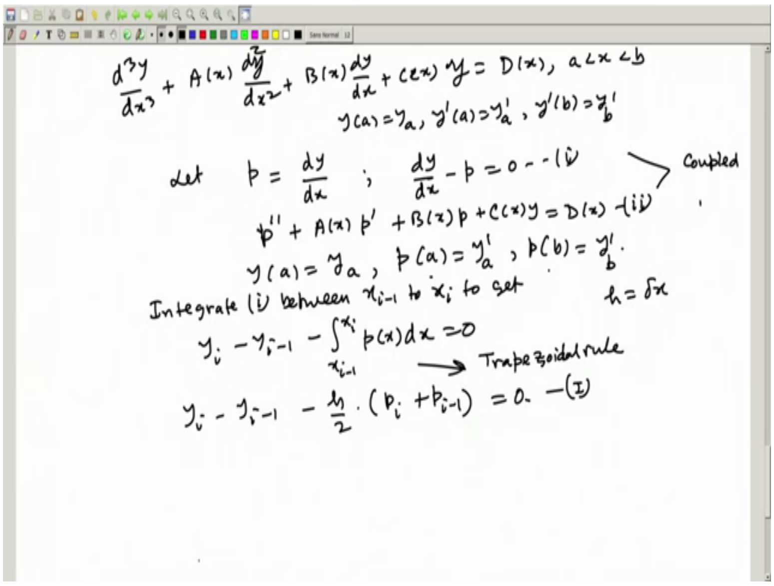

So, our problem was linear third order. So, d 3 by d x 3 plus A x dy d 2 y dx 2 C x y equal to D x and a less than x less than b y a is given y dash a is given y minus b is given. So, let we introduce another variable p equal to dy dx. So, we get a equation new equation now new equation can be given by this way dy dx minus p equal to 0 this is 1 equation another coupled with this equation second order equation.

So, p double dash plus A x p dash plus B x p plus C x y equal to D x so; obviously, these 2 equations are coupled. But they are linear coupled means one solution is depending on the other you cannot isolate one equation from the other. So, solution of the first one is involving p and the solution of the second one is involving y. So, that is the thing.

Now, what we do here that the second one is we are aware how to handle this one because what conditions we have is y a; y a is given to be this value is known. So, this is we are denoted as y a p a is given. So, that is y a dash and p b is given. So, p b is given to be y b dash. So, this is a second order the second one, the second order boundary value problem with the 2 boundary conditions. So obviously, if we discretize where central difference scheme it will lead to a tridiagonal one, but we have to because central difference if I use then it will be a second order accurate.

Now, here in this dy dx minus p if I use a 3 point a symmetric difference scheme so; that means, the last class whatever we are derived; obviously, first order is not possible because first order will be a first order one. So, you have to go for a 3 point asymmetric so; that means, involving either backward or forward. So, involving i plus 2 at the grid point i, i plus 2 i plus 1 and i or i minus 2 i minus 1 and I so; that means, you are again going beyond up at the tri diagonal situation at the grid point i. So; that means, this is not advisable to have a 3 point second order accurate first order derivative discretization.

So, what we do is integrate i; integrate i between xi minus 1 to x i to get. So, what I get is y i minus y i minus 1 minus xi xi minus 1 xi px dx equal to 0. Now, this is the exact; so, no approximation nothing as we done now approximation. Now we do because p is not known so; obviously, if p is unknown.

So, we cannot evaluate the integration exactly. So, what I do we replace this by trapezoidal formula trapezoidal rule. So that means, what I do is y i minus 1 minus h by 2 let us take h is the delta x h by 2 p i plus p i minus 1 and that is equal to 0 this is the discretization trapezoidal rule. So, the here we have done the approximation.

Now, what is the order of accuracy or trapezoidal rule is third order h cube. So, it is compatible of our requirement. So, this is the discretization of the equation I let us call this as I. So, integration is evaluated by the trapezoidal formula and the other one second one we discretize by equation II discretize by central difference scheme.

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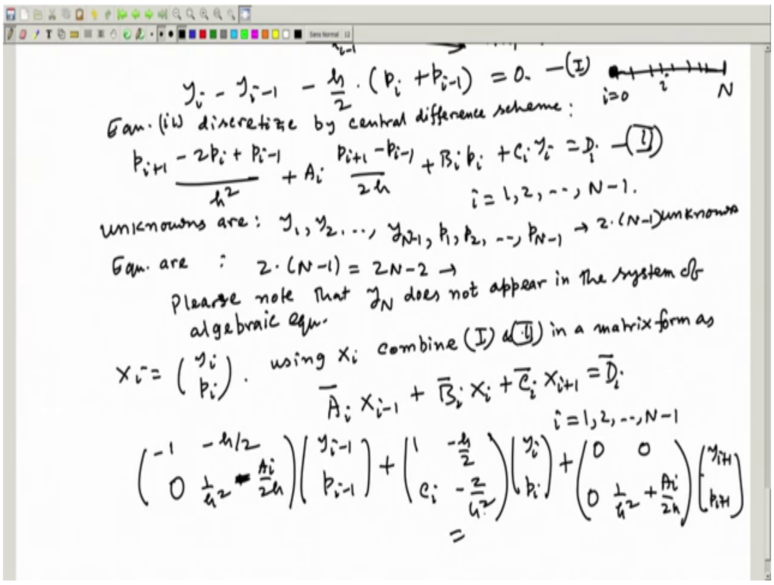

So, if I discretize with central difference scheme. So, what I get is p i plus 1 minus 2 pi plus pi minus 1 by delta x square or h square in this case should not confused; then Ai pi plus 1 minus p i minus 1 by 2 h plus Bi pi plus Ci y i equal to Di. So, I and II are I can vary i from and 2 etcetera up to N minus 1 because our grid points except the boundaries are i equal to 0.

So, this is i equal to 0 is the first boundary point and this is our i equal to N the last boundary point this is the N and in between all these are so, this is our grid point i. So, if we vary i equal to 1 to N minus 1. So, these leads to now how many unknowns we have? Now, we have doubled the known because not only the y is unknown we have the derivative p which is y i dash that is p i is also becoming a coming into the picture.

So, unknown are y 1 y 2 y n p 1 p 2 p N minus 1. Now and equations 2 into N minus 1 so; that means, 2 N minus 2 number of equations. Now shows till it shows that there is a lack of one. Now one important thing is to see is that y N does not come into the picture so; that means, y N does not appear in the equation 2 into N minus 1 unknowns number of equation. So, please note that y N does not appear in the system, in the system of algebraic equation y say y here y is appearing as y i. Now, I am adding I equal to 1 to n minus 1 so; obviously, there is no chance of y n to appear ruled out.

Now, here the first one y i, y i minus 1 so, if I vary i from 1 to n minus 1. So, y 0 appears, but no y n so, on so; that means, we are through so, same number of equations and same number of unknowns. So, if we solve we get a unique solution. Now the thing is we have for a grid point i we have two set of unknowns and two set of equations. So, we have learnt a coefficient matrix which is having a entry as single elements.

Now, what we do is we introduce a we first combine these equation I and II a single equation by introducing a variable say let us call this is the variable at each grid point our unknown is X i p i. So, using these using X i combine I and II in a matrix form; in a matrix form as let us combine this way A i X i minus 1 A i let us called A i bar because A i already we have used B i X i plus C i X i plus 1 let us call this is D i coefficient D i bar.

So, this I and II are two set of linear algebraic equation which I combine to form a single equation and what is the i is varying from 1 to N minus 1 because this equations are varied for i equal to 1 to N minus 1. Now what is A i bar? A i bar I can write. So, the first row of A i bar is the coefficient or first entry of A i bar is the coefficient of y i minus 1 in the first equation.

So; that means, this is nothing, but minus 1 next 1 coefficient of pi minus 1 in the first equation. So, that is h by 2 and the second row is coefficient of y i minus 1 in the second equation which is 0 and y i minus 1 in the second equation. So, which is 1 by h square minus A i by 2 h.

So, this is y i minus 1 pi minus 1 then the same way I can write this is the coefficient of first entry is the coefficient of y i in the first equation and coefficient of pi in the first equation and coefficient of y i in the second equation and coefficient of pi in the second equation I believe this will be 2 by h square. So, this is y i pi plus same way. So, this is 0 there is no chance of having y i plus 1 as well as no pi plus 1 and here also y i plus 1 is always 0 and pi plus 1 is 1 by h square plus A i by 2 h this is into y i plus 1 pi plus 1 and the coefficient matrix the unknown vectors is sorry known vectors are 0, now this is h square and here it is 0 and D i.

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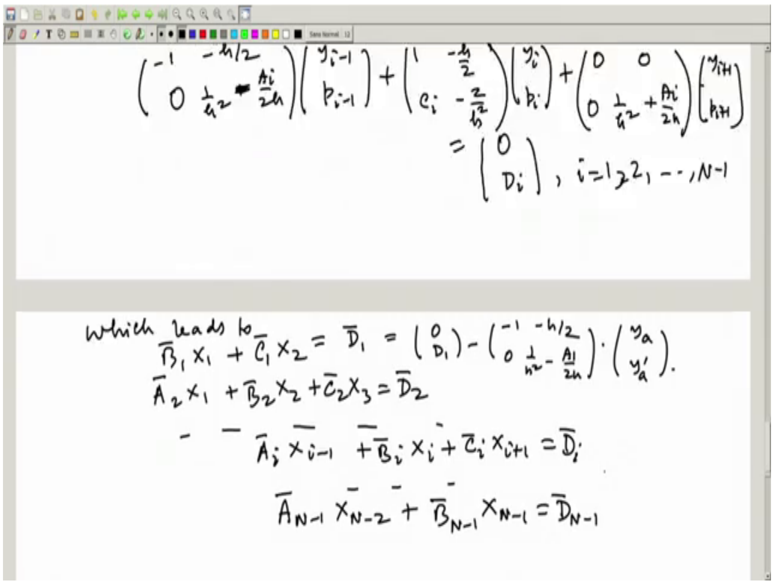

Now, one thing is this is varying from i equal to 1 1 to n minus 1. Now if I put i equal to 1 if i equal to 1. So, this becomes y 0 p 0 which is known. So, it goes to the other side. So, this will be transferred to the right side similarly if I put i equal to N minus 1. So, this pn is known so; that means, this term is known. So, that will go to the left right side. So, so that manner if I put the i 1 by 1. So, I get a set of equations like this way. So, which leads to B 1 X 1 plus C 1 X 2 equal to let us call that as D 1 bar because this is D 1 bar will be what?

D 1 bar is 0 D 1 minus minus 1 minus h by 2 0 1 by h square minus a 1 by 2 h into y i minus 1 which is given to be y a pi minus 1 one is p 0 this is given to be y a dash. So, all are unknown quantity. So, I call as D 1 bar. Next one there is a known things i equal to 2 if I put. So, if I put i equal to 2 here this equation. So, this gives you let us put bar.

So, i equal to 2 if I put. So, this is A bar 2 X 1 plus B bar 2 X 2 plus C bar 2 X 3 equal to D 2 and so, on like that way I have any ith equation is the same as say minus 1 plus B bar i X i plus C bar i X i plus 1 equal to D bar i and the last one is if I put here i equal to N minus 1. So, this guy X n is known.

So; that means, I get A N minus 1 X N minus 2 plus A bar plus B bar N minus 1 X n minus 1 equal to let us call this as the D bar N minus 1 so, which is nothing, but whatever will come up. Now one thing is that X n X n is y n is not known, but it makes no difference because position of Y n is fixed to be 0 whether y i plus 1 so; that means, if

I put i equal to N minus 1. So, this position of y n is coefficient is 0. So, y n does not appear. So, I have to just give the value of p n.

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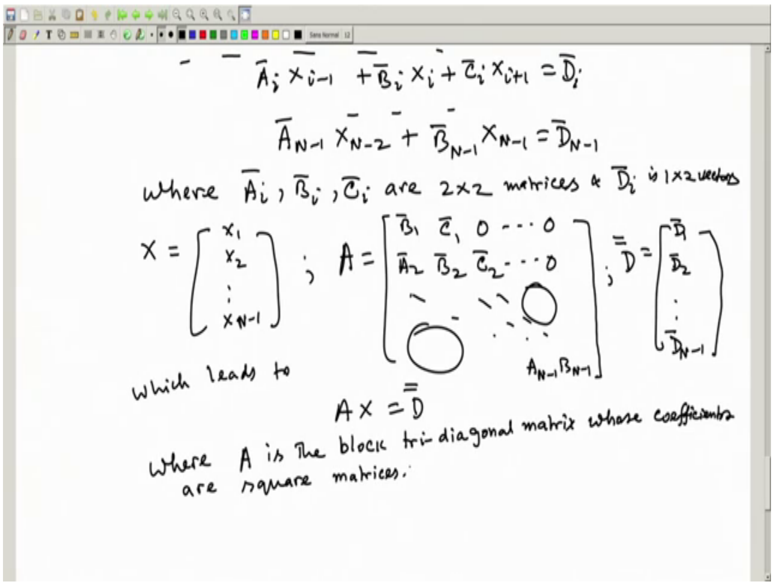

So, now if I now all this where A i B bar i C bar i are 2 by 2 in this case matrices and D bar i is 1 cross 2 vector vectors ok. Now, if I introduce a variable say x which is nothing, but X 1 whose components are X 1 X 2 X N minus 1 and a coefficient matrix in this manner the coefficient matrix B 1 bar C 1 bar 0 0 0 all are 0 and null element rather and A 2 bar B 2 bar C 2 bar all are null rest of the elements are null element; so, here all 0 entries.

So, let us write here this will be A N minus 1 and B N minus 1. So, I can put here as this matrix and obviously, this if I call D double bar is nothing, but these coefficients D bar 1 D bar 2 D bar N minus 1. So, with that with that with that we get which leads to a form as Ax equal to D bar, where this A is a tri diagonal type, but entries of A are not exactly the same as the ordinary matrix because here entries or coefficients of A the matrix A is again a 2 by 2 matrix.

So, this kind of matrix is referred as a block tri diagonal matrix A is a block structure. So, physical meaning of the matrix which is we define in like other properties of matrix. So, all of these are not exactly extendable for the block tri diagonal, but it is useful to consider a block tri diagonal matrix to solve a system of equation. So, that is why where A is called A A is the block tri diagonal matrix, whose coefficients are square matrices.

So, if I can solve this block tri diagonal and X is the vector of vectors and the components of X are vectors.

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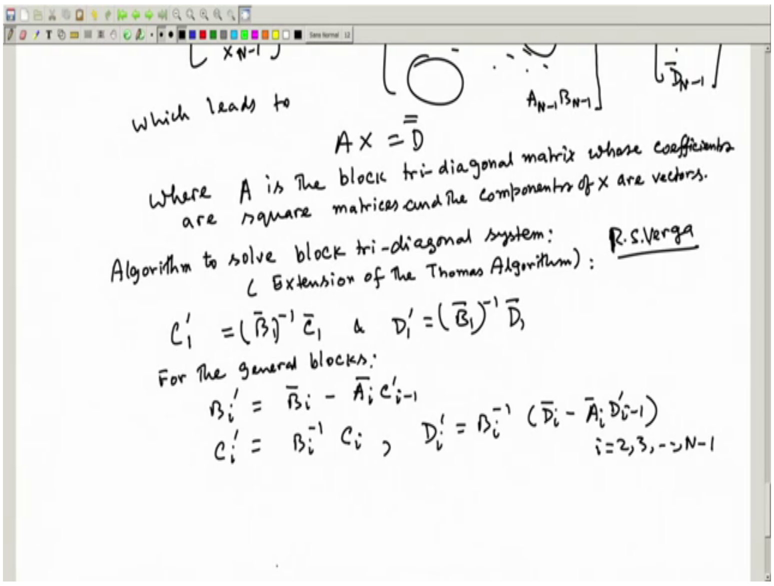

So, I can solve. So, I get the solution. If I solve I get the solution now is there is nothing to explain about it, but how to solve this that is the situation. Now this is a quite complicated set of equations situation. So, there are several algorithms. So, one of these easiest simpler one is the same that as we have done for Thomas algorithm. So, that can be extended for this case. So, which is called the same as the Thomas algorithm; so, we call as the algorithm to invert block tri diagonal block system.

So, it is a kind of extension of; so, there are more simply white or less time consuming or less time computational complexity are less are algorithms are there. So, do not want to discuss that, there is a book by R S Verga. So, the they that provides several other algorithm. Now the simple extension of Thomas algorithm for this kind of structure can be done this way.

So, let us define the C bar dash C 1 dash as B 1 inverse C 1. So, instead of division it is inversion and D bar 1 dash is B bar 1 inverse D bar 1 that gives you the first entries, and general blocks what do you have is B i dash equal to B i minus minus A i C dash i minus 1 and plus C i dash equal to B i inverse C i its be inverse C i and Di dash Di dash because it is the new one.

So, di dash equal to B i inverse and D i minus Ai B dash i minus 1. So, this operation we are doing for i equal to 2 3 up to N minus 1.

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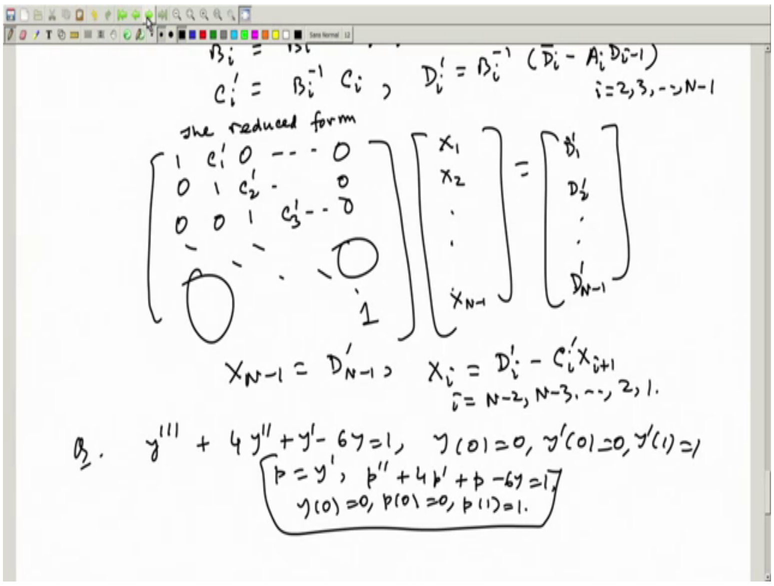

So, once I did this. So, the reduced form that is by the block elimination I get the reduced form as 1 C 1 dash 0 1 C 2 dash etcetera all are null 0 this is 1 all the diagonal entries are identity element not 1 exactly this is a 2 by 2 in this case. So, C 3 dash like that way and last one will be 1 is this is 0 and this is 0.

So, all these entries are 0 0 and here we have X 1 X 2 X N minus 1 and this is nothing, but X N minus 1 and here we have is D 1 dash D 2 dash D N minus 1 dash. So, the last element gives you X N minus 1 I just get D N minus 1 dash and generally I can write X i equal to D dash i minus C dash i X i plus X i plus 1 i equal to N minus 2 N minus 3 like that way 2 1.

So, this by Braggs substitution I get the solution of the block tri diagonal system. So, that is how the higher order situation can be handled third order particularly now this block structured things is helpful even for solving the set of equations. So, if you if you have a couple set of boundary value problem. So, in that case also we will come across a set of block tri diagonal system, there is one equation is depending on the other. So, a those cases can be handled by block tri diagonal situations.

So, let us take this example say y is 0 equal to 0, y dash 0 equal to 0 y dash 1 equal to 1. So, in this case what we do is we replace these by a p equal to y dash and this one gives you p double dash plus 4 p dash plus p minus 6 y equal to 1. So, these two set of equations with conditions as y 0 equal to 0 p 0 is 0 p 1 is 1 and the same way as we did for the previous one can be used.

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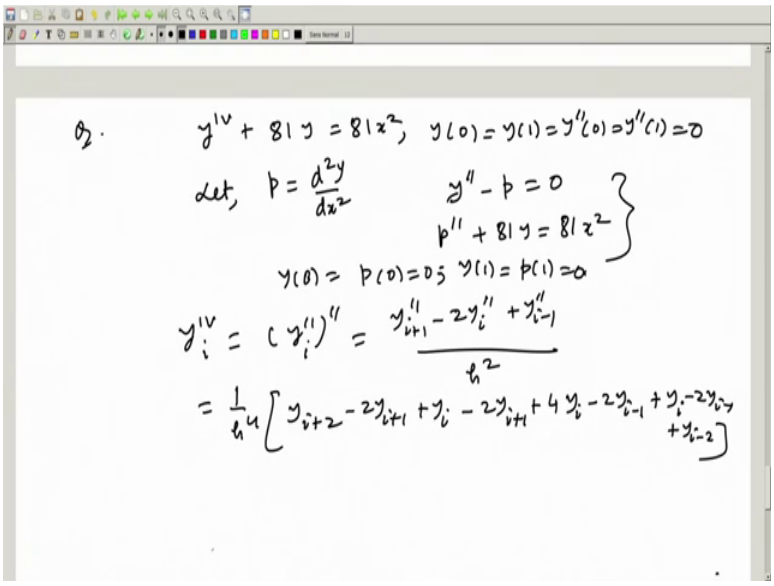

Now, I will one more example and conclude for the higher order situation. So, suppose you have a equation like this way x square y 0 is equal to y 1 equal to y double dash 0 equal to y double dash 1 equal to 0. So, in this case it is easy to solve this that if I put p I considered as d 2 y dx 2 let this so; that means, we have a coupled set of equation this one with the other one is p double dash plus 81 y equal to 81 x square.

So, this is a coupled set of equation is formed and the corresponding boundary condition is p 0 is 0 y 0 is 0 and also y 1 equal to p 1 equal to 0. So, this can be expressed in a form of a block diagonal structure and can be solved. So, these are the method because there are of course, in the ordinary way that is straight for order not exactly ordinary well what I mean is that straightforward way, that is if I discretized by the central difference scheme. Say for example, this y four can be written as y double dash double dash.

So, at the grid point i so; that means, y i double dash minus y i plus 1 double dash minus twice y i double dash close y i minus 1 double dash equal to h square. So, again if I discretize that is function of a function. So, y i double the fourth derivative, I considered as y i double prime of a double derivative say if I discretize this is the form is coming. So now, again here if I discretize y i plus 2 minus twice y plus 1, plus y i minus 2 y i plus 1 minus; so, plus 4 y i minus 2 y i minus 1 plus y i minus 2 y i minus 1 plus y i minus 2 minus 1 is there.

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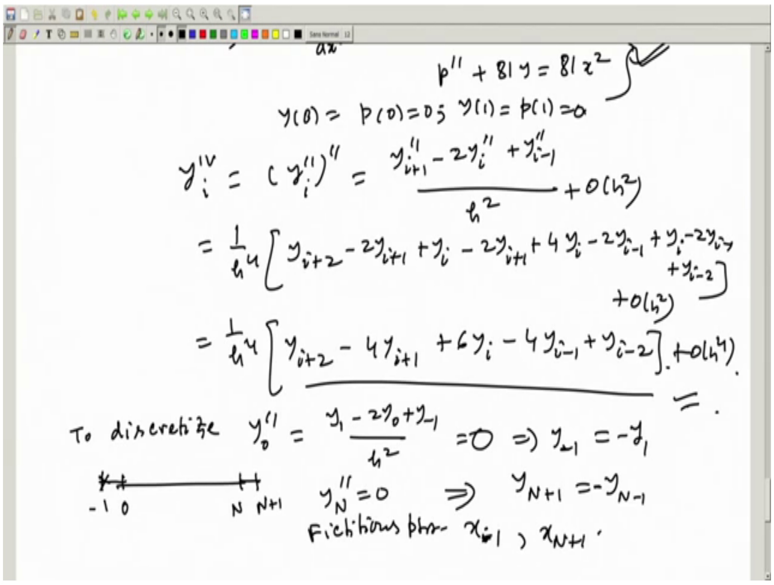

So, in this case again this is order h square; obviously, it is evident. So, this is h to the power 4 y plus 2 minus 4 y i plus 1 i anything and y i 5 6. So, 6 y i and the y i minus 1 is minus 4 y i minus 1 plus y i minus 2. So, order is 2 to the power 4. So, this discretization now if I put substitute there should I get a a equation like this way. Now, difficulty in handling this kind of irregular situation is that it is not forming any block tri diagonal whether it will be a bit diagonal because five elements are involved, i plus 2 i plus 1 i i minus 1 i minus 2.

So, five elements are involved there is no such direct algorithm like Thomas algorithm to invert this. Now in that process we have to go by either Gauss elimination which is a very slow process or some iterative method like Gauss Seidel or any conjugate gradient method. So, complicated as well as slow. So, in, but accuracy wise we are not having any benefit that if I reduce to 2 second order system as we proposed in the in this case club coupled set of equation this one.

So, this gives you a more better way that we lead to a tri diagonal block tri diagonal structure and we have an algorithm directly algorithm to solve that. But here it is solvable, but it is a complicated one and also we have to introduce fictitious points to discretize this second order derivative y double dash 0 y double dash 1. Say for example, if I y double dash 0 to discretize this, to discretize because this is at the boundary.

So, if I introduced symmetrically to more symmetric point another symmetric 1 minus 1 and similarly at N I introduce another point ahead of that. So, before of this 1 point and ahead of this another point. So, if I do that y 0 double dash can be written as y 1 minus 2 y 0 plus y minus 1 by h square which is some value say here in this case 0 and this is also y 0 is 0 this implies y minus 1 equal to minus y 1 and the same way I can write as y n plus 1 equal to y minus n minus 1 as y n double dash equal to 0.

So, we have introduced to fictitious point xi minus x minus 1 and x N plus 1 which has no physical significance. Now for a non-linear case this may lead to instability because this has no physical significance and it is not influenced by the boundary condition. So, that is why this is not advisable particularly for non-linear set of non-linear boundary problem to introduce fictitious point and solve by this type of algorithm.

So, this advantageous to reduce to a situation where we get a we confined to our attention within the grid ways physically meaningful grid whatever we are defined. So, as we pointed out here that in that process it leads to a block tridiagonal structure and we get can be solved by the algorithms which are available for handling a block today construction itself ok.

So, next we will talk about the non-linear boundary value problem which has not been introduced so far, but non-linear means there will have to have a iteration. So that means, we do not have any we cannot solve it directly so, that we will discuss in the next lecture.

Thank you.