**Mathematical Methods For Boundary Value Problem**
**Prof. Somnath Bhattacharya**
**Department of Mathematics**
**Indian Institute of Technology, Kharagpur**

**Lecture - 14**
**Finite Difference Method for Higher - Order BVP (Contd.)**

Welcome. Now we have introduced the Finite Difference Method for linear boundary value problem. Now in that what we did is we have the coefficients of the derivatives and the function was considered to be continuous. So, we have taken a uniform grid and uniform type of discretization for every grid points.

Now situation can be difficult when these coefficients are not continuous instead it can be piece wise continuous; that means, it can have kind of jump discontinuity in certain number of points or we may have to go for different grid size, different step size can be different it may vary from grid to grid.

So, in those cases it is not advisable to consider that uniform discretization irrespective of the grid point itself. So, another concept which is particularly in which deals with this physical phenomena like based on some principles, which are based on conservation principles say fluid flow equation or heat transfer which are based on conservation of momentum or heat transfer or flux heat flux.

So, in those cases what we need is any control volume over which the differential equation is integrated the conservation principle should hold good over there. Now so, for this we have to do a little bit of different formulation. So, I am not going to do a huge discussion on this, it just a conceptual outline in respect to a ordinary differential equation that will be thing will be first I will introduced. So, control volume formulation.

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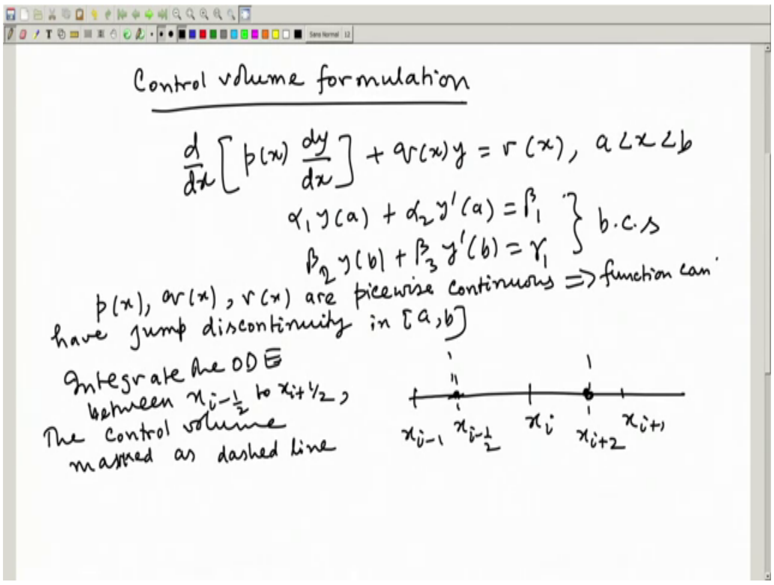


So, let us give the topic name as control volume formulation. So,. So, we have a differential equation or is the boundary value problem linear say p x dy dx plus say qx y equal to say r x and a less than x less than b. Now here we can have these p q and r can be piece wise continuous function. Now say boundary conditions are given by say alpha 1 y a plus alpha 2 y dash a equal to say beta 1 and a whatever beta 2 yb plus beta 3 y dash b equal to gamma 1. So, this kind of boundary conditions are they are. So, basically what it means is the mixed boundary condition.

Now px q x rx are piece wise continuous that means function implies functions can have can have jump discontinuity in a b. So, let us consider say the grid point i x i this is say x i minus 1 this is say x i plus 1. Now we choose a control volume this is the one we are calling. So, we choose a points x i minus 2 and x i plus 2. So, now, within these interval x i minus 2 and x i plus 2 we integrate the equation integrate the ODE between x i minus half to x i plus half to get.

So, we call this control volume the that is the or rather between x i the control volume marked as test line. So, of course, it is a interval because we have a 1 dimensional situation. So, it is an interval. So, this interval if we have a 3 dimension or 2 dimension case then it is a control volume situation.

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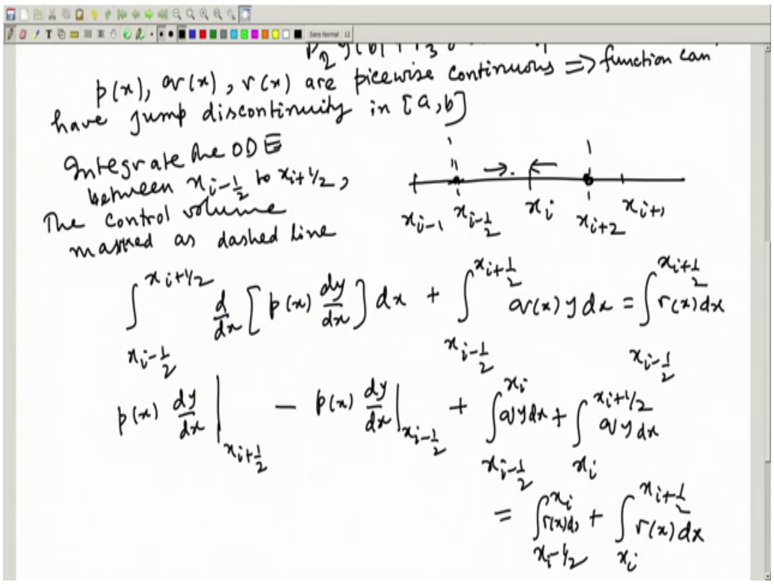


So, this we are now integrating between x i minus 2 to x i plus 2 of this equation sorry x i minus half to x i plus of d dx of px dy dx dx plus x i minus half to x i plus of x i minus half to x i plus half qx y dx equal to x i minus half to x i plus half r x dx.

So, basically what this method allows that the, it can have jump discontinuity at x i so; that means, the if I approach from this side; that means, in the increasing side towards x i so, that value and if I approach from the right side so, that may have a difference. So, that difference is allowed if we have a piece wise discontinuity so; that means, either this is we call as the say q i minus and this is I call as the qi plus so; that means, minus side we are approaching and this is the plus side of at x i we are approaching.

So, this can value can differ. So, if I now integrate this is there is no problem I can simply write dy dx this is at x i minus plus half minus px dy dx at x i and this is x i minus half sorry plus half and this is x i minus half and here since qx can have jump discontinuity. So, we divide into 2 separate integral x i 2 x i plus half and this is the qx qy dx q y dx and the same way x i minus half minus x i minus half to x i plus x i to x i plus half rx dx.

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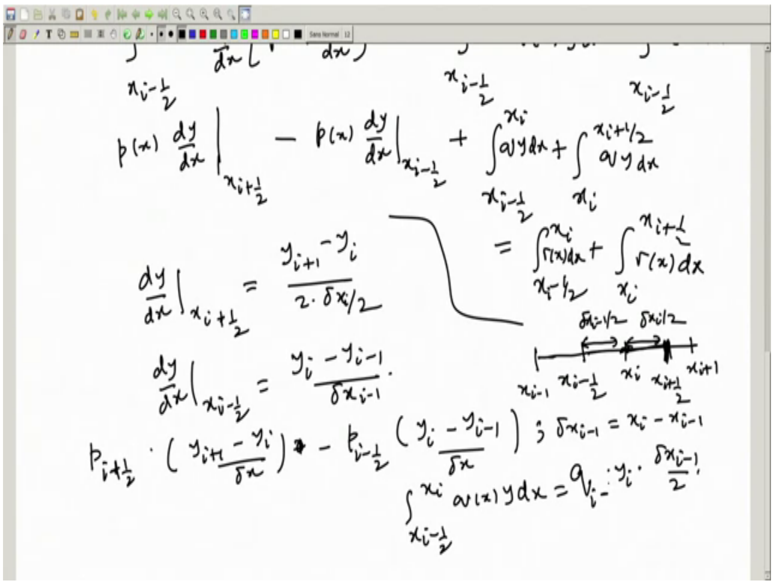

Now, what we do here this dy dx at x i plus half plus minus half we can approximate by central difference formula. So, as y i plus 1 y i plus 1 I think this is not a into 2 into delta x by 2.

So; that means, if I consider these. So, if I consider x i plus half and if I use the central difference scheme which steps I say delta x by 2. So, this is yi plus half plus half means yi plus 1 minus y i plus half minus half. So, that is yi minus y i. So, this is 2 delta x and same way I can write that dy dx at x i minus half also the same yi by yi minus 1 delta x. So, these are the two discretization of these derivatives here.

So, this one we write as pi plus half and this 1 is yi plus 1 minus y i by delta x and the other 1 comes out to be. Sorry, this is minus p i minus half yi minus yi minus 1 by delta x and this integral what we do is in each subinterval. So, we this is a very finite subinterval because xi is are very clustered a dense grid or cluster points. So, this is the x i and x i plus. So, x i plus 1 say. So, this distance is this step size step size is considered to be very small.

So, these within this say this is x i plus half. So, here what we do is we approximate the function by the function value at this stage at x i so; that means, what we do here is x i minus half to x i q x y dx we replace by q since it is got a i. So, this is i minus into y i and this is delta if I call this is the half step size. So, like here if I allow different steps i. So, so far what we have taken is this step size is the same as this step size x i minus half. So, we have taken this is and this distance are same. So, this may not be the equal because as I said in the very beginning, that we can have a different step size. So, depending on the grid points so, we can have a variable step size.

So,. So, this step size a x i minus 1 and x i and the x i x i plus 1 may be different. So, that is why we denote it by d x i minus 1 by 2 and this one we denote it by d x i by 2. So, if we call this distance x i minus 1 x i minus x i minus 1 as delta x i. So, what we did what we are denoting is delta x i minus 1 delta x i minus 1 as x i minus xi minus 1. So, if this is the case. So, this will be x i and this is x i minus 1. So, this allows you to consider a variable step size. Of course, what I wrote in the very beginning its still remain correct if I choose the step size as equispaced point.

So, now if I consider this variable step size; so this distance x i x i minus half is nothing, but delta x i minus 1 by 2. So, this is i minus I am writing this is q i minus I am writing if we have a jump discontinuity. So, this is q i minus and if we do not have if it is a continuous then this function where is the same as qi minus equal to qi plus.

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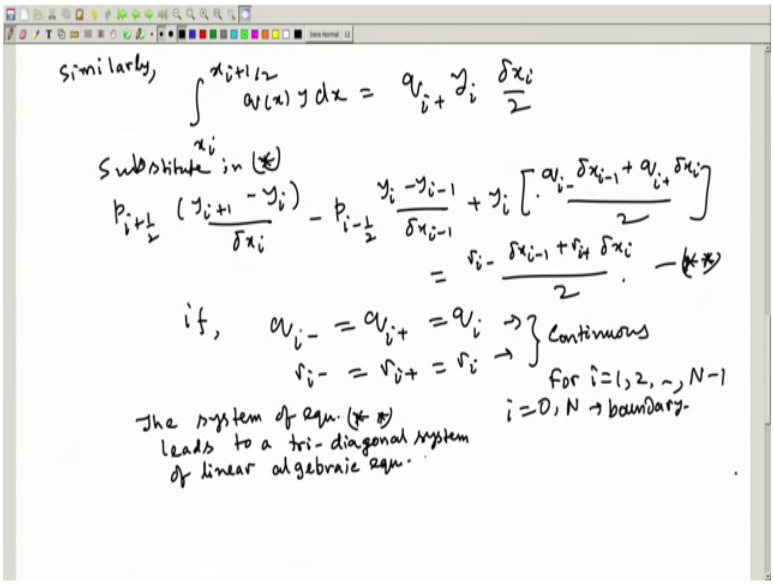


The same way; similarly x i to x i plus half q x y dx can be written as q i plus y i delta x i by 2.

Now, if I substitute this in the expression. So, what we get here is. So, this is I think one should and just note the equation. So, let us write this way. So, if I substitute this in the equation say star, substitute in the discritize intergrated equation substitute in star. So, we get p i plus half y i plus 1. So, this is delta x i this is delta x i minus 1. So, yi yi plus 1 minus y i yi plus 1 minus y i by into delta xi minus p i minus half yi minus yi minus 1 by delta x i minus 1 plus y i into qi minus delta x i minus 1 plus qi plus delta x i by 2 this the integral that is right and this is ri minus delta x i minus 1 same way plus ri plus delta x i by 2.

Now; obviously if it is a continuous. So, q i minus and qi plus are same and if it is a uniform grid delta x equi-spaced points. So, delta x i minus 1 and delta x i plus are all same. So, it will be nothing, but y i qi delta x i by 2; now, what I find that this kind of discretization which allows the.

So, if what I was telling is qi minus equal to qi plus equal to nothing, but qi if on ri minus equal to ri ri plus ri plus equal to ri if continuous. So, let us call this star double star. So, if I vary for i equal to I can choose i equal to 1 wto n minus 1 because if I call i equal to 0 i equal to 0 and n are boundary.

So, within this in between the boundaries we get this set of equations. So, this system of equation leads to a system of equations double star leads to a tri diagonal system tri diagonal system of matrix equation system of linear algebraic equation.

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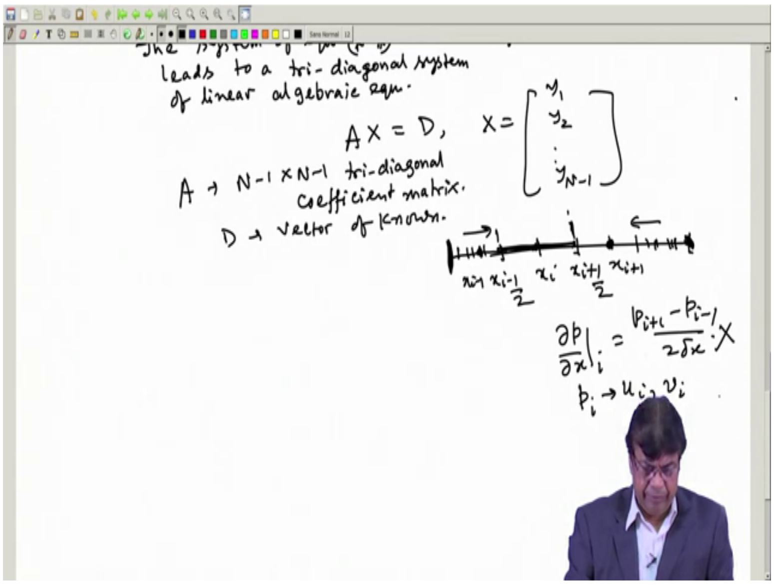

So, basically this leads to a x equal to some d. So, x i call as y 1 y 2 y n minus 1 and a is n minus 1 cross n minus 1 tri diagonal system coefficient matrix. So, this is the situation which takes care of D is the vector of knowns vector of knowns. Now this situation takes care of variable grid size so; that means, you have a situation where the points are not equi-spaced so; that means, x i xi plus 1.

So, in some cases where say near the boundary say suppose some heat transfer from wall. So, what happen is that from the heat transfer of a wall. So, you have a larger change in heat transfer rate when you consider near the wall. So, as you move away what do you have the heat transfer rate becomes slow down. So, we can have a larger grid size. So, in that case we will have a dense grid near the wall one boundary and as we approach as we move away either of this, you can have the 2 boundaries here. So, near the boundary I can have a denser grid and as I move away I can have a course grid.

So, this kind of situation can take care and also another important thing what I have I was told in I told in the very beginning there is the control volume approach that is the conservation principles hold good so; that means, every neighborhood in x i within this neighborhood or control volume the conservation principle is satisfied. So, this kind of discretization is very helpful when you have a jump discontinuity kind of situation arise.

So, I do not want to go further details on these, this is very helpful particularly for solving the Navier Stokes equations and where can have a advection dominated flow and on. Another important aspect I want to just mention over here in respect to develop the solver for Navier Stokes equation. Now here what we do is we have a grid point say x i and all this is influenced this x i the solution is influenced by the by this control volume of x i minus half and x i plus half.

So; that means, at this we can call this cell phase what we do is we have taken a average between x i and x i plus 1 the; that means, the cell value or the values at these grid points x i and x i plus 1 or x i and x i minus 1. Now, when you have a fluid flow problem; so, there what do you have in the Navier Stokes equation if you remember.

So, we have a first order derivative in the pressure. Now if I want to give a central difference here at the grid point i. So, what will happen is that. So, at i if I write this ratio p i plus 1 minus p i minus 1 by 2 delta x so; that means, the point i or the grid point i which is influenced by the pressure p i. So, this pi is governing the flow at say ui or vi, but it is not coming into that impact is not directly reflected by this kind of discretization.

So; that means, what we are taking the pressure that is at the point grid point i which is influencing the flow is the pressure measured at i plus 1 and i minus 1; so, which may not be a correct way of discretizing the situation. So, that is why the control volume approach is become very useful. So, for this kind of situation where at every control volume the conservation principles are satisfied and we can handle the piecewise discontinued a piecewise continuity situations and other ok.

With that now we go to the situation when we have a higher order boundary value problem. So; that means, so, far we have what we had did is 2 point boundary value problem and boundary value problem means it would be always of the order higher than two. So, we so, for what we considered is a second order differential equation and that has been discretized by several methods.

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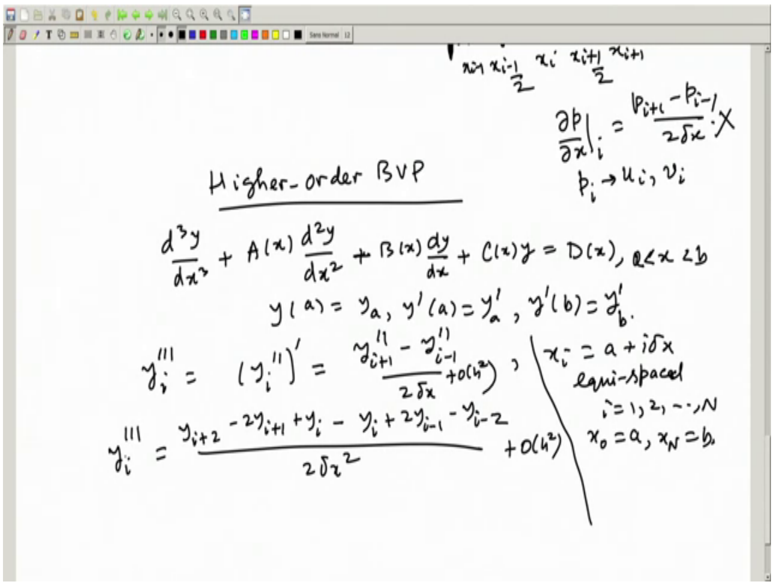


Now, let us see that if we have a situation which is higher than 2 higher order BVP then what the difficulty may come across. So, consider this next advanced one so; that means, 2 to 3. So, consider this boundary value problem third order boundary value problem let us keep linear. So, this is the one and Bx dy dx plus Cx y equal to say Dx and say let us call this is a less than x something like that 0 or a less than x less than b this is our boundary and conditions 3 conditions to be given at 2 different points. So, say suppose y a is given and y dash a is given say y a dash and the third condition this is y b. So, let us call this is y b dash these 3 conditions are given at 2 different points, with that this is sufficient to solve the boundary value problem.

Now if I discretized a third order derivative. So, this I can write as I need not have to go all the way by Taylor series expansion and all, if I know the how to discretize the first order or second order that will be good enough for me to a discretization. So, I can write this yi double dash as dash so; that means, y this is the function as second derivative. So, if I discretize this. So, this is yi double dash minus yi minus 1 double dash by 2 delta x, we are taking a equispace point so; that means, we call as x i equal to a plus i delta x all are equispaced say i equal to 1 to N x 0 equal to a and xN equal to b these are the 2 grid points.

Now, this I can write as yi double yi plus 1 double dash as this will be y i plus 2 minus 2 y i plus 1 plus y i, then this one is yi minus. So, this is plus 2 y i minus 1 minus y i minus 2 by 2 delta x square. So, this is order h square correct accurate order h square no problem in it so; obviously, if I now substitute to this third order bond derivative discretization so; obviously, it is tri diagonalilty is lost because at the grid point x i what we have is i plus 2 i plus 1 i i minus 1.

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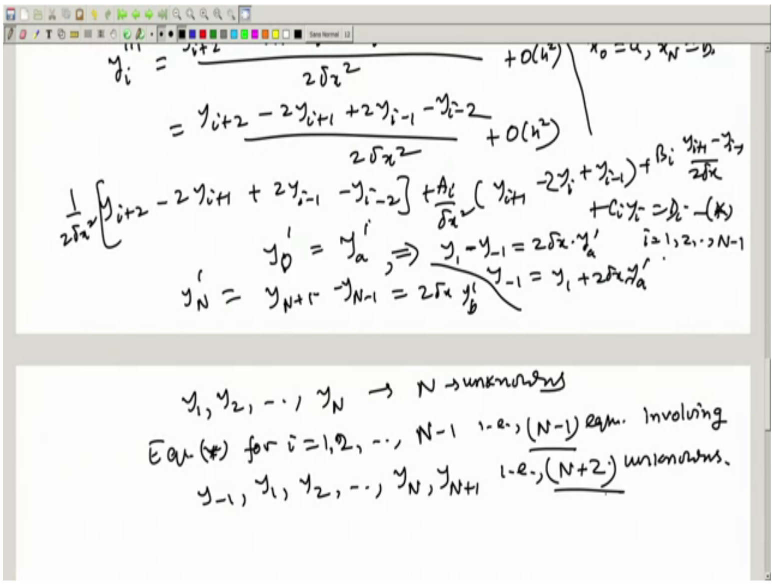


So, formula looks like little simplification.

So, 2 yi plus 1 plus oh this get cancelled so, 2 y i minus 1 minus yi minus 2 y 2 delta x square plus order h square. So, now, if I substitute what I get is a situation yi plus 2 minus 2 yi plus 1. So, 1 by 2 delta x square yi plus 2 plus 2 yi 1 minus yi minus 2 plus ai by delta x square yi plus 1 minus yi minus 2 yi plus y minus 1 plus Bi y i plus 1 minus y i minus 1 by 2 delta x plus Ci y i equal to Di. Now I can use this formula from I equal to 0 to n now why I can use this i equal to 0 to n is that, if I if I discretize this y i. So, what I have is yi dash yi 0 dash is given to be y i dash.

So, this I can write as y plus 1 minus y. So, this implies that this implies because this is the first boundary condition. So, this implies y i 1 minus i minus 1 equal to 2 delta x into y a dash. So, y minus 1 can be written as y 1 plus 2 delta x y a dash so; that means, y minus 1 can be replaced, similarly I can write this y N plus 1 minus y N minus 1 equal to 2 delta x y b dash 2 delta x this is your the other boundary condition y b dash equal to. So, or rather y N dash. So, so this process this is the y N dash. So, in this process what it shows that I can consider the boundary value problem now I can. So, unknowns are in this case y 1 y 2 up to y N because yN is also not given.

And I have to get this so; that means, this is N unknowns because y 0 is given now how many equations we have? Equations now these equations if I write i equal to 0 I cannot write, I can write I equal to minus 1 because we have yi minus 2. So, y i minus 2 means if I put i equal to minus 1. So, what I get is y i equal to 1 if I get so; that means, I can write this equation for i 1 2 N minus 1 because if I put i equal to 1 I get the least 1 is y minus 1 and if I put i equal to N minus 1. So, the highest one will be y N minus 1 means this is n N minus 1. So, N plus 1.

So,. So, let us call this is the equation star. So, equation star for i equal to 1 2 n minus 1 that is how many equations n minus equations involving how many unknowns involving how many unknowns we have is first if I put i equal to 1. So, y minus 1 y 0 I am not considering as an unknown. So, y 1 y 2 y N y N plus 1 so; that means, that is N plus 1 plus; so, 1 to N plus 1 plus 2; so, N plus 2 unknown unknowns now N minus 1 equation N plus 2 unknowns, but we have two more conditions. So, that gives you the 2 more conditions.

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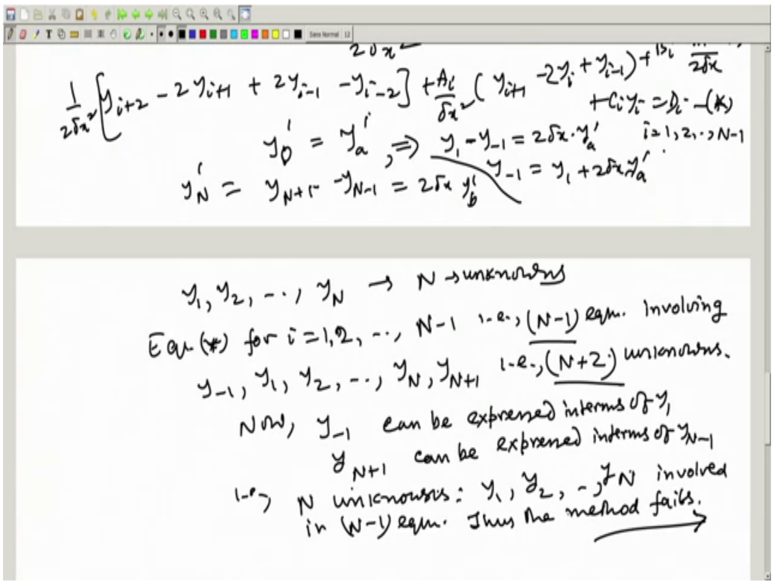


Now, y minus 1 is replaced by can be expressed or I would say y minus 1 can be expressed in terms of y 1 and similarly y n plus 1 can be expressed. So, they are no longer treated to be in terms of y N minus 1. So, these two are out. So, these two if strikeout. So, still that is N number of N number of N unknowns N unknowns involved in N minus 1 equation. So, N unknowns who are the N unknowns y 1 y 2 y 1 y 2 up to y N. So, N unknowns involved in N minus 1 equation.

So, thus the method fails because, it does not cannot give a unique solution. So, the next class I will talk about how to. So, this procedure does not work because we are lacking by 1 equation. So; that means, one of the variable has to be taken free and we get a infinite number of solution. So, that is not a practically useful way strategy. So, in the next class we will talk about how to handle this kind of situation where the number of unknowns, or where the third order and higher order boundary value problem arise ok.