Mathematical Methods For Boundary Value Problem
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Lecture - 13
Finite Difference Method for Higher Order BVP

Welcome back. Now we are reducing the second order linear Boundary Value Problem to a linear system of equation; that means algebraic equations.

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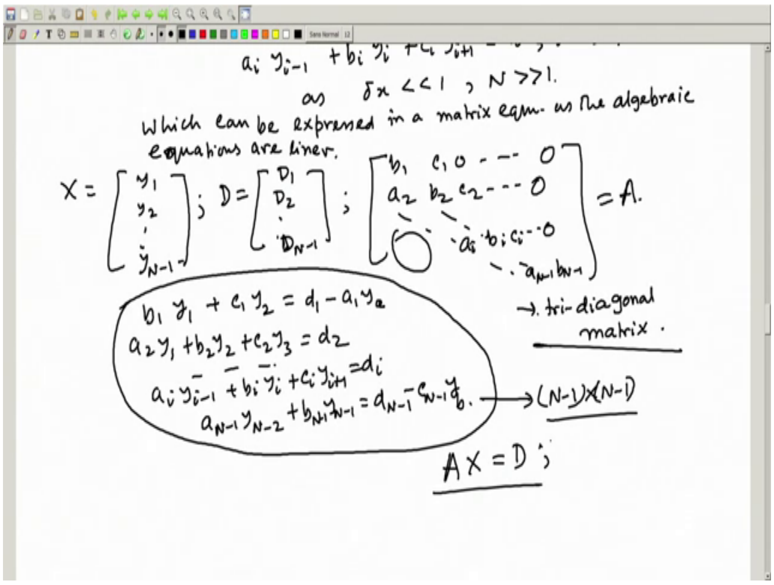

So, we get a algebraic system of equations A X equal to D given by this manner and what we talked about is this A has a special features, A is a tri diagonal situation, so; that means, what we have A is that all the diagonal elements diagonal positions there is an entry and just before the diagonal position or if I write it little more clearly here I think.

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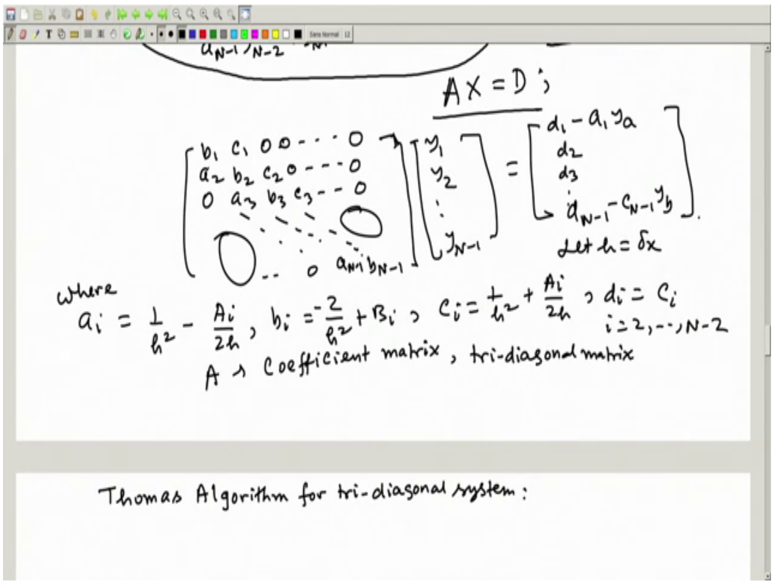

So, A is given by, so first entry is b 1, so now, what we have let us forget about matrix form if I write x equal to D form. So, b 1, c 1 0 0 etcetera 0, a 2, b 2, c 2 all entries are 0 like that way a 3, b 3, c 3 all are 0 this way you are going down; going down here all 0. So, the last element is b N minus 1 a N minus 1 and these are all 0 set and is multiplied with this multiplied with the unknown vector.

So, in this case the unknown vector is y 1 multiplied with y 1 y 2 etcetera, y N minus 1 equal to all these known quantities. Now, known quantities from here it is what we have is d 1 minus a 1 y a, d 2 d 3 etcetera this is d N minus 1 minus c N minus 1 y b now, so this is a matrix system. Now, what these a i, b i, c i are? So, that will be obtained from here, so one can write the form of the expression for a i, b i and c i all these coefficients can be very easily expressed in this way.

So, here a i where a i equal to 1 by h square minus A i by 2 h basically this is a coefficient of y i minus 1 and b i is 2 by h square minus or plus B i minus 2 h square plus B i if we look into this. So, B i is the coefficient of y i, so y i coefficient is minus 2 delta x square plus B i. So, I have written h equal to delta x let because instead of writing in the big form h equal to delta x is better way of presenting the or simple way of presenting the coefficients and this is A i by 2 h and what I have is d i equal to nothing, but the non homogeneous term c i.

But d i apart from this is i equal to 2 to N minus 2 at d 1 the first one and last one is little different because the y a and y b are known; that means, y 0 and y N are provided. So, that is why we can take these wrong the left side to the right side. So, the solution can be obtained by inverting this tri diagonal system.

So, this matrix A, which we call the A the coefficient matrix here which forms a tri diagonal system; tri diagonal matrix; now, there is some simple algorithm direct algorithm to solve the tri diagonal. Now tri diagonal means its very simple situations to handle because, one can do is that if we like gauss elimination method; that means, we can reduce this tri diagonal to a triangular matrix.

So, once we have a triangular matrix life is very simple because, triangular matrix means we can obtain the solution in a straightforward manner. So, there is a algorithm for these and that algorithm is if was given by Thomas. So, this is called the Thomas algorithm for tri diagonal matrix tri diagonal system.

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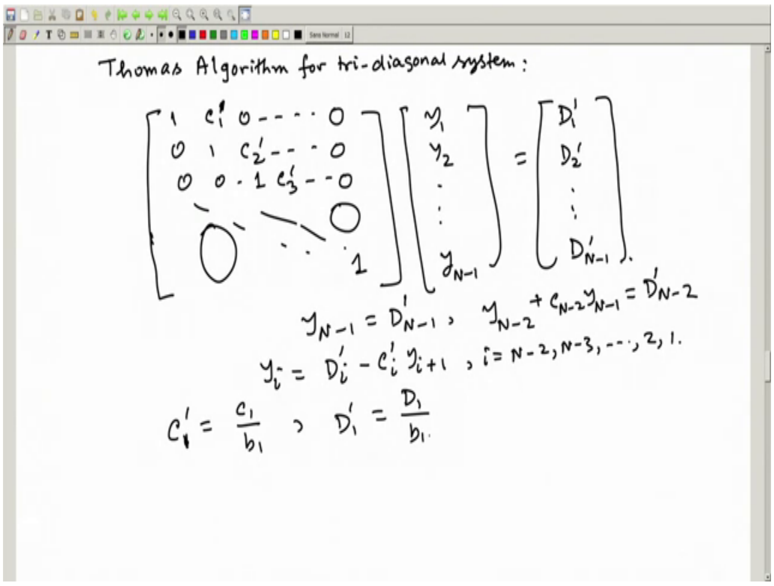

So, idea in the Thomas algorithm is that in the matrix A what we do is we in the Thomas algorithm that we bring one to the diagonal position and a i position we eliminate all this by doing a series of row operation and column operation, what we do is we bring diagonal elements to the one in the diagonal elements and a i positions are eliminated.

So, a i are all met with 0 so; obviously, in that process the c i, so will be change as well as these d i elements also will be changed. So, we do this elementary row operation that is subtracting one row in the other and that process. So, a reduced form will look like that 1 c 1 dash 0 0 0 it cannot be the same as they want sure it is a change form let us call that c 1 dash, then 0 1 c 2 dash and remaining positions are should be all 0 itself. So, that should not be disturbed.

So, and then the last element is 1 because that is the diagonal element. So, let us make this is the reduced form and unknown vector is y 1, y 2, y N minus 1 and this side they also get changed. So, I call this as it was D capital D was termed sure let us call this is as D 1 dash D 2 dash etcetera D N minus 1 dash. Now; obviously, very simple situation now, if I know this c i dash and d i dash I can obtain the solution very straight away.

So, last equation will give you y N minus 1 equal to D N minus 1 dash and then next one will be next one is diagonal position 1 and next to diagonal is y N c N minus 2. So, what do you have is rather c N minus 1, so you have is 1 when y N minus 2 plus c N minus 2 y N minus 1 equal to D N minus 2.

Now, already y N minus 1 is known, so I can get the y N minus 2. So, in other words we go by back substitution; that means, I can write the solution as y i equal to D i dash minus c i dash y i plus 1 and i I substitute as N minus 2, N minus 3 and so on 2, 1 etcetera.

So, with that we get the complete solution, but question remains that how to get this c i dash D i dash because a i, b i, c i and d i this is already declared. So, that we obtain from the discretized equation, so that is fixed now through the knowledge of this a i, b i, c i and d i we have to get the this D i dash and c i dash itself. So, what we that is what the Thomas algorithm is.

So, you can write this c 1 dash equal to c 1 by b 1 this is very simple because in the first equation you have b 1 and c 1 know a. So, I just divide by b 1 and I get c 1 dash equal to c 1 by b 1 and D 1 dash equal to D 1 by b 1.

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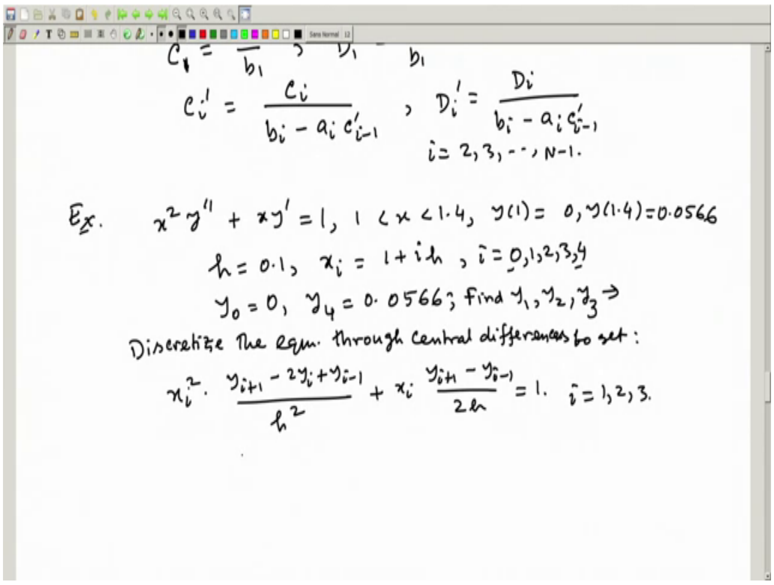

The first one is not at all the problem, but the next one is c i dash equal to d i dash minus v i plus 1 v i sorry I made a mistake here c i dash next 1 which is not 1 is c i by b i minus a i c dash i minus 1 and D i dash equal to D i by b i minus a i c dash i minus 1 denominators itself c i and D i. So, this we put the value as i 2, 3 etcetera N minus 1 what about the range of the or of the subscript i s i 1, 2, 3 1, 2, 3 N minus 1. So, this is all about the Thomas algorithm.

So, we have when we have a discretized equation we already know what is c i, b i, a i s are c i, a i, b i, c i and D i s are already declared. So, through that we compute these all this c i dash and D i dash. So, once I know these I come to the solution part by the back substitution I get the solution like this way because this is the forming a lower triangular matrix I guess this is triangular matrix.

So, from the triangular matrix without any hazard we get the solution in a straightforward manner, so this is the general principle. So; obviously, one has to write a computer program to get the solution because as we have noted that delta x because our order of accuracy is second order.

So, delta x is quite small very very small quantity, so that implies that capital N is quite large. So, a large set of grid points are involved, so; that means, the order of the matrix A is quite large. So, even if my boundary value problem the boundary is 1 of length 1 that is b minus a is 1 and I choose delta xs point 01 I get 99 points because N minus N is 100 we get a 99 number of points.

So, let us illustrate these by a simple example. So, consider this one x square y double dash plus xy dash equal to 1, 1 less than x less than 1.4 now very small interval 0 and y at 1.4 is given to be 0.0566. Now, first step is discretize this equation. So, we now first step is choosing the step size say let us take step size and then accordingly define the grid points. So, this is grid points are 1 plus i h, so i is varying from 0 that is the first point 1, 2, 3, 4.

So, 0 and 4 are the boundary points, so y 0 is given to be 0 and y 4 is given to be 0.0566. So, find y 1, y 2, y 3 these three unknowns to be obtained for this case. So, if I discretize this equation; discretize the equation through central differences to get. So; that means, what we are doing is we are replacing these y double dash and y dash by the central difference formula. So, what I get is x i square y i plus 1 minus 2 y i plus y N minus 1 by h square plus x i, y i plus 1 minus y i minus 1 by 2 h equal to 1.

So, i is ranging from 1, 2, 3 because we have three unknowns we need three equations. So, I have put the value of i as 1, 2, 3. So, if I write the put the value of i, so first equation will be or if you do little manipulation. So, what I get a set of equations.

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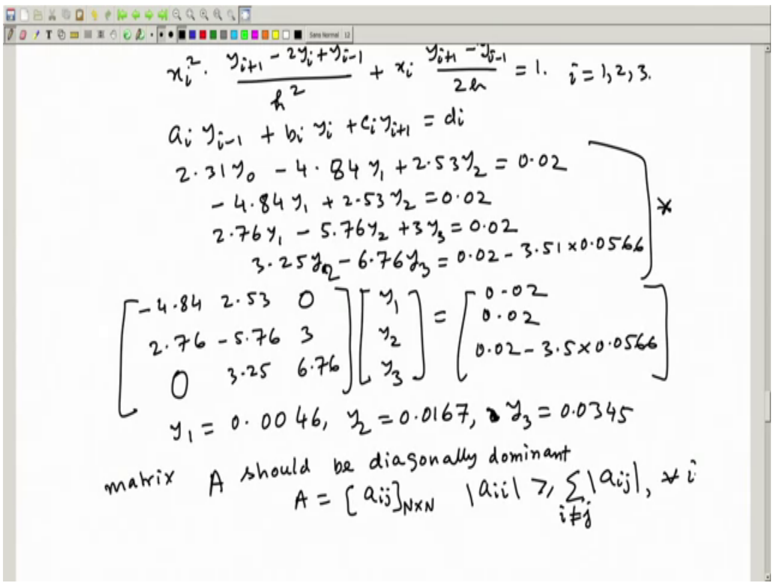

As 2.31 y is 0 minus because if I put i equal to 1 they will be y 0 is appearing here no y i minus 1. So, if I put i equal to 1 y 0; y 0 is appearing and so first what you have to do instead of writing like this way. So, first what I do is we write this as a i, y i minus 1 plus b i y i plus c i y i plus 1 equal to d i. So, i equal to 1 this is y 0 goes to the other side, so we get 2.31 y 0 minus four 4.84 y 1 plus 2.53 y 2 equal to 0.02 next one. So, this if I transfer to the other side using the boundary condition y 0 is 0, so there is no value to this. So, 84 y 1 plus 2.53 y 2 equal to 0.02, next equation gives you 2.76 y 1 5.76 y 2 plus 3 y 3 equal to 0.02

And the last equation is 3.25 y 1 minus 6.76 y 3 equal to 0.02 minus 3.51 into 0.0566 this is the boundary condition. So, we get it a set of equation 3 by 3 equation like this way. So; obviously, this set 3 by 3 equation is not a big problem, but if I want to for a practice purpose if I want to put in a matrix form, so it will look like minus 4.84 the entries are 2.53 0 3 by 3 matrix.

So, minus 2.76 here this will be 2.76 minus 5.76 3 and sorry this will be y 2 not y 3 this is y 2 and this location is 0 3.25 6.76. So, this is multiplied with y 1, y 2, y 3 and the unknown known constants are 0.02, 0.02, 0.02 minus whatever comes 0.566. So, choices yours you can apply the Thomas algorithm or you can do by basically Thomas algorithm is also a kind of gauss elimination method, but in a systematic manner because this is the direct method there is no iterations or any approximations are involved here. So, this is a direct elimination method.

So, if I do also the elimination and this for 3 by 3 one can work out simply and we get the solution 0046, y 2 equal to 0.0167, y 3 will be 0.0345. So, for a small system this is a very simple procedure one can do the elimination and get the set of solution, but if its a large one, so; obviously, we have to write a computer program and by using the Thomas algorithm to get the correct solution algorithm wise. Now, few things are there one is here stability issue does not arise because we are not repeating the process; that means, there is no marching for what procedure like time dependent situations; that means, one step and step by step we are not doing.

So, we are solving in one stroke itself. So, the stability is not issue, but we have the solution should be bounded and that is possible if A is matrix A should be diagonally dominant. Diagonally dominant means basically what it means is the absolute value of all the non diagonal elements in a row should be bounded by the diagonal element, so; that means, if say generally if I call a i j say sum in course in matrix, then what should be for any row a i i the diagonal element should be greater than equal to sigma a i j for all I not equal to j.

So, this is the stands for the diagonal dominance, so; that means, the for all I which are not equal to j for all this will happen for any choice of i. So, this is situation, so; that means, all the absolute value of the non diam entry diagonal entries sum of absolute values of all the non diagonal entries are bounded by the diagonal elements.

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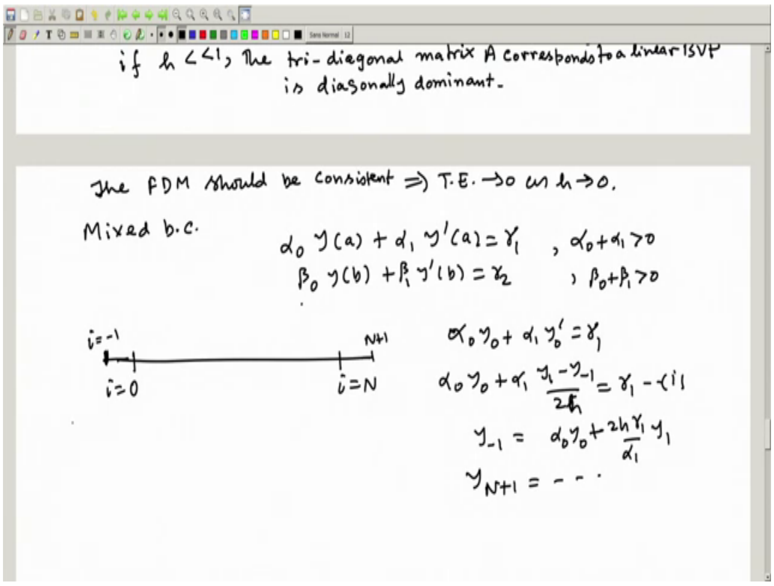

Now, if h is sufficiently small the tri diagonal matrix matrix A which we of course, one A corresponds to a linear BVP is diagonally dominant that can be shown very easily. So, what we require for this case e is that method should be consistent; method should be consistent, so; that means, the truncation error; so, the scheme rather or the finite difference method or the definite difference scheme. So, the if finite difference should be consistent.

And that is the case this implies truncation error should tends to 0 as h tends to 0. So, that is if we have a problem which has a solution, then and you are using a consistent numerical scheme that is good enough to have a converge solution this implies a converged solution. Now, one assumption we made that we have taken the boundary conditions to be given by the function value itself, but the boundary condition can be a mixed time. So, you have alpha 0 y a plus alpha 1 y dash a equal to gamma 1 say beta 0 y a b plus beta 1 y b equal to gamma 2.

So, in that case if all these y 0 y 1 are not sorry all these alpha 0 alpha 1 cannot be 0. So, this alpha 0 plus alpha 1 should be greater than 0, if I consider this coefficients are positive a limit as 0 plus beta 1 should be greater than 0 both cannot be 0 at a time, so this is the bottleneck. Now, there is a two ways if I choose a fictitious point, so; that means, say we have at the point 0 i equal to 0 and this is i equal to N.

So; that means, the derivative, so what do you have is alpha 0 y 0 plus alpha 1 y 0 dash equal to gamma 1, but what I need is y 0 which is not given straight away. So, y 0 dash has to disgrace now, y 0 dash you cannot discretize by central difference because you have now definition beyond this. So, one remedy is you introduce a fictitious point as fictitious grid point as i minus 1 and here you introduce another one as N plus 1. So, if I do that, so now, our grid points is from minus 1 to N plus 1.

So, the symmetrically we have just taken i equal to 0 is the same step size i go back and get i minus 1 and this one we get at i N plus 1. So, if I do that I can write as y 0 plus alpha 1 y 1 minus y minus 1 by 2 delta 2 h equal to gamma 1. So, this is one formula and from here we can write y minus 1 as something like minus alpha 0 y is 0 and whatever this gamma 2 h gamma 1 by alpha 1 y 1 know I think this the both will be positive because this is, so this is positive no minus sign. And same way y N plus 1 also can be expressed in this way. Now, so we discretize this now instead of, so our unknowns are in this case is y 0, y 1, y, so N plus 1 unknowns.

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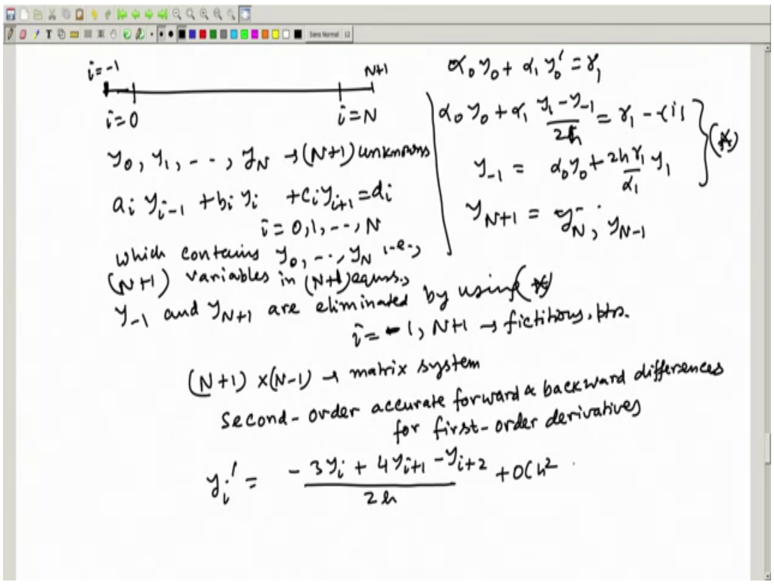

So, if I discretize is a question for all the N plus 1 points, so; that means, now I write this equation a i, y i minus 1 plus b i y i plus 1 plus c i y i sorry b i y i c i y i plus 1 equal to d i I write this from i equal to 0, 1, 2 up to N. So, which contains all this y 0 to y N that is N plus 1 variables in N plus 1 equation same number of equation now equations.

Now, what I did not say is here the y minus 1 and y N plus 1 are eliminated by using; by using this b c, so let us call this star. So, y minus 1 and y plus 1 eliminated by; eliminated by using star because here we can write directly y minus 1 in terms of y 0 and y N y 1, y N plus 1 we can write in terms of we can write as y N and y N minus 1 because N minus 1.

So, in that process one can handle this situation, but this is a called the i minus 1; i minus 1 and N plus 1 are fictitious point fictitious points. So, fictitious point now this fictitious points are not advisable when you have a non-linear equation because that creates hampers the stability if you have a non-linear situation because that is not physically defined these.

So, that is why for linear set of equations this is good. Now, one another drawback in using these fictitious points that now instead of N minus 1 cross N minus 1, we have now N plus 1 cross N plus 1 system of equation the metric system. So, this gives a higher number of competitions.

Now, one remedy is we can derive second order accurate forward and backward difference formula differences for y for first order derivatives for first order derivatives. So, this formula is like this y i dash I can write as minus 3 y i plus 4 y i plus 1 minus y i plus 2 by 2 h which is order h square second order accurate one can verify by Taylor series expansion. So, this is a forward difference because all what we have used are forward to forward difference of i forward points and the same way.

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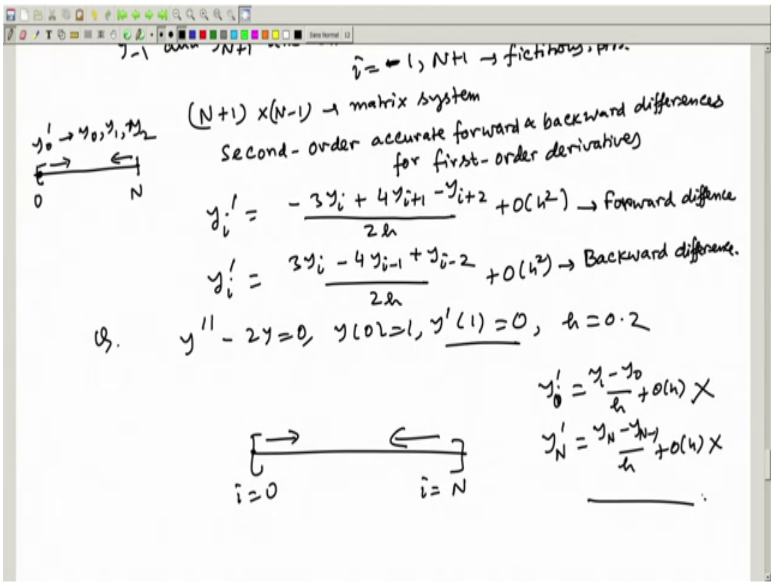

So, this I can use at the first point, so; that means, when I have this 0 y 0 dash, now I can write in terms of y 1 y rather y 0, y 1, y 2. Similarly, the when you are the other end, so; that means, so this is 0. So, you can go for 0 forward and for n you have to go backward. So, backward difference formula is same type over the h square this is backward difference formula. So, this form of representations are found to be much useful for handling this derivative conditions situation.

So, one of the example one can work out for this. So, what do you have is here the derivative at the last point, so; that means, we have to go backward; so; that means, there is no other when you are at the extreme points. So, you can go back only sure you are using this backward difference formula at the end point and when we are at i equal to 0 this is i equal to, so you have to use the forward difference formula.

So, this second order we could have done for the first order also, but that hampers the accuracy of the method. So, that is why we have not used the first order forward first order backward the same way one can do this once your y 0 dash can be written as y 1 minus y 0 by h which is order h. So, this is not advisable similarly y N dash can be written as y N minus y N minus 1 by h order h again this is also not advisable, first order formula because we are using other differences are the second order accurate ok.

So, this is all about the second order two point boundary value problem with second order derivatives or differential equations with two conditions are prescribed at two different points. So, now, will you go to the next lecture, we will talk about the higher order boundary value problems with complicated boundary conditions and so on.

Thank you.