Mathematical Methods For Boundary Value Problem
Prof. Somnath Bhattacharyya
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 12
Finite Difference Method Linear BVP (Contd.)

Welcome, now we will discuss the Boundary Value Problem to solve as a boundary value problem. So, the previous day we talked about the shooting method; there what we did is we have converted the boundary value problem to a equivalent initial value problem and then that initial value problem was solved. Now, first we start with a linear boundary value problem for the sake of simplicity let us take a 2 point boundary value problem which is linear.

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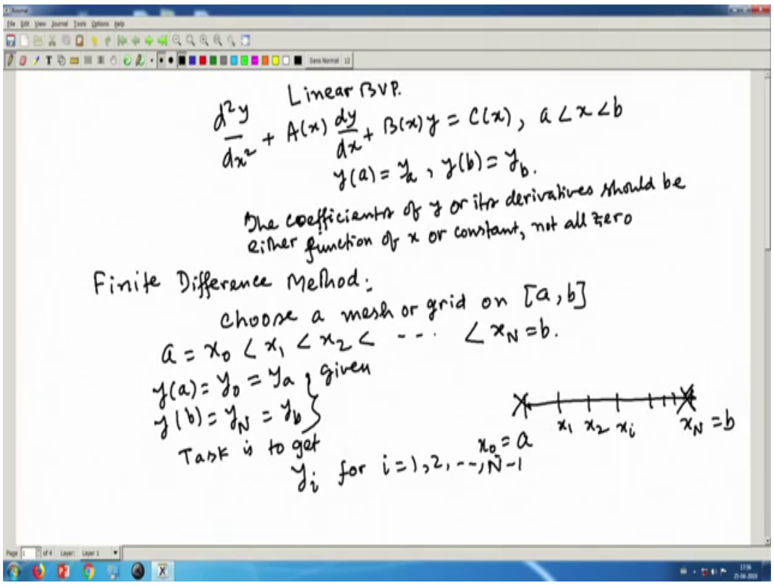

So, any linear boundary value problem can be expressed in this manner d 2 y dx 2 plus A x; dy dx plus B x; y equal to C x and say the boundaries between a to b. And what we have given is y a let us call this is y at a is y a and y at b is y b. Now if a; the coefficients in this case coefficients of y or its derivatives should be independent of should be either function of either function of x or constant; not all 0. So, this is for the linear BVP; so let us give the topic as linear BVP. So, a generally a linear BVP can be of this form; it cannot have any other form other than this.

Now, a the constant A x, B x, C x can vary it can have the constant or it can be function of x this coefficients can vary, but it cannot have y or its derivative as a variable in the argument of this coefficients A, B, C; so that is the idea. Now in finite difference method this is what we will be talking about; finite difference method. So, what we do is we choose a set of mesh points or discrete points within the interval a. So, what we do choose a mesh or grid on a; b.

So, let us define say a; which I call as the first grid point x 0. So, x 1 x 2 etcetera and this is the last one say capital X; x capital N; this is the b. So, basically we have this mesh point or grid points the discrete number of points. So, this is we call as x 0 which is equal to a; x 1, x 2 like this way. So, x i say and this is x N equal to b. Now, what is our task is now what we have given y at a. So, is y 0 which is given to be y a and this is given and y at b; y at b which is we call as y N this is given to be y b these two are given.

So, task is to get y i for i equal to 1, 2 all this intermediate grid points. So; that means, except this two boundary point where the function value is given. So, all this points are unknown to us. So, basically what we have is if we choose N number of a grid points between the two within the interval a b. So, and the boundary conditions are given at the two end points then we have to find out the solution at all this intermediate points.

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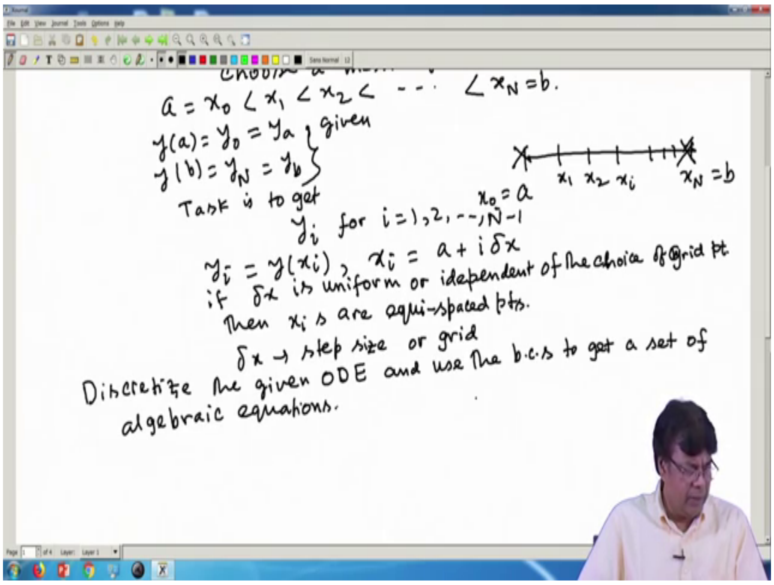

And how we define this xi's where y i is basically y at x i and x i is a plus i delta x. Now, in this case we are calling delta x if delta x is uniform; that means, it is independent of the location i. So, is in uniform or independent of the choice of i; choice of mesh grid point then xi's are called they xi's are equi spaced point; so; that means, we have all the grid points are equal interval apart. So, between any two we have the difference or the interval distance is delta x that is the thing is assumed for the time being. So, it may not be all the time for some practical purpose; this step size or delta x the grid size.

So, delta x 1 can call as step size or grid size; so, they may not be uniform, may be uniform; so for the time being, so let us assume it uniform. Now in the finite difference method this is the thing we have to do, but how we do? What we do is we discretize this equation; discretize the given ODE and use the boundary conditions to get a set of algebraic equations; set of algebraic equations. Now if it is a linear boundary value problem, so we get a linear set of equations; if it is non-linear it will be a non-linear.

So, if it is a linear then we will be lucky what we can do is; we can reduce these we can cast it into a matrix equation which can be inverted and get the solution; if it is linear. But if it is not linear or what is termed as non-linear, so in that case this has to be some other procedure or iterative procedure has to be adopted. Now this discretization, so discretization means as we mentioned before that what we will be doing is will we have a; this is the differential equation.

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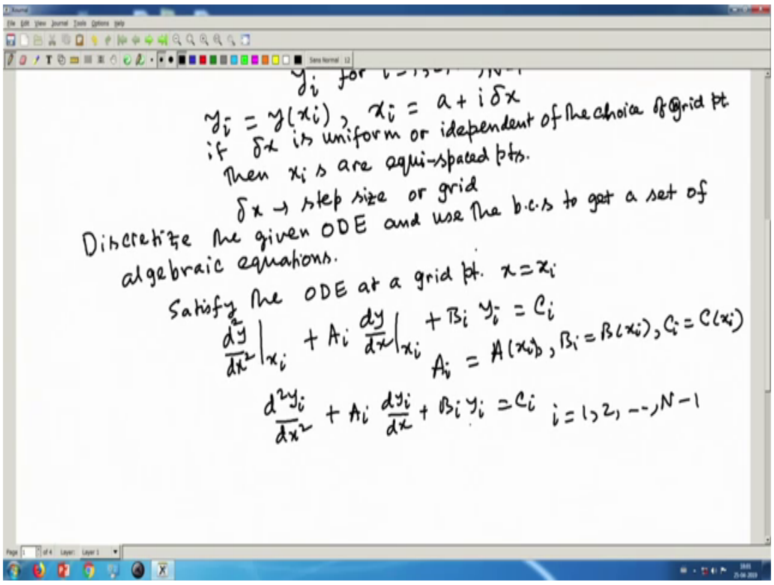

So, if we satisfy because the differential equation is valid between a to b. So obviously, it is valid at all this grid points x i. So, satisfy the ODE at a point say at a grid point arbitrarily; you can choose x equal to x i. So, we get d y i; d 2 y d x 2 at x i plus A i; d; dy dx at x i; x is fixed at x i, Bi and this is y i equal to C i, where A i is nothing, but Ai is A at x i. So, this is the notation we are using A; Ai is at xi; B; Bi is B at x i and C i; we denote C at x i. So, with that we can write this is as d 2; y i dx 2 plus Ai dyi; dx and B i; y i equal to C i.

So, this I can satisfy since the equation is valid between A to B; so this is satisfied for i equal to 1 to N minus 1. Now another thing is that what we want to find out is y i now how many y i's; y i we need to find except the two boundary points where the solution is provided. So, we need to find out at n minus 1 distinct points i equal to 1 to n minus 1. So, that is why I have satisfied this equation for i equal to 1 to N minus 1. So, that means, we have now n minus 1 equations and n minus 1 number of variables. So, but unfortunately what we have here it is variable y i is involved, but as well as its first and second derivatives are involved.

So, discretization is a procedure; there what we do we reduce these derivatives to a finite difference formula; that means, a difference of the function value at finite number of points. So, when we do that; so we come across a set of algebraic equation ok; so how to do about that one? Now there are several ways, so for the time being let us assume that Ai, Bi, Ci are continuous in interval A B.

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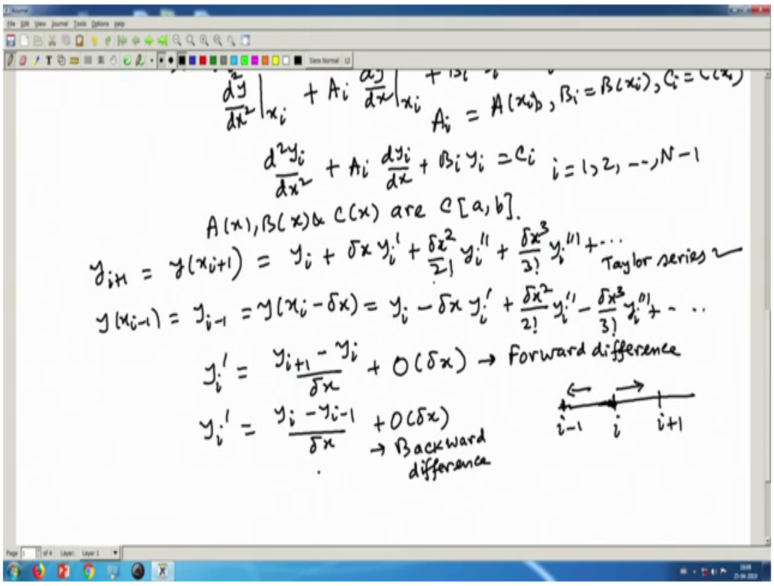

So, all this A; A x, B x and C x are say C a b; that means, it is continuous and maybe we can call it is a differentiable also if required. So, there is no discontinuity or jump discontinuity is not occurring. So, for the time being let us assume that these coefficients are all continuous function; no singularity and that means, no divided by 0 type of situation is appearing in this case.

Now, y i plus 1; we can write as this is what is y at x i. If I write my Taylor series expansion this is y i dash; delta x square by 2 y i double dash. So, this Taylor series expansion it relates the function value with its derivative; like this way. So, this is Taylor series expansion Taylor series about x i plus 1. So, this one is relating y i and its derivatives. Similarly, I can write y at x i minus 1 which is nothing, but our notation y i minus 1; now if i expand my Taylor series which I can write as y x i minus delta x.

So, if I expand by Taylor series; so instead of decrement, instead of increment in the previous one; here it is a decrement, so this is this will be minus. So, this is minus minus delta x y i dash, next one is delta x square y 2 factorial y i double dash then minus delta x cube by 3 factorial y i triple dash and so on ok. So, now we have two series expansion; y i plus 1 and y i minus 1. Now if I do one approximation as we did in the Euler method; that means, if I chop out the terms from the second order of delta x square delta x cube and all this.

So, then one simplification one reduced form we get is or one representation of y i dash can be obtained by this way. So, if I take only up to the linear order. So, what we will get these all the terms which will be of order delta x because delta x is getting divided. So, y i dash if I represent by difference between two points y i plus 1 and y i. So, then this is approximation from here because this Taylor series expansion; there is an approximation this is an infinite series.

So; obviously, we cannot handle this infinite series. So, that infinite series has to be truncated to a finite number of terms. So, if I am very lazy; I do not want to go for quadratic and cubic and all. So, I stopped at this stage that is up to the linear in delta x and from there I get a y i dash representation by given by this way. So, this is a representation of course, and this representation is order delta x; order delta x means whatever the term we have dropped. So, this is minimum order or minimum power of delta x is delta x to the power 1.

So, that is why we call it is the order of delta x as that will be delta x by 2, delta x square by 2; so the least order is 1. So, this is called the forward difference so; that means, what we did is here that this is the point say i; this is say i plus 1 and this is i minus 1. So, at this point i; I have used point forward to i; so i i plus 1.

The same way from the second one; I can write y i dash as y i minus y i minus 1 by delta x. Again it is a order of delta x and in this case it is a called the backward difference; backward difference because what why is that at this point I have used the point backward of i. So, that is why this for this difference is referred as the backward difference.

So, again these are all first order approximation so; that means, the representation will be handy or accurate provided to some extent accuracy; provided this delta x has to be quite small very very small. So, otherwise our accuracy will be very diluted.

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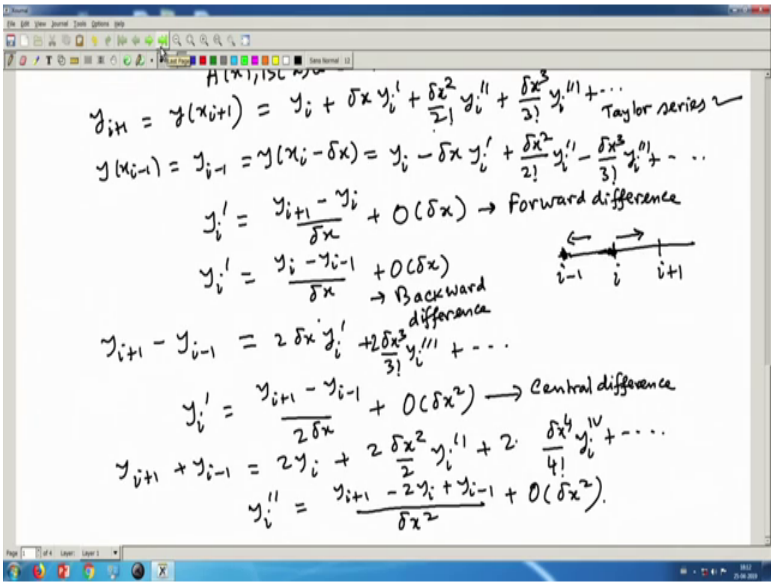

Now another representation if I want to go higher than that; so if I subtract y i plus 1 minus y i minus 1. So, if i subtract this two; so what I get here the term which are all positive they get cancelled. So, y i; y i get canceled; so what we left is delta x y i dash then next term delta x square get cancelled.

So, next term is delta x cube by 3 factorial; 2 delta x cube by 3 factorial y triple dash and so on. Now if I take a representation of y i dash by this manner y i minus 1 by 2 delta x.

So, the remaining term whatever we are dropping the term this is what is called the truncation; truncating the infinite series to a finite number of terms. So, that is why this is referred as a truncation error ok.

So, this truncation error is order of delta x square because the terms which is delta x square and higher are dropped ok. So, this is the one is referred as the central difference; central difference because you at i and you are using symmetrically two points i minus 1 and i plus 1. So, i the grid points is located centrally to this two grid points i plus 1 and i minus 1. So, what do you find that if we use the central difference; this is much better approximation, this order delta x square.

Whereas the others forward difference or backward difference we have a first order representation. So; obviously, it shows that if I use more number of points I get a more order accuracy ok. Now we have also y i dash and y double dash; now to get y i double dash what we do is we add this two. So, if I add we get y i plus 1 plus y i minus 1. So, here we get 2 y i and y i dash get cancelled. So, 2 delta x square by 2 y double dash, then next one will be delta x 2 delta x to the power 4 by 4 factorial.

So, in our notation delta x to the power 4; this is delta x to the power 4 not del should not be confused that del of x 2; x 4. So, this is y i fourth derivative because third derivative is out; so, and then mode number of terms. Now one thing you please note here that delta x is; obviously, a small quantity. So; that means, the terms are increasingly diminishing; as we move from move or and more number of terms this weightage of this terms or the contribution of this terms are diminishing because it is multiplied with the small quantity delta x power.

So, if suppose delta x is choose to be which is delta x is less than 1. So, delta x square is much lower than 1, delta x to power 4 is much lower than 1. So, as we go from terms this side, so I get the more lesser and lesser weightage in delta x. Now one representation for y i double dash can be put in this way; y i double dash is y i plus 1 minus 2 y i plus y i minus 1 by delta x square and this will be order delta x square because this is divided with delta x square; so delta x square.

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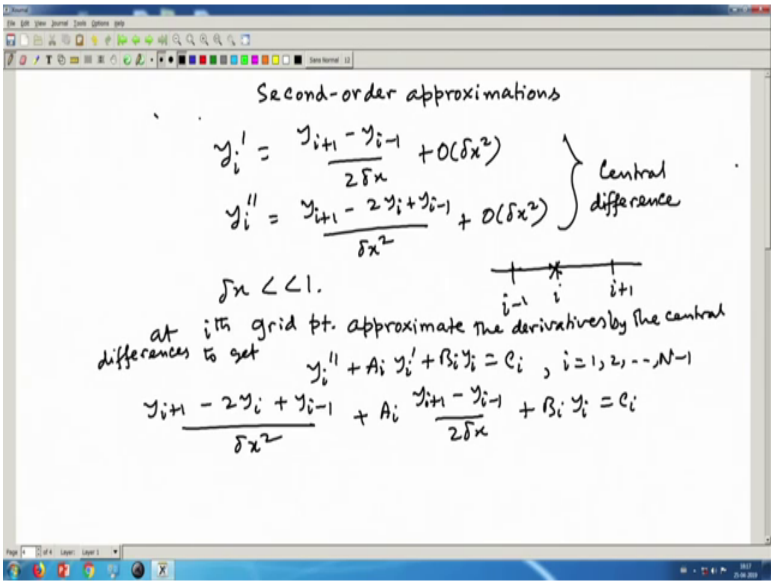

So, now we have the representation as y i dash, if I go for second order representation; second order approximations can be expressed in this way, second order approximation only. So, this is for the first order derivative y i double; y i dash is y i plus 1 minus y i minus 1 by 2 delta x, order delta x square and y i double dash is y i plus 1 minus 2 y i plus y i minus 1 by delta x square plus again order delta x square.

So, this is also a central difference; central difference formula these are all central difference representation, central differences is also can be said in a very say simple way as central difference. Now because our grid is i and we are using i minus 1 and i plus 1; now in finite difference method this most of the cases we will be very happy with having a central difference representation itself second order accuracy. Because if I choose delta x is delta x is normally is very very small.

So, say for example, delta x is 0.01; so delta x square will be 10 to the power minus 4 or so. Now our equation what which was satisfied at a grid point i; so our equation at at x equal to at the ith; x ith grid point or you can call ith grid point approximate the derivatives by the central differences; approximate the derivatives by the central differences to get this reminder that what our equation was looking like is y i double dash plus A i; y i dash plus B i y i equal to C i and we are writing is for i equal to one to n minus 1.

Now, we are replacing by this formula. So, what i get is y i plus 1 minus 2 y i plus y i minus 1 by delta x square plus A i; y i plus 1 minus y i minus 1 by 2 delta x plus B i y i equal to c i. So, this process is called the discretization; so this step is called the discretization.

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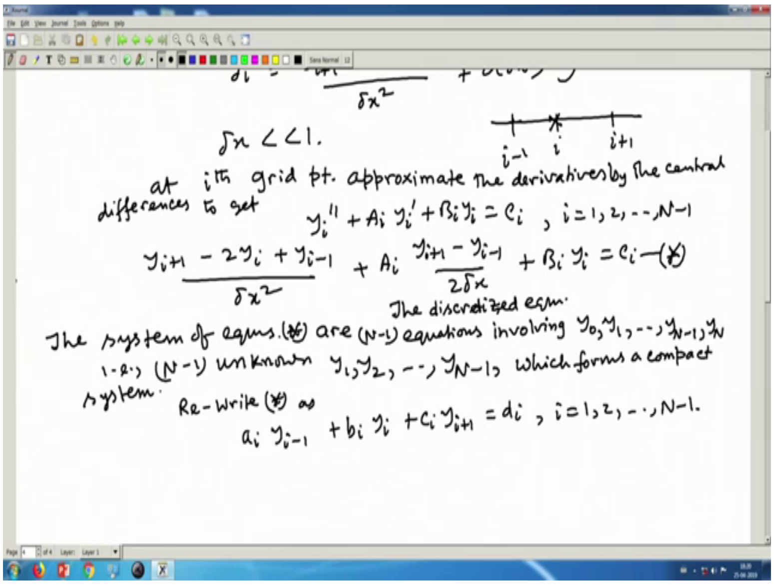

So, or the disretized; disretized equation; now so let us call this is a system of equations star I am varying i equal to 1 to n minus 1. Now so; that means, this equation star the equation star; are or the system of equation star the system of equations this is a system set of equations.

So, the star are how many equations? N minus 1 equations involving n minus 1 equations; which is involving y 0; say for example, if I put i equal to 1. So, i get y 0 y 1 and up to y N minus 1 and y N. So, since y 0 and y N are known that is n minus 1 unknown y 1, y 2, y N minus 1. So, which forms a compact system; compact system in the sense that same number of equation and same number of variables.

So, I can put in this way that rewrite the equation a i; y i minus 1 plus b i y i plus c i y i plus 1 equal to d i i equal to 1, 2 etcetera n minus 1 i equal to i is varying from 1 to n minus 1. So, this forms a set of equations; now this equation can be solved directly.

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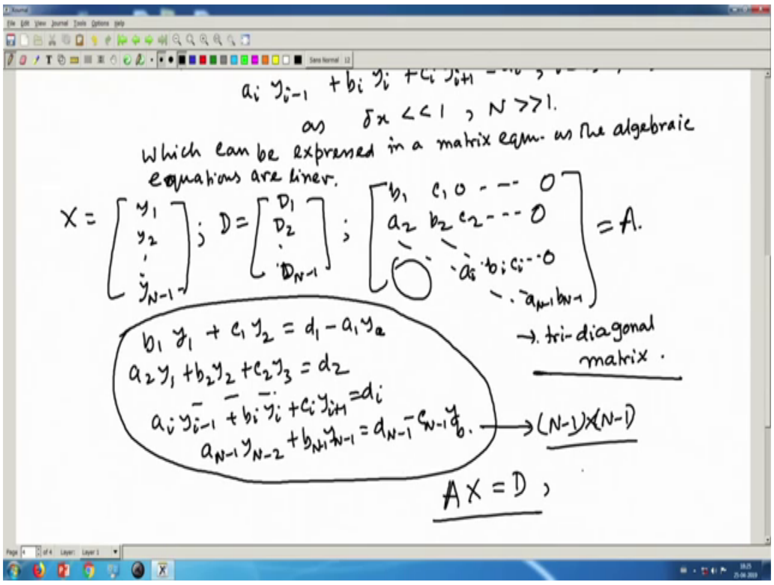

But, in the since we are choosing as delta x is small. So, n will be quite large so; obviously, just like that we cannot solve by elimination and all, some algorithm is required. Now, this is a set of which can be expressed in a matrix equation as the algebraic equations are linear algebraic equations are linear.

So, if I define a vector say X equal to say this vector y one y 2 y N minus 1 which is a vector of unknown and a vector say D which is the vector of all the known quantities d 1, d 2, d N minus 1. I would say this is d n this d may not be same; so most of the cases, so may not be same. So, let us take D 1, D 2, D n minus 1; so I can define this. So, first row will be i equal to 1; if i put i equal to 1, now this a i; y 0 now a i; y 0 is not a unknown.

So, let us look into the form how what will be. So, a i; y a 1 i equal to one a 1, y 0; y 0 is known; so that should be taken to the other side. So, what we have is unknown is b 1, y 1 plus c 1 y 2 equal to d 1 minus a 1 y 0; y 0 is given to be a. Next one is a 2; y 1 plus b 2; y 2 plus c 2 y 3 equal to d 2 nothing is known in this stage. Similarly all these things any ith has we have written a i; y i minus 1 plus b i y i plus c i y i plus 1 equal to d i. And the last one is a N minus 1; y N minus 2 plus b N minus 1; y N minus 1 and d N minus 1 minus c N minus 1; y b because that is y N which is given.

So, this is a set of equations which are forming which are N minus 1 set of equation. So, N minus 1 cross N minus 1 matrix is formed and this matrix most of the places will be 0. Because see first row have two unknown y 1, y 2; second one is y 1, y 2, y 3 third similarly like that way. So, what we have a if I write in a matrix form. So, this is b 1, c 1 all are 0 all are 0 then a 2, b 2, c 2 remaining portion are all 0.

So; that means, any generally we have the diagonal position say b i just before the diagonal position and next to the diagonal position. And remaining portions are all comes out to be 0 and the last one is a n minus 1; b n minus 1, so this is the coefficient matrix a. So, this form of the matrix is called the tri diagonal matrix. So, a tri diagonal matrix system is formed ok.

So, we have a system of set of equations now; this is the matrix equation AX equal to D, where A is the coefficient matrix given by this way. And if I; X can be obtained by inverting this by solving this matrix equation AX equal to D. So, the next class we will talk about how to solve this tri diagonal system an algorithm will be discussed.

Thank you.