Mathematical Methods For Boundary Value Problem
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Lecture - 11
Finite Difference Method Linear BVP

Well, so we are discussing on the shooting method; that means, a method by which we can transform a Boundary Value Problem to a initial value problem.

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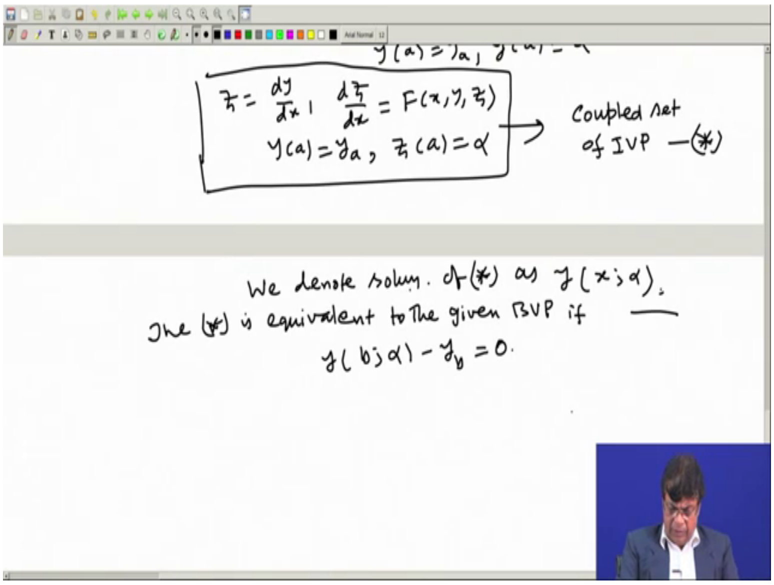

So, this was the thing we have discussed before that is we have reduced the given boundary value problem to a set of initial value problem given by this way. Now, the trick is how to choose alpha? Now, I need to choose alpha if I call this; obviously, the solution of this initial value problem depends on how the alpha is determined alpha is taken. So, in order to highlight that, we have denoted the solution as y x alpha.

So, the star is equivalent to the given BVP provided the other condition is satisfied, if y b alpha minus y b is equal to 0, y b alpha should be equal to 0. So, that is the condition need to met that is the solution should be equal to the given boundary condition at the other end. Now, so this gives us a clue that how to obtain this alpha; that means, alpha should be choosed in such a way that this equation let us call this double star is satisfied.

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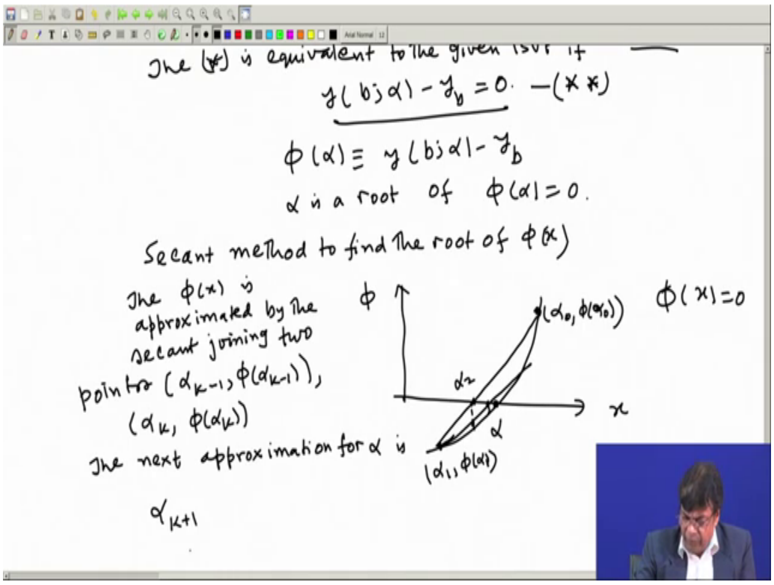

So, if I say this as a function of alpha which I call this as a y b alpha minus y b. So, alpha is a root of phi alpha equal to 0 correct. So, if I call this as the this difference as phi alpha because it is a function of alpha. So, alpha will be the root of phi root of this phi alpha.

Now; obviously, we do not have any function form of phi. So, we have to now need to find out this alpha the root of alpha; root of this phi alpha in some numerical procedure. Now, one of the numerical procedure is very simple one is the secant method to find the root of phi alpha. Now, what in the second method does is that see for example, if I draw in a, so this is a x and this is phi; so, phi x equal to 0 so; that means, where the phi is meeting the x, so this is our root alpha.

So, we need to a secant method to find the root of I would say phi x because alpha if I call as the root, so this is phi x; so at x equal to alpha this is equal to 0. So, in the secant method we choose two points to initial guess some methods bisection method and all interval where in between the phi has a change of sign. So; that means, so we consider that the root is lying within this interval. So, phi alpha 1 this is the point and this is alpha 0 phi alpha 0 and then what we do is we replace the graph by the secant by joining the straight line and then next guess for the root is where it meets the x axis.

So; that means, now the graph is replaced by the straight line or the secant joining alpha 1 phi alpha 1 point alpha 1 phi alpha 1 and alpha 0 phi alpha 0. So, the next point we say alpha 2 we call that. So, we get this point say something like this and then again we join by this two and wherever it meets the x axis, so this is meeting here. So, like that way we calculate the successive approximation for the root.

So; that means, the phi x is approximated by the secant joining two points; two points let me write points two points let us call alpha k minus 1 phi alpha k minus 1 this is the 1 point and alpha k phi alpha k this is at any intermediate stage. So, the next approximation for the root; so the next approximation for alpha is we call as alpha k plus 1. So, this will be the point where the secant meets the x axis.

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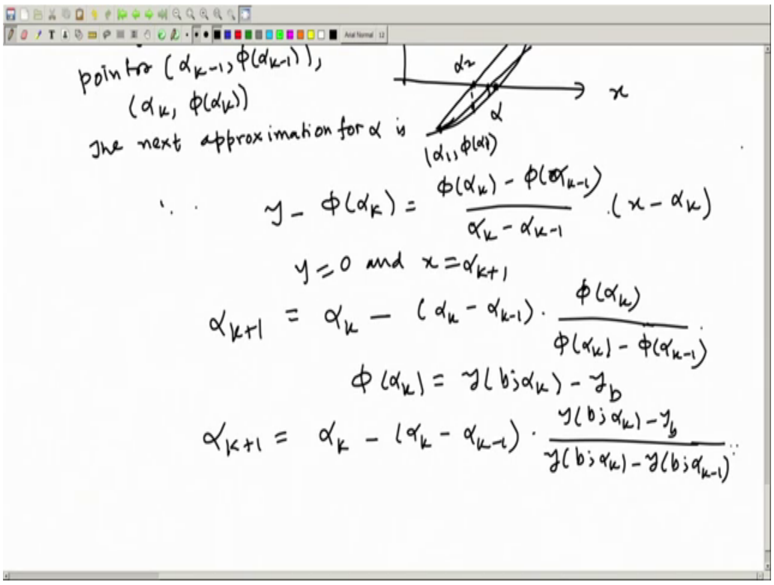

Now, how we draw the straight line, how I write the straight equation of the straight line is I can write the equation of the straight line as y minus say y minus phi alpha k equal to. So, phi alpha k minus phi alpha k minus 1; phi alpha k minus 1 by alpha k minus alpha k minus 1 this is the slope of the straight line and y equal to mx plus y minus y 1 into x minus x 1 in and that formula y minus y 1 equal to y 1 minus y 0 x 1 minus x 0 into x minus x 1.

Now, so y equal to 0 and x equal to alpha k plus 1. So, that gives you alpha k plus 1 equal to, so where wherever this is becoming 0 we call as the alpha k plus 1. So, this gives us if this is alpha k plus 1. So, this is alpha k alpha k plus 1 alpha k and this is alpha k and this is 0, so minus alpha k minus alpha k minus 1 in to phi alpha k by phi alpha k minus; phi alpha k minus 1 right.

This is the formula I hope I have done correctly this should be the formula x is. So, alpha k plus 1 no I think this is plus we will write be because this is going alpha k plus no because this is y 0, so this is minus; minus was correct; so this is minus. Now, what the definition of phi alpha is given by this ratio phi alpha k basically, phi alpha k is the solution of the IVP with missing initial condition as alpha k minus at the point x v. So, if I substitute this what I get alpha k minus; alpha k minus; alpha k minus 1; alpha k minus alpha k minus 1 and this will be phi alpha k is y b alpha k minus y b by y b alpha k minus y b alpha k minus 1.

So, from this formula by using the two successive approximation, we can construct the formula to successive approximation alpha k and alpha k minus 1 we get the alpha k plus 1. So, to get these what we need to do is we have to compute the initial value problem for y b alpha k and y b alpha k minus 1, so; that means, the process to be repeated.

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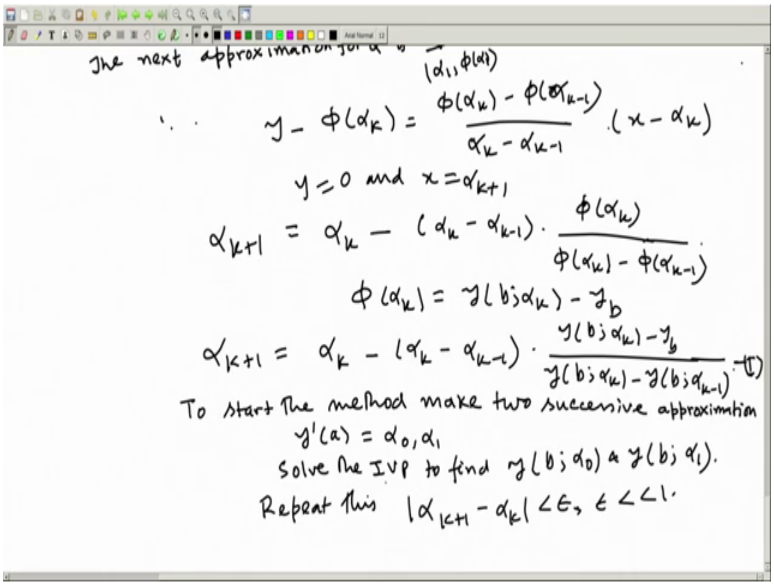

So, to start the method; start the method make two successive approximation for missing initial condition in this case is y dash a which we denote as alpha 0 alpha 1. Then solve the IVP till and solve the IVP to get y b alpha k and y b alpha k minus 1.

And then solve the IVP to find y b alpha 0 and y b alpha 1. So; that means, you are starting from x 0 and then going to a rather x 0 a; a to b from x 0 is a to b and in that process you are finding this end and then you are coming here this stage to find the next approximation alpha k plus 1 and this process will repeat. So, repeat this till we get alpha k plus 1 minus alpha k is less than epsilon; epsilon is a some very very small quantity greater than 0 and this is the Cauchy's criteria. So; that means, basically what we did is this process we are generating a sequence of iterates.

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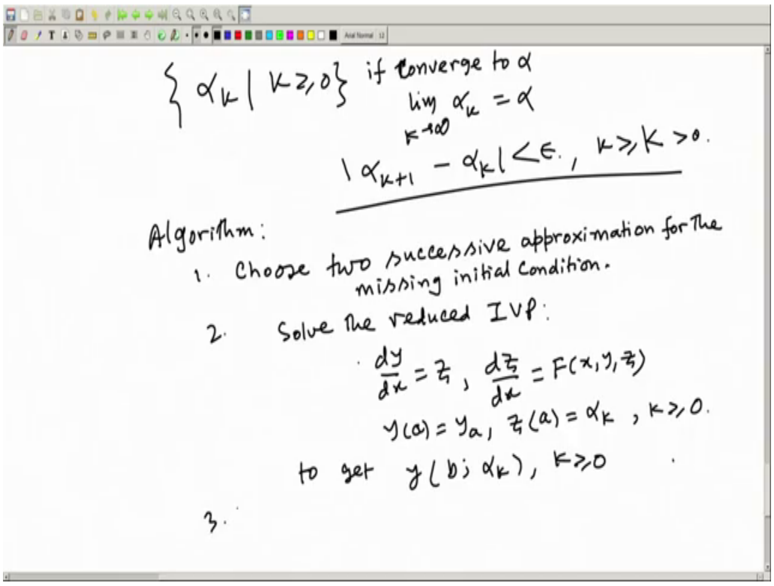

So, we have generates this alpha k; k greater than equal to 0. So, if it converge; if it if converge to alpha, so; that means, limit k tends to infinity alpha k equal to alpha; obviously, we cannot say what is k equal to infinity. So, what we can test is the this is the criteria that, if it converge then at certain stage for k greater than equal to 0 or rather to be more mathematically correct k greater than some capital k greater than 0.

So, after certain number of iteration you get a satisfaction or criteria of convergence is satisfied. So, what do you are doing here the algorithm if I want to say in a short manner. So, algorithm is for the shooting method its very simple you have a boundary value problem. So, first choose the choose two success not the; choose two successive approximation for the missing initial condition; missing initial condition. Then what we do is solve the IVP; solve the reduced IVP for.

So, this reduced IVP let us rewrite the reduced IVP as dy dx equal to z and dz dx equal to F x, y, z, y 0 or y a is some value is given and z a is alpha k. So, k greater than equal to 0 to start with k equal to 0 1 and solve this reduced IVP to get y at b alpha k for k greater than equal to 0.

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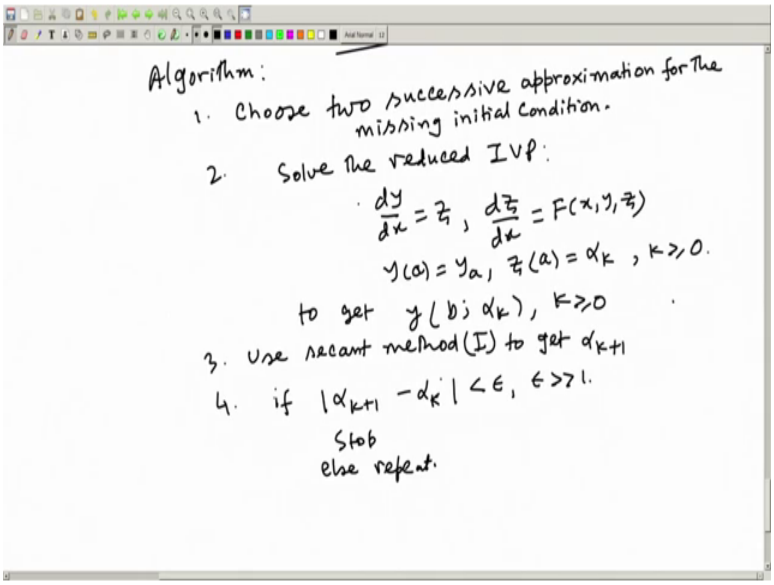

Then 3rd use the second method let us call this as one use secant method, one to get alpha k plus 1. 4 is check if this is happening alpha k is very very small, then stop else repeat the process repeat or rather I would say that repeat the process till this is happening. So, this is what is the shooting method is all about.

So, in this case we have assumed that the missing initial condition is y dash 0 the derivative condition can be other way also and also the same thing can be generalized for higher orders also. So; obviously, things are complicated when we have a higher order situation, so; that means, if the derivatives more than second or a series of boundary value problem or a fourth order or third order differential equation. So, in that case we will have a set up initially value problems are quite huge number. Now, why this is referred as a shooting method?

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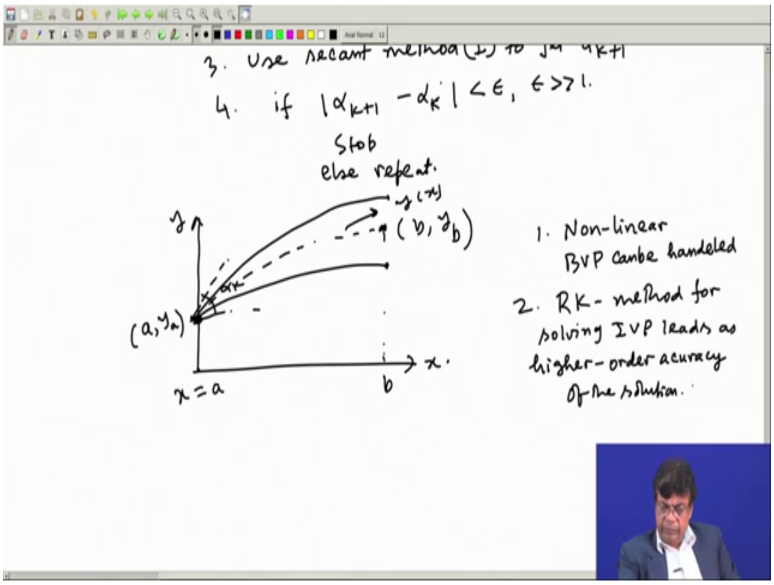

Now, basically what we are doing is here say this is x and this is y say this is x equal to a and you are fixed at a y a and this is the 1 say x equal to b y b. So, assume that you have a cannon which is fixed at the point a y a and you want to hit the target b y b and what you can do is you can rotate the cannon because rotate means you have the choice of different y dash a, so slope.

So, first you hit the target somewhere wrong, so this is a this is the one is alpha 0, then you reduce somewhere it gone. So, for from every hitting or every shooting you are getting a correction that what which one will be the proper choice of alpha k is; so, the one which hits the target, so this is our y x.

So, that is why this method is referred as the shooting method that is shooting an object or the target from a fixed point by rotating the angle of the angle of inclination. So, this is a very popular, so one there are few good points. Now, number 1 is we can handle non-linear BVP can be handled because the method does not care about the whether it is a your boundary value problem is linear or non-linear it does not make any difference because finally, you are reducing to first order the initial value problem.

And another thing is that if I use the Runge Kutta method, so if I use RK method; so, for RK method for solving IVP leads to a higher accuracy; higher order accuracy of the solution; accuracy of the solution. So, these are the very good points and that point there are plenty one of the wet points which I can we can readily see is that lot of work to be done for this. So, let us take, so far we do not have any option that how to solve the boundary value problem by other method. So, we have only learned the shooting method so far.

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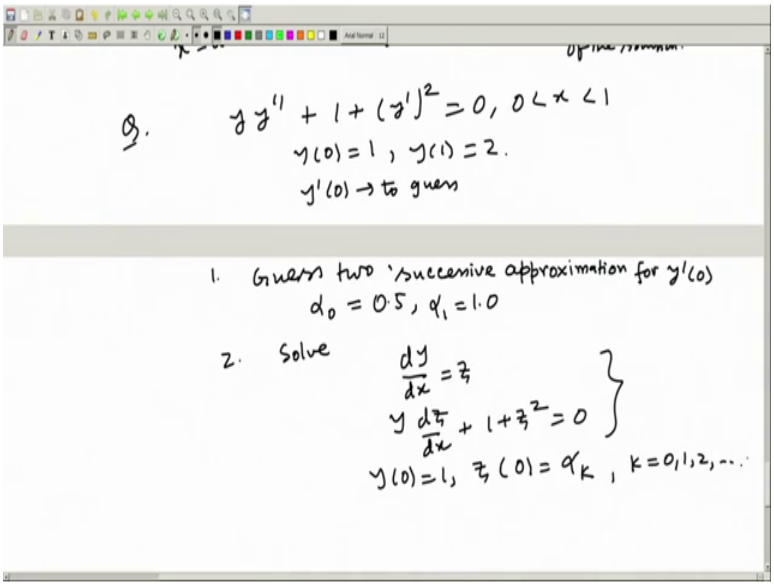

So, let us take one example say we have the problem as y y double dash plus 1 plus y dash square equal to 0 1 less than x less than 2 is the boundary variable are given I am sorry not the 1. So, this is varying between 0 to 1, so 0 to 1 y 0 is 1 and y 1 equal to 2 this is the boundary condition. Now, here; obviously, the machine initial condition y dash 0 to guess. So, first step guess two values two successive not two values two these two values is misleading two successive approximation for y dash 0, so we say this is alpha 0.

So, let us take alpha 0 is 0.5 and alpha 1 equal to 1, then solve the reduced equation what will be the reduced equation is d y d x equal to z this is standard because we have to just reduce the derivative second order to first order. So, the next one is y dz dx plus 1 plus z square equal to 0. So, this two along with the condition as y 0 equal to 1 z 0 equal to alpha k; k is 0, 1, 2 etcetera, so solve this x is from 0.

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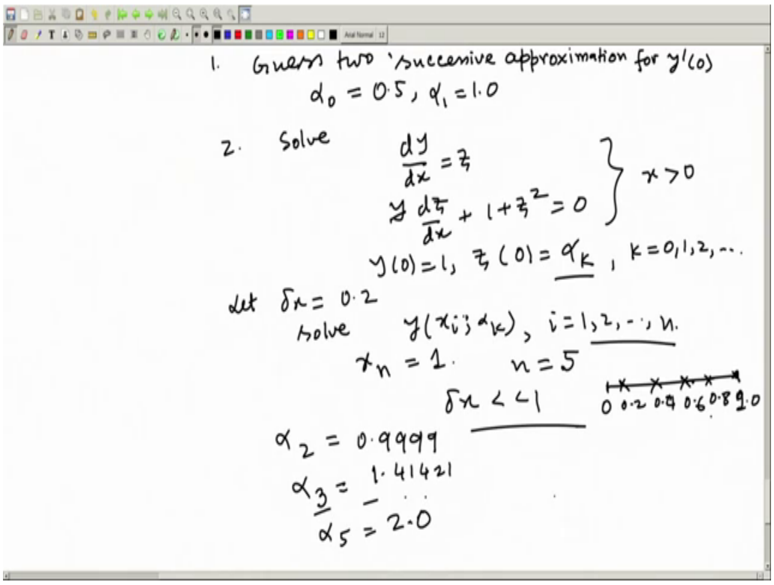

So, choose some step size let delta x equal to say which we some step size say 0.2 or something. So, you have to solve for solve y i y i. So, y i of course, here it is y i is a somewhat the way we are defining is not exactly y i, we are defining as y x i alpha k. So, y x i alpha k, i to be varying from 0 is the initial condition 1, 2 say n minus n; n equal to such that x n equal to the other point the last point 1. So, if I take point 2, so n equal to 5 in this case, so depending on the choice of delta x.

So, x n; n equal to 5 and like that way depending on the choice of delta x and we get the of course, another thing is that delta x small means we have a higher accuracy because it is order delta x to the or 5 4 for the Runge Kutta fourth order method. So, if I choose delta x small we get a higher accuracy.

So, delta x is always to be quite lower than 1. Now, so if we do this what we can find the next step that alpha 2 can be obtained as 0.9999; obviously, its not very simple to calculate by calculator. So, even if I have to calculate I have to solve this two successive boundary value initial value problem for how many times 5 times; 5 steps, so; that means, I starts from x this is 0, so 0 is given. So, then we have to go to 0.2, 0.4, 0.6, 0.8 and then 0.1 1.0. So, up to this you have to obtain the solution.

So, once you obtain the solution you go back to that second method step to get this alpha 2. So, alpha 2 comes to be this value and then once alpha 2 is obtained again you come back here; that means, this is now alpha 2, solve this from all these steps and then come back to the second method stage and you get the alpha 3 as 1.41421. And like that way finally, what you will find that maybe alpha 5 and all is nothing, but 2 2.0 is the correct solution, so this is the solution.

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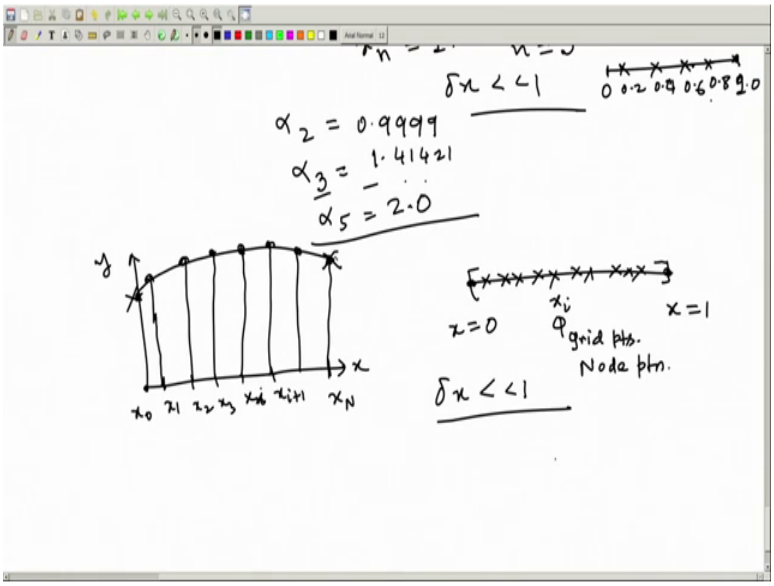

So, every stage we are covering, so what we are doing is we are covering see this is the interval boundary is x equal to 0 and x equal to 1 and we have choose this is the boundary two points where the function value is given. So, these are the points are xi grid points, so this I will call as a grid point some point some time because it is only one dimension. So, instead of grid point somebody called node points also, but we do not want to distinguish between the one dimension and other dimension I can simply call its a grid points.

So, basically what we are doing that every stage we are finding the solution at all these grid points. So, once I come out with these convergence criteria I have already the solution, so that is what the solution of the boundary value problem stands for. So; that means, this is your y this is, so you are finding the solution at some choice able number of points which we call as the grid points x 1, x 2, x 3 like that way, so x n rather x i I would say. So, let us make it a little bigger. So, this is x i plus 1 and this is our end point say x N something like this.

So, at this two points or rather I would say these two points the solution is given and in the remaining points we are finding the solutions and we are getting an approximate representation of the function. So, this is the graphical representation of the unknown function. So, this is our unknown function this two values are given and this are in between points grid points we are finding and then joining by some cardboard and drawing a plot making a plot to get the form of this unknown function.

So; obviously, one thing is pretty clear that delta x as delta x is reduced the situation improves. So; obviously, the shooting method is not the very handy in the sense that it is cannot be an ultimate one though it has the good advantages, but it is a huge task and more bottleneck is the successive approximation. Now, most of the cases we have a complicated set of equation for which we have no idea about what is the solution to be. So, we have to make a initial approximation to successive initial approximation.

But some cases it is advantageous like if we have some say heat transfer problem or some radiation flux and all and we have an experimental data for a simplified situation. So, those experimental data points can serve as the initial approximation to start the method.

And another thing is that if the method may not converge if the approximation for the initial solutions is way away from the proper solution itself that is the drawback of the second method that you have to have a closeness with the root, so; that means, the root should be captured within the two initial approximation.

So, there should be close enough to have a to contain the actual root. So, these are the drawbacks of the shooting method. So, that is why shooting method cannot be the ultimate one, but though it has the advantages as we pointed out that it has nothing to it can handle any non-linear situation and it can have a higher order of approximation because where using Runge Kutta methods and all these things. So, with that we stop here and next we will talk about how to solve the boundary value problem as a boundary value problem that will be by finite difference method.

Thank you.