Mathematical Methods For Boundary Value Problem
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Lecture - 10
Numerical Techniques for IVP; Shooting Method for BVP (Contd.)

So, we will come back. Now we will consider the higher order scheme for the initial value problem and then subsequently we will go to the boundary value problem solutions.

Now there are a method to solve the boundary value problem by converting the boundary value problem to a equivalent initial value problem, that is the thing we will be emphasize today's this lecture. Now, we stopped in the first order Euler equation, Euler scheme.

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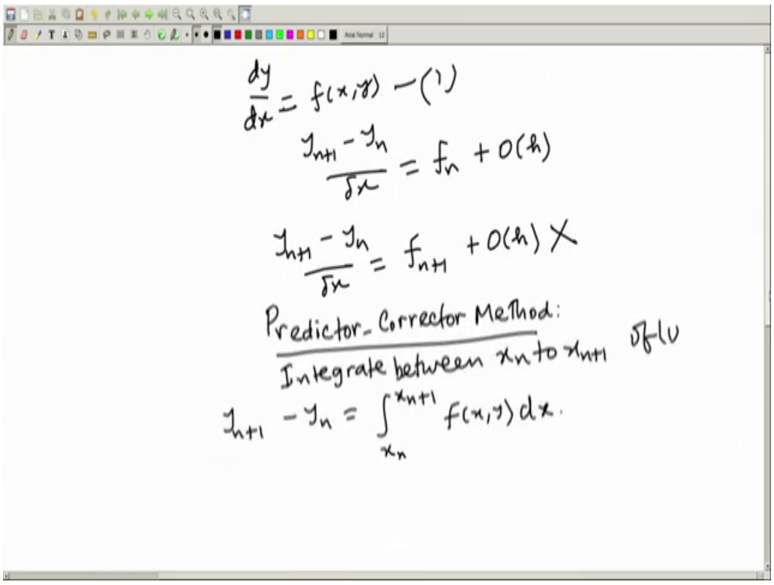

So, this was like dy dx equal to f x y. So, what I did is at n plus 1 this is the scheme what we proposed and this is a order h if I do the analysis that is series Taylor series expansion of y and so, the truncation error is order h and it is consistent and we have shown that it is stable provided delta x is that condition we have derived and also we have shown in the previous example, that a implicit scheme is stable.

Now, implicit scheme is unconditionally stable. Now the implicit this is explicit. So, implicit scheme was yn minus y n delta x equal to fn plus 1. Now this is not a very useful one because the unknown here is yn plus 1 and it is involved in a function form f inside the function f. So, we cannot obtain the yn plus 1 straight away. So, this is that is why it is not a useful scheme though it is a stable one.

Now, one remedy there are several remedies one class of method, this is call the predictor corrector method. So, my intention is not to talk much about the initial value problem. So, I suggest that remaining other methods, one can look into the other literatures. So, the predictor corrector method the basic idea is say one extension or one improvement of the Euler's scheme, we can propose this manner. Say if we integrate the equation say this is one integrate between xn to xn plus 1 of 1.

So, what I get is yn plus 1 minus yn equal to xn to xn plus 1 f of xy dx. So, this is a exact representation, we have not make any approximation as it is. So, now, what we do this integration of course, its not a very straight away we can integrate it.

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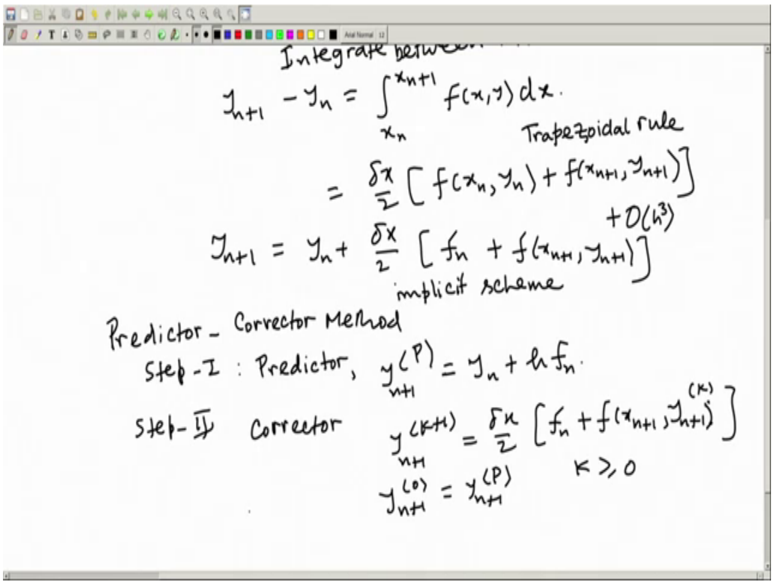

So, what we do we use trapezoidal rule. So, if I use trapezoidal rule for integrating this, what I get is delta x by 2 f xn yn plus f xn plus 1 yn plus 1. So, this is an implicit scheme, but advantage is that it is a very high accuracy order h cube.

So, we can have a truncation error of order h square, that is if we replace that dy dx by this form. So, the yn plus 1 can be expressed in this manner delta x by 2 now as. So, this is delta x by 2 sometimes it is also written as h. So, f n plus f x n plus 1 y n plus 1. Now this cannot be used straight away. So, this is a implicit one. So, what we do a two step procedure is adopted the two step is that a predictor corrector. So, predictor step predictor corrector method which is basically a two step method.

So, in the step I what you do is a predict which is called a predictor step. So, we do a prediction for yn plus 1, we call as p and that is simple Euler method if we use yn plus h f n. So, this is a first order prediction for the unknown y n plus 1 now we come to the step II and that has to be done in a iterative manner.

So, this is yn plus 1 k plus 1 equal to delta x by 2 fn plus f x n plus 1 y n plus 1 k and what is yn plus 1 k is yn plus 1 k we call this as iteration, where k greater than equal to 0 and y n plus 1 0 is yn plus 1 P. And this process repeat step II till we have a.

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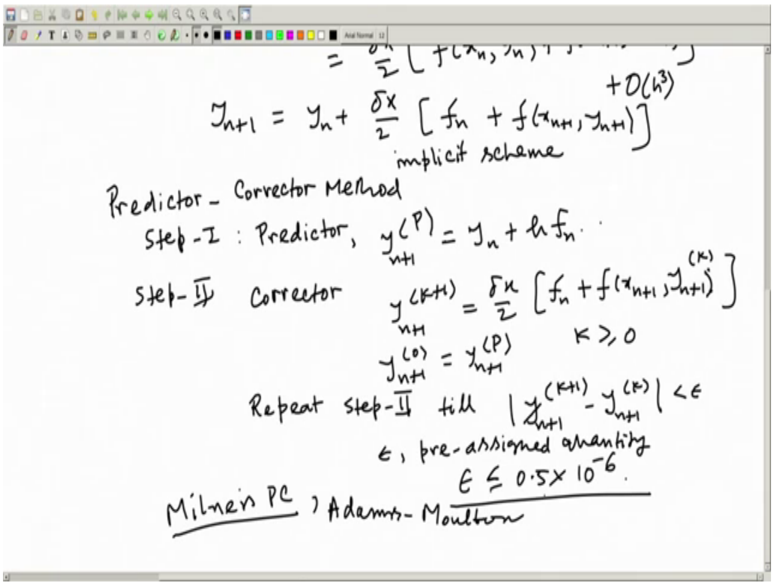

So, you need to repeat step II till we get that two successive iterates are very close; that means, y n plus 1 k is less than epsilon, epsilon is pre assigned its pre assigned quantity. So, epsilon is a something like 0.5 into 10 to the power minus 6 or so, or even less than that. So, that is depending on the accuracy what we are looking for. So, this is the way the predictor corrector step method goes.

So; that means, from a simple formula we used to predict the solution, a simple formula we use to get a predicted value; that means, this is the first step and then we use this first step to get the correction at every iterative manner. So, once this is achieved we get the solution. So, this is one of the class of predictor correctors method and there are several others I am not going to discuss about that the Milnes predictor corrector method Milnes PC method Adams Moulton methods and so, on. So, these are basically a class of method which is developed while integrating this one.

So, here we have used only the h key trapezoidal formula; that means, trapezoidal formula means what we did is we had taken a linear approximation of f that is using two points. So, if you use the Simpson predicted Simpson formula, so, it will be giving you higher accuracy and so, on that is what Milnes predictor corrector method is. So, I leave this one now. So, this is again a multi-step implicit skip. So, of course, this is a single step. So, far, but this Milnes predictor corrector method or Adams Moulton predictor corrector method. So, they are of multi-step methods.

So, now a predictor corrector method is also we have to write the algorithm twice and all these things. So, there are several one of the most useful method. So, what we need a method which is high accurate self-starting so; that means, it is a single step; that means, by using x n we can got obtain the solution at xn plus 1 and so, on.

So, if this kind of situation is there then it is a single step and if it is higher order. So, single step higher order is the one which will be most useful for obtaining the solution. So, one of them is luckily one can derive is the one proposed by Runge Kutta. So, that method is I will described now. In the Runge Kutta method what we do. So, like what we did for the Taylor series expansion the same way

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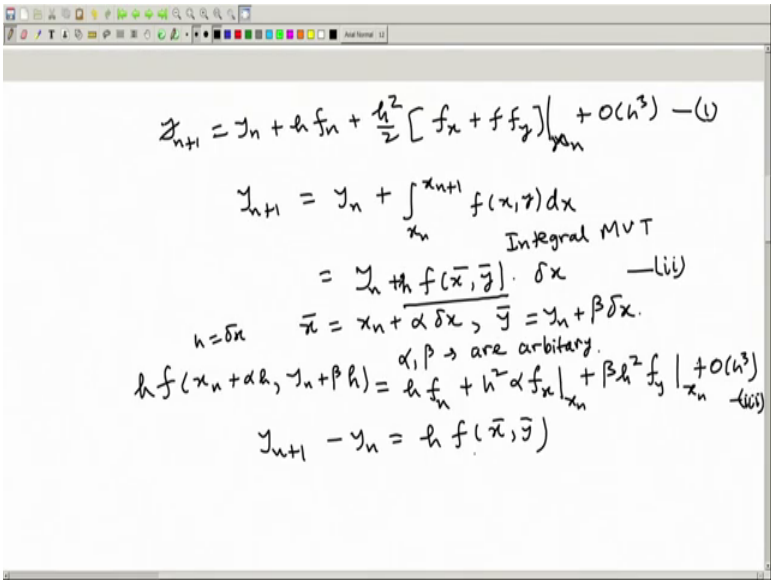

So, what I do is we make an expansion say, y n plus 1 equal to yn plus h dy dx so, that I can write as fn plus h square by 2 d 2 f. So, it is a x we are writing. So, fx plus f fy and that is evaluated at xn at n that is the better way of representing n plus order h cube, n stands for x n rather. Now this is one formula up to order h cube.

Now, if I integrate this one, yn plus 1 equal to yn plus xn to xn plus 1 f xy dx. Now this if I apply the integral mean value theorem integral mean value theorem. So, integral mean value theorem means what I do is we have to take a intermediate value and then this will be taken out. So, yn plus some fx bar y bar we denote this way and this is delta x, where x bar we can call as xn plus alpha delta x and y bar equal to yn plus beta delta x alpha beta are arbitrary constants.

So, we have two relations like that way let us called this is two. Now if I expand these f xn plus alpha h alpha h let us write as delta x and y n plus beta h by Taylor series again. So, what I get is h f n. So, we have here sorry here it will be h will be multiplied because xn into xn plus 1. So, h f n I can write as h this is the hf x bar y bar.

So, this term we are expanding by Taylor series. So, this is h fn plus h into h square alpha and fx at x n plus beta h square f y because this is the one at x n two variables Taylor series expansion next term will be all order h cube because already h is multiplied this is a second order.

So, this is basically a linear expansion, but since h is multiplied. So, the next term will be of order h cube. So, now, we are keeping all in the order h cube form h cube terms. So, now, yn plus 1 minus yn is our h f x bar y bar. So, now, if I substitute this if I compare. So, what I get here is if I compare this two. So, and the other one in the very beginning we have the Taylor series expansion.

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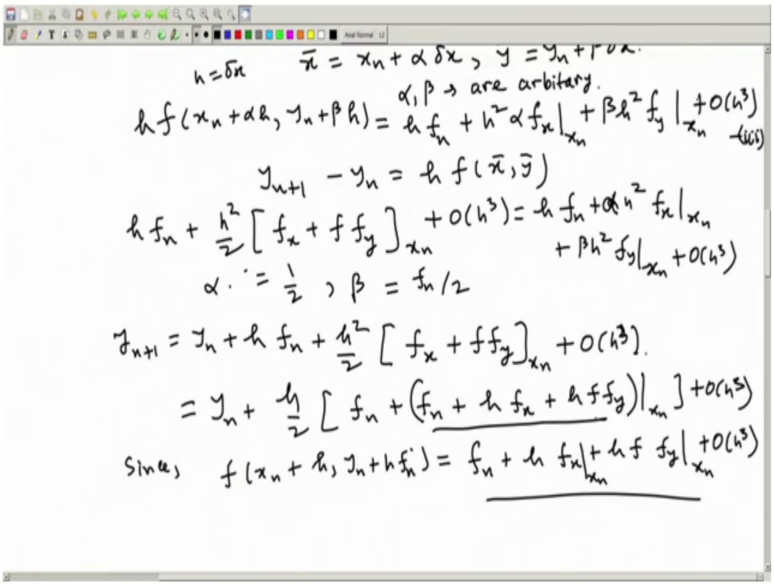

So, what I get is h fn plus h square by 2 fx plus f f y at xn order h cube this is equal to yn minus yn plus 1 minus yn and that is equal to this h f x n and I can write as h f n plus alpha h square fx x n plus beta h square fy x n plus order h cube both site this means this is the 1 and this 3.

So, now because this is happening for any choice of h for all values of h whatever the h I choose this is happening. So, if I equate because this is a polynomial. So, if I equate the like powers of h, I compare this like powers of h I can now say that this implies h of n h of n get cancelled. So, this implies fx alpha h square equal to alpha h s square equal to h square by 2. So, alpha becomes half and from here what I get is beta, I think this is I can just say alpha equal to half and beta becomes beta h square. So, fn by 2. So, beta equal to fn by 2.

So, this two coefficients which was arbitrarily arbitrary constants, what was used for integral mean value theorem can be now said as half n fn by 2. So, now, from the equation, what we have here yn plus 1 equal to y h fx. So, this one and also from these h fx bar y bar. Now if I substitute this. So, I get yn plus 1 equal to yn plus h fx bar y bar or this one. So, h fn plus h square by 2 fx plus fy f fy because beta and this is at xn plus order h cube order h cube. So, this is I am getting by from here by putting the value of alpha and beta.

Now, I can write this one as yn plus; now I can write this yn plus h by 2 fn 2 fn is coming h by 2 if I take common and then fn plus all these terms h fx plus h f f y at xn, order h cube and also if you look into the since if xn plus alpha h yn plus beta h which was our fx bar. So, f xn plus alpha h and ok. So, this is the one.

So, the putting the value of alpha and beta. So, what I know is fn plus hh if I take out. So, h fx plus h f n fy at xn of course, this is at xn. So, f fy basically no this is not alpha this is this is the one is h beta n, h hf. Now, xn plus h yn h plus fn. So, this is what is this quantity is.

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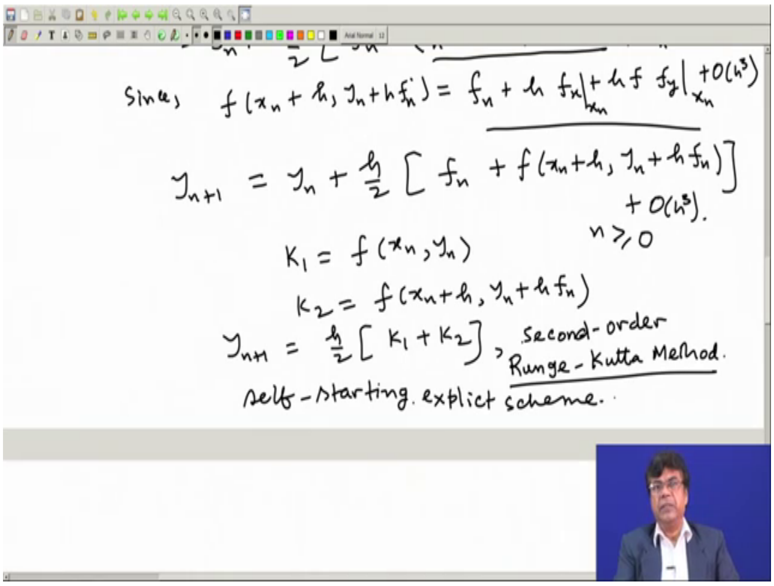

So, what I can write now that yn plus 1 can be written as yn plus h by 2 fn plus f xn plus h yn plus h f n. So, order h cube. So, this is what is the second order Runge Kutta method is. The this form now if I evaluate the yn plus 1 by this method. So, this is a self-starting method; that means, you have x n values is provided and using this xn value I can or using this yn value or there is a y at xn, I can get the solution at n plus 1.

So; that means, n equal to 0 which is our initial condition that is y 0. So, that is why it is also referred as the self-starting. Now the for the formula wise what is written as let us call this k 1 equal to if I take out the h outside. So, this is xn yn and k 2 is equal to f of xn plus h yn plus h f n; then yn plus 1 equal to h by 2 k 1 plus k 2 which is called the second order Runge Kutta method.

Now, Runge Kutta are to German mathematician and they derived they first proposed this one fourth order fifth order. So, higher ordered Runge Kutta methods, but one of the way to derive the Runge Kutta method or to understand that how to obtain the Runge Kutta procedure is the one I explained there can be several other ways also for the second order, but if it is a fourth order or higher orders that becomes very complicated to derive. So, but advantage is that, this is has all the good thing; that means, it is higher order accurate and self-starting means you need to give that self-starting explicit scheme.

So, you need to give only the you need to provide only the solution at initial condition x equal to 0 and once that is provided. So, that is enough to obtain the subsequent solution. So, one of the generalization of this Runge Kutta second order method is the fourth order Runge Kutta method, which can be expressed in this way hm.

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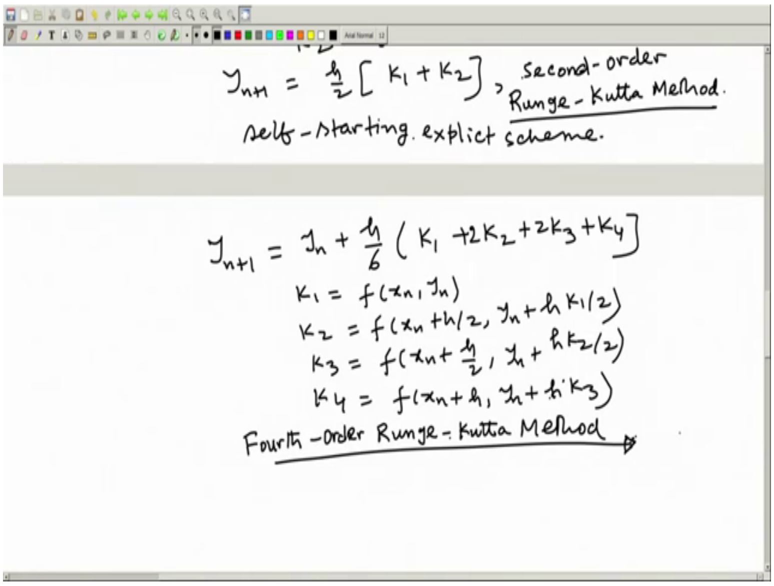

So, their fourth order Runge Kutta method is written this way y n plus 1 equal to yn plus h by 6 k 1 plus k 2; 2 k 2 plus 2 k 3 plus k 4 where k 1 is nothing, but f xn yn k 2 is f xn plus h by 2 yn plus yn plus k 1 by 2 h in this case because sometime h is multiplied outside. So, here I have not done that.

So, hk 1 by 2 k 3 is some type h by 2 yn plus h k 2 by 2 and k 4 is little bit different. This is f xn plus h yn plus h k 3 h k 3. So, this is called the fourth order Runge Kutta method. So, this is a most popular again the same reason and it is of order h to the power 4. So,. So, this is most popular as we stated it is a explicit and self-starting method and higher order scheme.

So, that is why in most of the cases the Runge Kutta methods are used to solve the initial value problem any kind of initial value problem you have. So, this is all about the discussion about the initial value problem for numerical solutions I wanted to make now. Why I brought this initial value problem to talk in the boundary value problem situation is now, what we will do is in certain cases that we can replace the given boundary value problem to a initial value problem.

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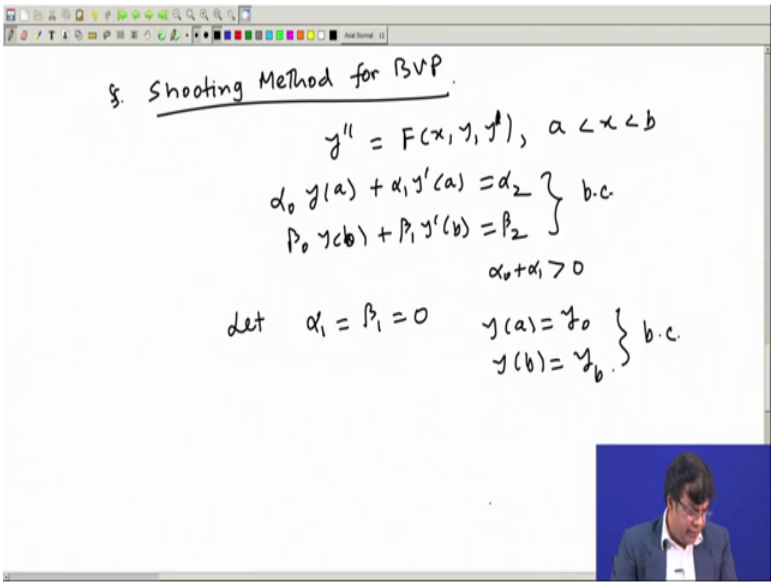

So, that kind of method is there is a class of method is referred as shooting method shooting method for BVP. So, this is the thing we will be talking now.

Now, a boundary value problem as I have said before is something like this, f x y y dash x a bearing from a to b or 0 to a wherever you like and what conditions are can be given in different way, one of the way is; say in general it can be a form like alpha 0 alpha 1 y dash a equal to alpha 2 beta 0 y a y b plus beta 1 y dash b equal to beta 2. So, this is the boundary conditions bcs.

Now, in this boundary conditions both the function value as well as function derivative are present. Now we first start with a simpler one. So, which normally that both alpha 0 and alpha 1 cannot be 0 so; that means, you should have this greater than 0. So, either of them alpha 0 can be 0 or alpha 1 can be 0, but both cannot be 0 if both is 0 then you do not have any boundary condition. So, that is ruled out now first we consider a situation where you have given the boundary conditions so; that means, let us take alpha 1 equal to beta 1 equal to 0; that means, a simpler situation.

So, what you have is y a is given to be some value y 0 and y b is given to be a value say yb. So, this is the bc for simplicity let this is the one.

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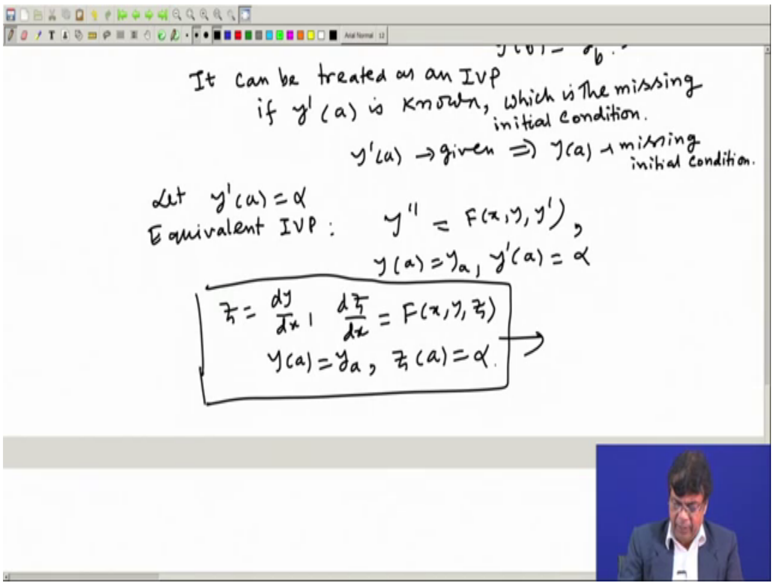

So, if this simple situation occurs now I can convert it to a IVP it can be treated as a IVP as an IVP Initial Value Problem if y dash a was known is known y dash a if this is this was given. So, could have been treated as IVP. So, this can be identified which we will referred as which is the missing initial condition.

Now, it may so, happen that you have given the solution at the two bound[ary]- on the derivatives are given at the two boundaries. So, in that case the function value itself is missing. So, so; that means, if you have given y dash a and y dash a. So, in that case you can you have to choose if it is given, then y a is missing initial condition. Now if I assume something. So, let y dash a equal to alpha, then how the problem looks like.

You have now. So, equivalent IVP initial value problem is y double dash equal to f x y y dash and y a is some value y a and y dash a is given to be some alpha. So, now, if I substitute z equal to dy dx. So, I get dz dx equal to f x y z and y a equal to some y a and z a equal to alpha. So, now, one can obtain a solution for this initial value problem because you have two set of equation, but they are coupled.

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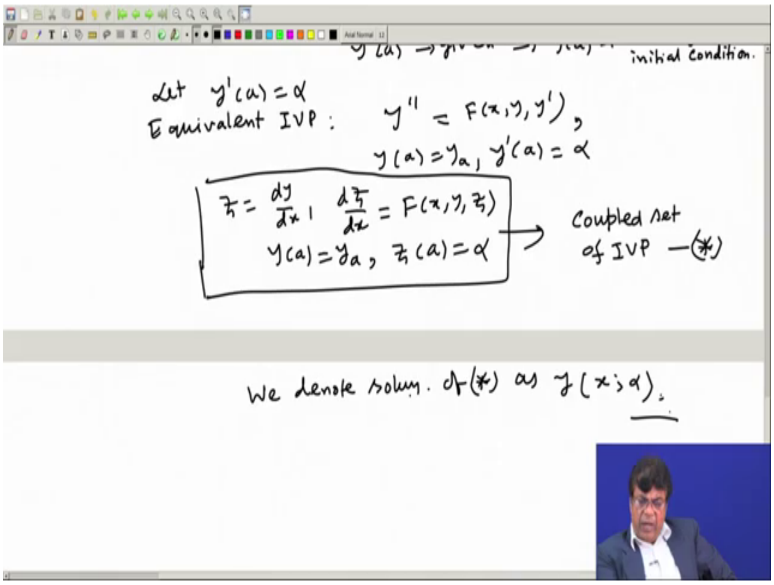

So, coupled set of IVP. So, coupled means one solution is depending on the other, you cannot isolate the one from the other and need to be solved simultaneously both together.

So, now if I find the solution. So, I choose any arbitrary and I get a solution, but that cannot be the one which we are looking for the boundary value problem. So, what we need that it will be the equivalent initial value problem provided the solution whatever we obtain from this initial value problem equivalent initial value problem, must satisfy the other boundary condition.

So, that means, the solution of this if I call this star we denote solution of star as y x alpha because the solution; obviously, depends on how the alpha was choosed. So, next lecture we will discuss that how alpha to be determined. So, that will be the procedure that is the one will provide us to solve the modern value problem by converting to a equivalent initial value problem.

Thank you.