Mathematical Methods For Boundary Value Problem
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Lecture - 01
Strum-Liouville Problems, Linear BVP

Welcome everybody. So, this is the first lecture for the Boundary Value Problem this course. So, now here what we will be doing is boundary or solving solution techniques for boundary value problem involving both ordinary differential equations and as well as partial differential equations. So, first we will start with the simple one ordinary differential equations; that means, the independent variable is single.

So now before that little bit of discussion: on some prerequisite, not exactly prerequisite some preliminary concepts which we will be using in the solution techniques. Now, one of these is that we will come across several complicated function.

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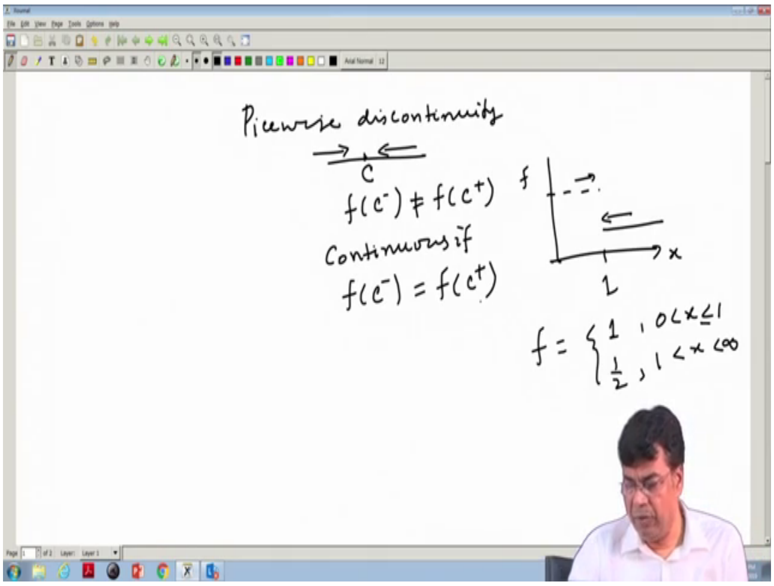

So, complicated function which may have a piecewise discontinuity piecewise discontinuity means say at any point; piecewise discontinuity. So, what it implies that at any point size c now at this point a function if you approach from this site that which is a increasing site; so this we call say f c minus.

And, if I approach from the other site that is we call as fc plus. So, if these are not exactly equal both are valid so, but it is not exactly equal; so then it is a jump discontinuity. So, any step function say some function which is some value say at 1 given by say f is like this. This is x and at say f is f is a x for 0 less than x less than 1 and it is a half I can say no this is 1. So, this is 1 say 1 and this is half for 1 less than x less than some other value infinity or something; so, less than equal to 1.

So; that means, from 1; so this is half. So, if I approach from here f value is half at the point 1; from here if I approach that is 1; so there is a jump discontinuity is occurring. So, this kind of piecewise discontinuous or if it is continuous; continuous means both why is it is same both fc minus equal to fc plus.

But this kind of function is still integrable; it may not be differentiable, but it is integrable now we may have come across this kind of function. Now one of the simple way is to express this function in terms of a smooth functions that is well behaved function.

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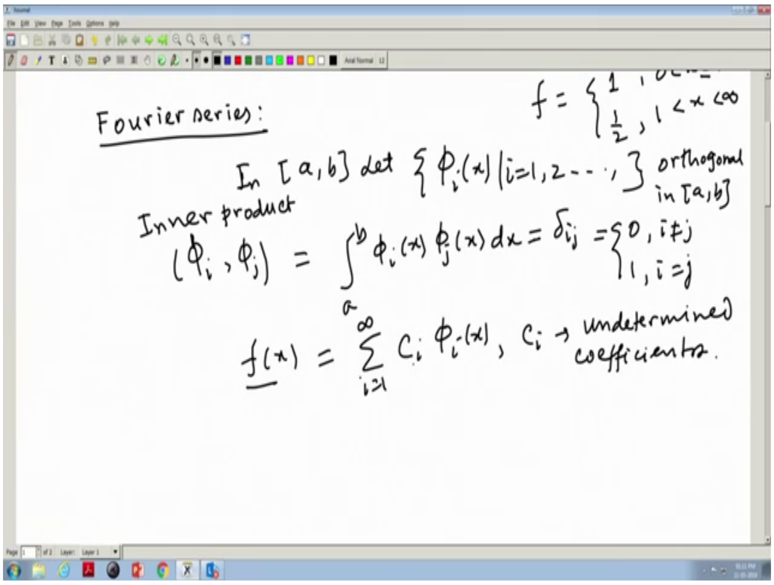

So, one of these is called the Fourier series. So, Fourier the; I believe is a French engineer. So, basically if Fourier started this Fourier series he developed the Fourier series for solving the heat transport equation.

Now, in the Fourier series now see that is one of the kind. So, in general we can say suppose in interval a, b in a say any interval a, b let a set of function phi x say i equal to 1, 2 etcetera they are orthogonal a sequence are orthogonal. So, orthogonal means what I have is orthogonal in a, b in the interval a, b.

So; that means, the inner product of this two function phi x; i x and phi j x; dx is equal to delta ij that is the usual. So, we can write this as phi i phi j which is called the inner product or dot product of this two function. If delta ij means it is equal to 0; if i not equal to j and this is 1, if i equal to j. So, this kind of situation occurs; so then it is referred as the orthogonal function.

So, if that is the thing then f x can be expressed as a linear combination i equal to say 1 to infinity; C i phi i x in terms of this functions C i are undetermined; yet undetermined function undetermined coefficients which can be determined through the orthogonal properties.

So, this expansion now what; what is the advantage that if the f x is discount jump discontinuity is there. So, but may not be differentiable, but what I have here in these series; it is well we have function because this phi i x can be a smooth function differentiable and everything. So, I can express f x in terms of a series of differentiable functions; so that is the advantage.

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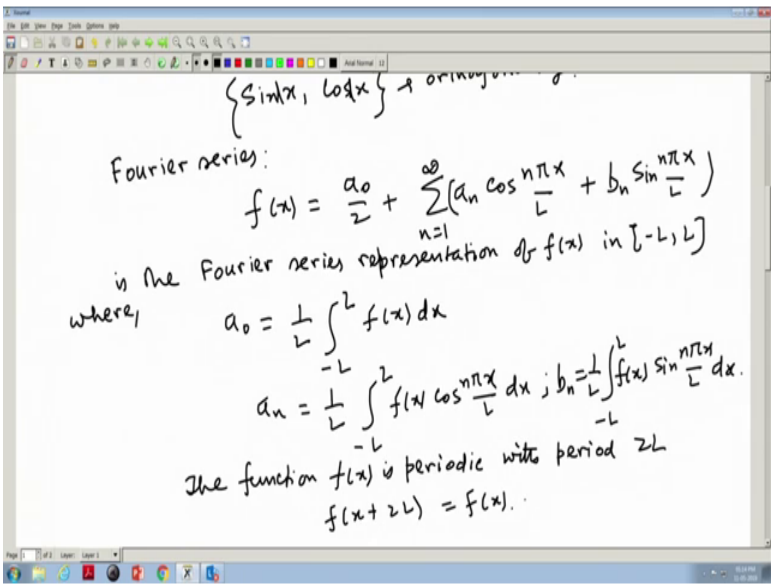

Now Fourier what he did? He explored the condition this say the sine function; sin x cos x they are orthogonal; so, orthogonality of this function; orthogonality of this functions. So, sin let us called lx; lx some coefficient, so orthogonality and from there what I can write a Fourier series say let a function f x; if I represent by this way a 0 by 2 plus sigma n equal to 1 to infinity n cos; n pi x by L plus bn sin n pi x by L, in rather I would say is the Fourier series representation of f x in minus L to L; within this zone Fourier series presentation in minus L to L f x.

So, where a 0 I can; I can be obtained as 1 by L minus L to L; f x, dx; where these are called the Fourier series coefficients a n equal to 1 by L. Again this is all by the orthogonality property of the sine function and cosine functions. And, B n can be written as 1 by L minus L to L f x sin n pi x by l d x. Now, so within the interval minus L to L; so what we have assumed that the function f x is periodic with period 2 L. So; that means, we have extended the function f x beyond minus L to L by this manner; f x plus 2 L is equal to f x.

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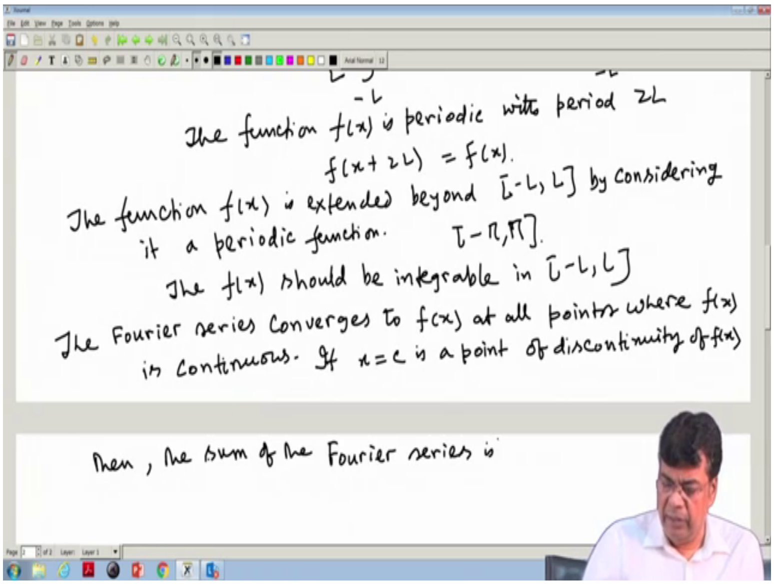

So, by that way extend or rather I would say the function is extended the function f x is extended beyond minus L to L; by considering it a periodic function. Now, this minus L to L can be replaced to any other say minus pi to pi or 0 to pi and other by simple transformation.

So, now one important thing is that the function; f x it can have discontinuities. So, but the function to be the important thing is that the function f x; the f x should be integrable in minus L to L this is the only requirement; that the function has to be integrable no need for any differentiable or continuity and other things, what I need is the function to be integrable within the interval minus L to L. If that is the thing is occurs; if that is the thing is guaranteed then this a 0, a 1; a 0, a 1, a 2, a n and etcetera B 0, B 1, B 0 is 0, B n etcetera can be determined.

And, how this summation behaves is that we can say that the; if f x is integrable, so f x equal to the summation of this. So, at any inter-point f x and the function value are same. So, I can say this way the Fourier series converges to f x at all points, where f x is continuous. If x equal to c is a point of discontinuity of f x; but still integrable; then the sum of the Fourier series then the Fourier some of the Fourier series is can be written as half of f x at c not x.

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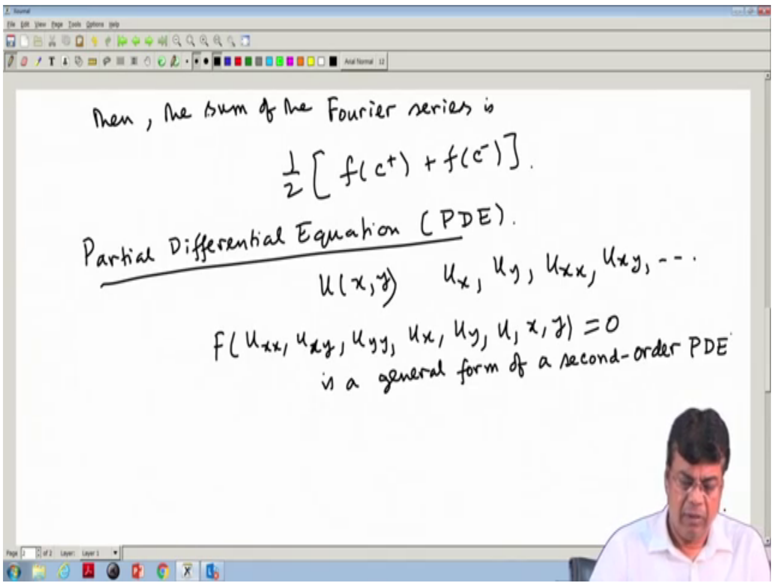

So, f c plus plus f c minus; so we know that both because it is only allowed is a jump discontinuity; if you have a situation where the function is integrable did not be differentiable. So, in that case you can only allow a jump discontinuity. So, the Fourier series will converge to the average of f c plus and c minus in this case ok.

So, there is the about the Fourier series which concept has been introduced in the subsequent discussion for solving the boundary value problem. Another important thing is that partial differential equation; so; obviously, I am not here to talk about all the characteristics of the partial differential equation and just in classification of partial differential equation how we classify that one.

Now partial differential equation means what we have is a function you say; if it is a more than one variable. So, say if you have a more than one variable as independent variable then a PDE arise; that means, you have u x, u i, u xx, u xy etcetera. So, this kind of things appears; so these are called a partial derivative; so instead of the ordinary differential equation.

So, a situation which involved so, a PDE can be written as second order PDE is a u x let me write this way u xy, u yy, u x, u y, u xy equal to 0 is a general form of a second order PDE; a general form of a second order PDE any combination; F is any function a known function.

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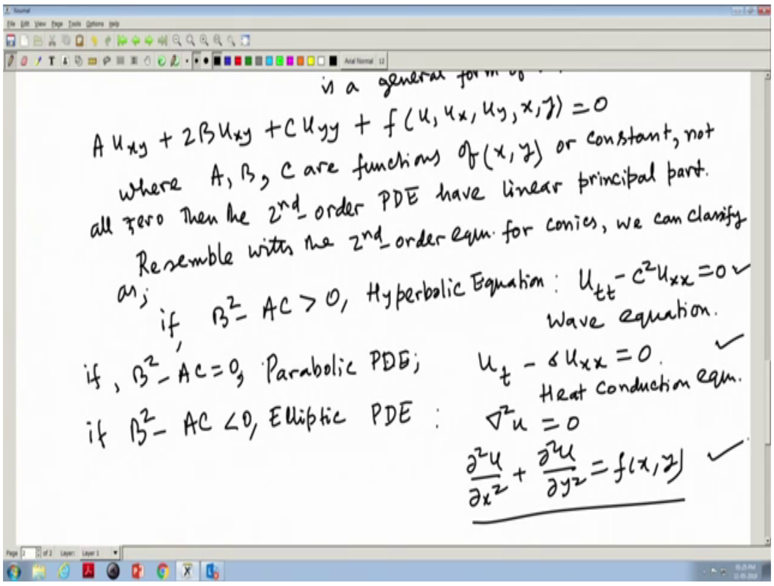

Now, if I consider a PDE given by this A u xy plus 2 B ux y plus C u yy plus say some small f which contains all these first order and u and etcetera, where A, B, C are function of x, y or constant, it can be constant or function of either x or y or not all 0; all cannot be 0.

So, then the 2nd order PDE have a linear principle part so; that means, the principal part which is the second order part is linear. So, in that case we can make a some classification which resembles with the ordinary the conical equations. So, resemble with the equation second order homogeneous equation for conics.

Second order equation for conics we can classify we can classify as a if B square minus AC is greater than 0 this is called the high parabolic equation. Again it has nothing to do with the solution form; it is just resemble or resemblance with the second order conic equation. So, if you have a second order homogeneous conic equation for the conic.

So, which are termed as which are termed as the hyperbolic; if B square minus AC greater than 0. So, classical example is this equation u tt minus c square equal to u xx; wave equation this is referred as the wave equation; any kind of wave propagation which is governed by this second order hyperbolic equation. Same way if B square minus AC equal to 0, this is called the parabolic PDE; parabolic PDE. Classic example ut minus sigma u xx equal to 0; we will be talking about this equation how to solve this kind of equations in the subsequent lectures.

So, that is why I am introducing it here; we have termed this as a parabolic equation. So, it is a heat conduction equation heat conduction equation; it is very well known equation which have this characteristics. Then third case which can be is B square minus AC less than 0; these are terms as elliptic PDE elliptic PDE. So, one of the; I think you know tried with in clearly; one of the example of elliptic PDE is this Laplace equation or Poisson equation.

So, that is del 2 u del x 2 plus del 2 u del y 2 equal to f x y. So, this kind of any diffusion process is governed by this way. So, this is the boundary value problem this is a pure boundary value problem because x y are the space coordinate; so it has a definite boundary. So, our emphasis will be this class of PDEs because this class of PDEs are the forming the whole many in mathematical modelling; we come across either a hyperbolic type PDE, parabolic PDE and elliptic PDE.

Now the way we have defined, so it is very obvious that the nature of the equation can change as x y are varied. So, when the domain is varied; so we can have the nature of overlapped; that means, which was hyperbolic it can be elliptic, it can be parabolic and so on. Now, we will talk about how to solve the boundary value problem by this Sturm level problem will be discussed.

I will be now talking about the boundary value problem how to deal with the boundary value problem with those; preambles whatever we talked in the previous lecture, the importance of the boundary value problem in solving several mathematical modelling and engineering problems and realistic situations. Now boundary value problem as these statements suggest that there is a boundary; so, the all the conditions are to be imposed in different boundary points.

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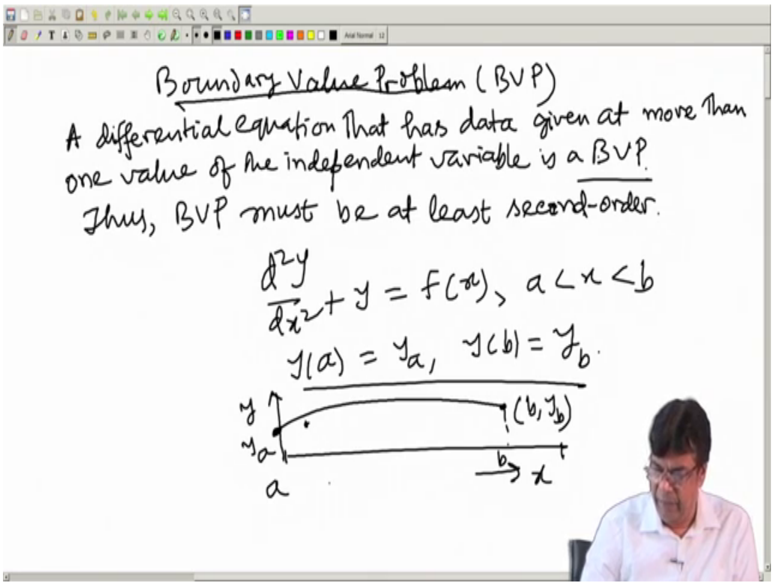

So, we can define a boundary value problem by this manner; boundary value problem we call as abbreviation as BVP. So, a differential equation now this differential equation can be a ordinary differential equation or partial differential equation; that has data given at more than more than one value of the independent variable independent variable is a BVP.

So, thus BVP must be at least second order, at least second order second order, it can be higher as well that is not a issue. Say for example, one of these boundary value problem we can write say d 2 y dx 2 plus y equal to say some f x. And x is varying from a to b and the conditions are two conditions second order equation.

So, two conditions are given at two different point y a and y b say y b. So; obviously, this is the situation that you have a variable x is varying from a to a point another point B. And what you have is given is the y and y is a suppose at this point y is given some value y a and the and it is taking some value at say this is the point let us call B.

So, this is your b y b; so this kind of situation; so this is x axis. So, this kind of situation is referred as the boundary value problem. Now in the contrast to initial value problem which are relatively easier to handle initial; value problem which abbreviated as IVP.

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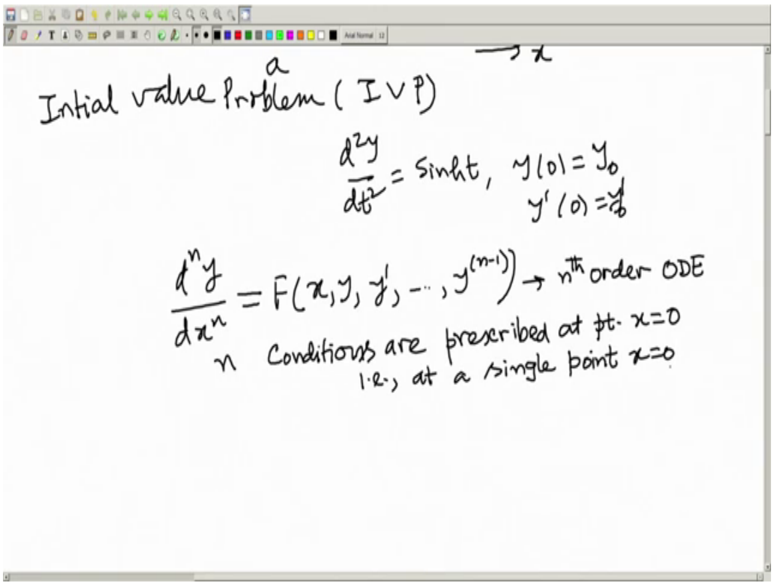

So, in the IVP; what happen is all the conditions are given at a single point; all the points all the conditions or whatever the conditions are imposed they are imposed at a single point and that point is referred as the initial point. So, for example, d 2 y normally this will be of time derivative.

So, sin hyperbolic t; so all the conditions say y 0 is given to be some value and y dash 0 is given say some other conditions is prescribed. So, two conditions one as a function value at t equal to 0 and the function derivative at t equal to 0; so let us call this y 0 and y dash 0. Now this IVP has a advantage is that one can reduce the initial value problem to a single first order ordinary differential equation or first order differential equation.

So, for example, if I have a nth order IVP. So, suppose your equation is given like this way d n y; d x n equal to say f some function of x y; y dash y n minus 1 and derivative. So, we have a nth order ODE n th order Ordinary Differential Equation and the conditions are all given conditions. So, since it is a nth order ODE showing it n conditions.

So, n conditions n number of conditions are prescribed at a point x equal to 0s; all the conditions are prescribed at a single point; that is at a single point x equal to 0. It can be any other x equal to 0 or x equal to l or x equal to whatever we can bring it down to x equal to 0.

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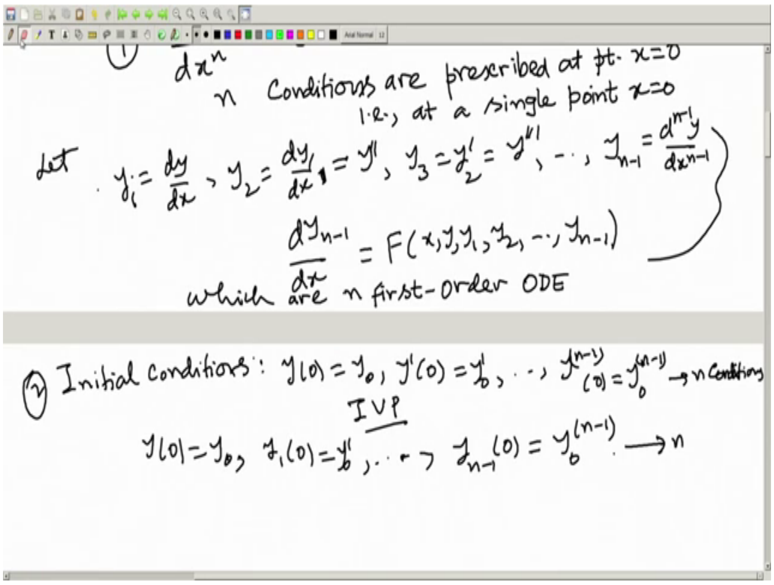

Now the advantage is that now if I do this simple manipulation that is suppose if I define say z dash equal to a rather first we let us say z. So, let us call z equal to dy dx; then say w equal to instead of the instead of this way because over to it will be will be exhausted with the alphabet.

So, let us call this is a y 1; so if I call y 1 is d y dx; y 2 is dy 1; d x y etcetera; so dy 1 dx. So, in other words it is y double dash. So, y 3 if I call as y 2 dash; so basically this is y triple dash and like that way, so I can go up to say y n minus 1; which is basically d n minus 1 d x n minus 1. So, if I substitute let these are the variable we introduced; we have introduced the variables n number of variables n minus 1 number of variables, y 1, y 2 y 3 yn minus 1 and our y.

So, if I now substitute there; so what I get is d n y dx n becomes y n minus 1 d x. So, there should be this is n minus 1; I mean, so yn minus 1 dx equal to F x, y, y 1, y 2 etcetera y n minus 1. So, these constitute n first order which are; which are n number of n first order ODE. In this case it is ODE because y is depending on a single variable independent variable is x only.

So, and accordingly our conditions can be expressed in this way; say all these conditions what are the conditions which was prescribed is all the conditions are prescribed at x equal to 0. So, the conditions say it is given like this way y 0 initial condition we call as initial ok; I think should rub this. So, let us call this is initial condition; initial conditions are which are given is y 0 say some value y dash 0; say let us call as y 0 dash etcetera y n minus 1 at 0; there is the n minus 1 derivative we call this as n minus 1 like that way.

So, these are the n initial conditions; n conditions n conditions are prescribed to solve this n th order ordinary differential initial value problem. So, this along with this 1 and 2 are the constituting the IVP and now this IVP what we are doing is we have introduced n minus variable n minus number of variables like y 1, y 2, yn minus 1.

And with that we have constructed n first order ordinary differential equations with conditions given by y 0 is y 0, y 1 0 is y 0 dash etcetera y this is n minus 1 0 is equal to y 0 n minus 1. So, these are the initial conditions are so n conditions and n ah. So, these are the n number of conditions are reduced.

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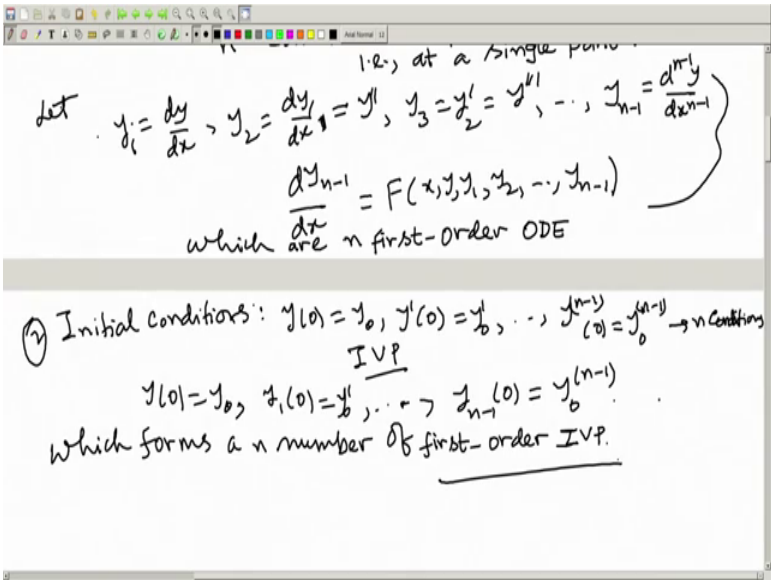

So, basically we can say a system of which forms; which forms a n number of first order IVP yeah; it is coupled, they cannot be separated out, we cannot take out one and solve and separately. So, there couple solution of y 1 is depending on y solution of y 2 is depending on y 1 and so on. So, they are coupled set of n first order initial value problem, but; obviously, this is not possible this trick does not work for the boundary value problem. Because as we have defined the boundary conditions say for example, even for the second order boundary value problem; here we have the conditions at two different point y a and y b; we even this tricks will not work over here.

So that means, the boundary value problem cannot be may reduced to a first order situation. So, we have to handle a at least with the second order differential equation. Now a first we start with the homogeneous boundary value problem; now homogeneous means we have the all the terms are depending or involving the variable y. There is no term which is independent of or rather no term which is free of y are involved.

If there is some term which are free of y then that becomes a non homogeneous situation. So, first we talk about the homogeneous and first this can be classified in a form which is termed in the name of Sturm and Liouville.

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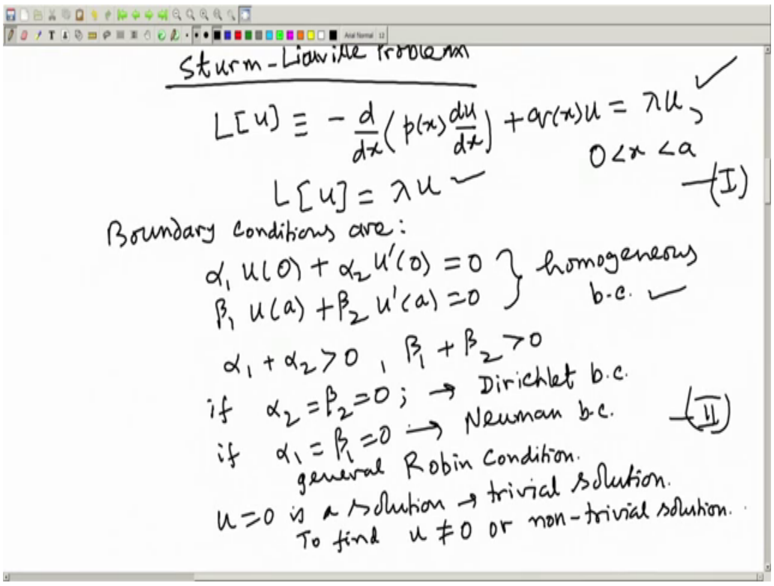

So, we considered this Sturm Liouville problem; now this is in general classified as Sturm Liouville problem. So, we define a operator L u by this manner minus d dx of px say d u dx plus qx; u equal to say lambda a constant u. And this is the operator we call this operator as L u; so this can be expressed in this way; L u equal to lambda u. And boundary conditions are given by this manner say alpha 1; u let us take the x is varying from 0 to say a, it can be a to be, but I can make a shift and reduce the situation between 0 to a.

So, we can put in this way alpha 1 u 0 plus alpha 2 u dash 0 equal to 0 and beta 1 u 0; u a plus beta 2 u dash a equal to 0. So, again this is a homogeneous boundary condition; homogeneous bc. So, we have a homogeneous equation given by this way L u equal to lambda u and homogeneous boundary condition. And this alpha 1, alpha 2, beta 1, beta 2 are such that this cannot be let us take these are all positive and not all 0. So, if I put this condition; that means, it cannot be all positive. So, if we take alpha 2, beta 2 equal to 0; alpha 2 if alpha 2 and beta 2 equal to 0; alpha 2 and beta 2 equal to 0.

So, in that case this is called the basic usual again beta 2 I mean. So; that means, there is no derivative in the boundary condition; this u dash 0 or u dash, so only you have a prescription for u 0 and u a. So, this is called a Dirichlet BC boundary condition in that case.

And if alpha 1 equal to beta 1 and both are 0 this is called the Neumann boundary condition and if both are present; so then call it the generally general one we can call that Robin boundary condition. So, Robin condition these names are not very important. So, in other words what we have is a combination of function value along with his derivative in a combination form.

So, either way we can have only the situation where the function value is given or its derivative is given or can be in a both form. So; obviously, what we can see is that this is say equation 1 and this is boundary condition as equation 2. So, this is a homogeneous Sturm Liouville problem; so u is a 0 is a solution. So, u 0 is a solution and we call this solution as the trivial solution.

So, basically what we looking for now; how to find out in one trivial solution for this kind of things; that means, to find u not equal to 0 or non trivial solution. Now, first few property or few characteristics of this Sturm Liouville problem or this operator L; we can look into it one is this operator is self ad joint.

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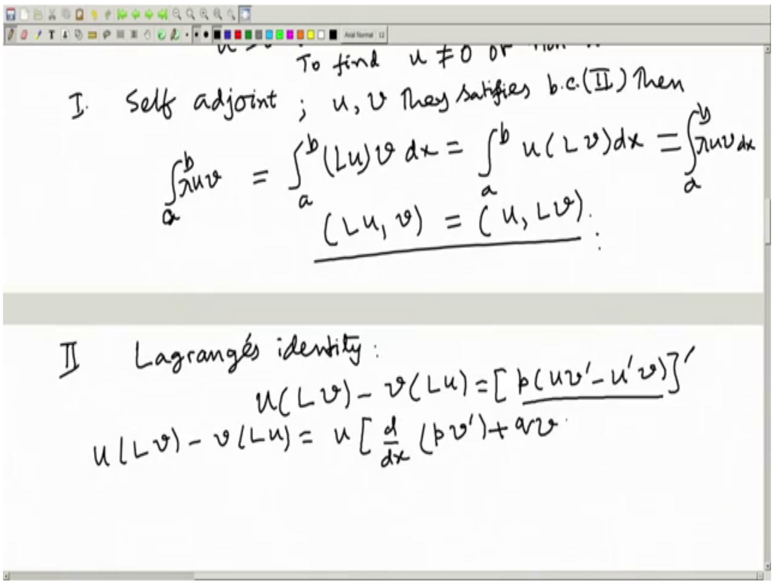

That means suppose you have two u and v; they satisfy the boundary conditions, they satisfies bc given by II, bc II then what I have is a to b; L u; v d x equal to a to b u L v; d x.

So, as we have discussed in the introduced in the very beginning that this can be stated as the inner product. So; that means, L u v this is the inner product in domain a to b and this is can be written as u L v. So, if their operator is such then we call this operator as a self ad joint operator and it is very easy to say that. So, what we have chose is that uv to satisfy the boundary conditions.

Now, now this we can write as sigma a to be lambda u v dx and in the same way this is also can be written as a to b because L u equal to lambda u; so we get lambda u v; so this implies these conditions. So, this operator one can say that their self ad joint. Then another characteristics of this operator is given by this way this is called the Lagrange's identity. Again the name is not important to remember; only these characteristics are important to keep it in mind. So, u L v minus v L u equal to p; u v dash minus v dash u; no u dash v; p u v dash minus u dash v derivative.

So, if I can say that this is a constant. So, in other words this implies that if u v are satisfying the Sturm Liouville problem, then we can have that this is s constant because this side is 0. Now what I get from here is u L v minus v L u equal to u d dx of p this is v. So, v dash minus or what we have is plus q v. Now, in the operator we have introduced here is a minus sign show that can we minus and without that also; there is a sometime in this customary to consider this minus sign.

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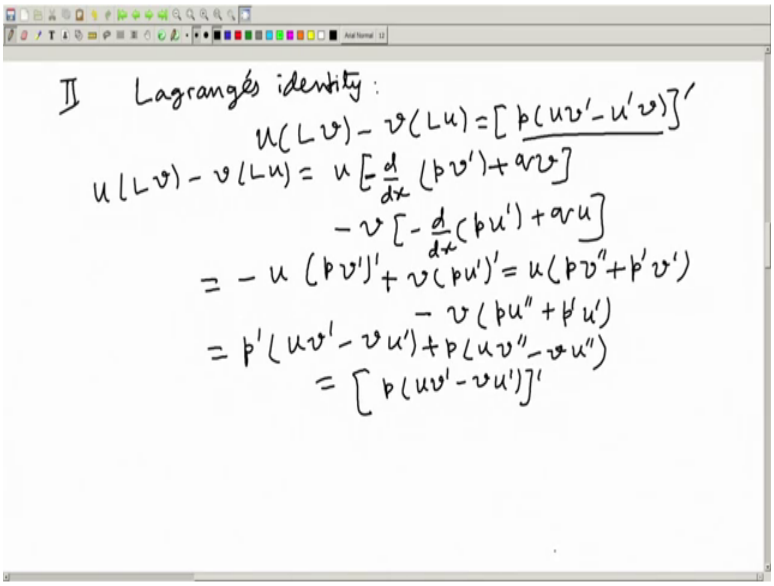

So, if we keep this minus here and then we have v and this is minus d dx of p; u dash plus q u. Now, if we do the operation here, so what I get here this u q u v; so these terms are cancelled out and what is left is minus u minus u; p v dash dash minus.

So, plus v p u dash dash and this is becoming u into p v double dash; even to p double dash plus p dash v dash minus v into p u double dash plus p dash, u dash. So, if we do this manipulation; so finally you will get a situation as because what we will have is p dash; p dash will be there.

So, p; so p dash will be there. So, what I have is p dash into what we have here uv dash and minus v u dash and plus p into uv double dash and minus v u double dash. So, this together can be said as p u; v dash minus v u dash derivative because the other term will get cancelled automatically. So, this is the two important characteristics of this; these non homogeneous Sturm Liouville problem. Now before we leave for this part; so I just introduce what is called the eigen values.

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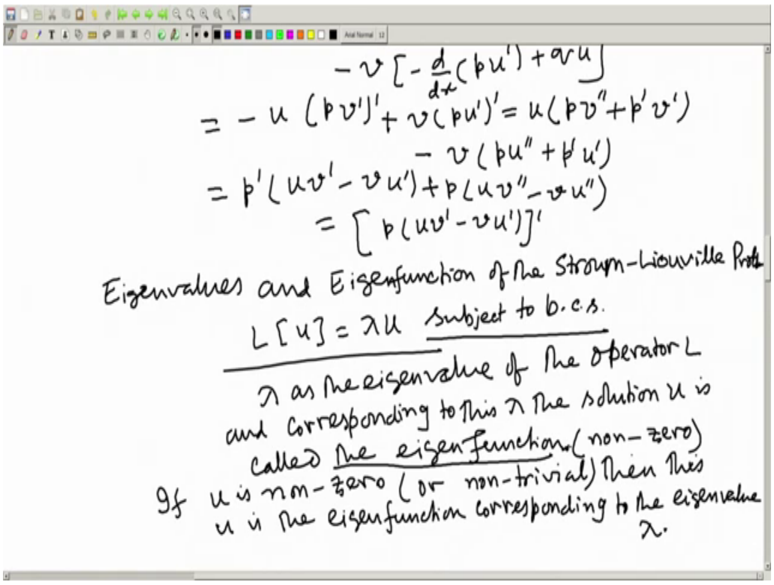

And, eigen functions; now what we have found eigen functions of the; this Sturm Liouville problem Stroum. So, basically this L is a linear operator and it is leading to L u when it is operated on u, we get it we are getting back this term lambda u. So, we call this lambda as the eigen value of L of the operator L. And, corresponding to this lambda lambda the to this lambda; corresponding to this lambda the solution u is called the eigen function. Now; obviously, what of course, it is subject to a certain boundary condition; subject to boundary conditions.

So, as usual the boundary conditions; so lambda is a eigen value, now given lambda we can solve this subject to this solution of solution u. So, solution u means it is not only satisfying this equation is also satisfying the governing boundary conditions. So, this u must satisfy the not only this equation; is also satisfying the given boundary conditions.

So, if u is non zero or non trivial, then this u is the eigen function corresponding to the eigen value lambda the; let us call this instead of here; this u is always can be 0 is a solution, 0 is always a trivial solution. So, this eigen function when non-zero solution. So, we call the eigen function provided u is non-zero over there.

So, we stop for this one for this lecture now; we will continue next.