Statistical Inference

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Lecture – 42
UMP Unbiased Tests – II

Let us consider example here binomial distribution.

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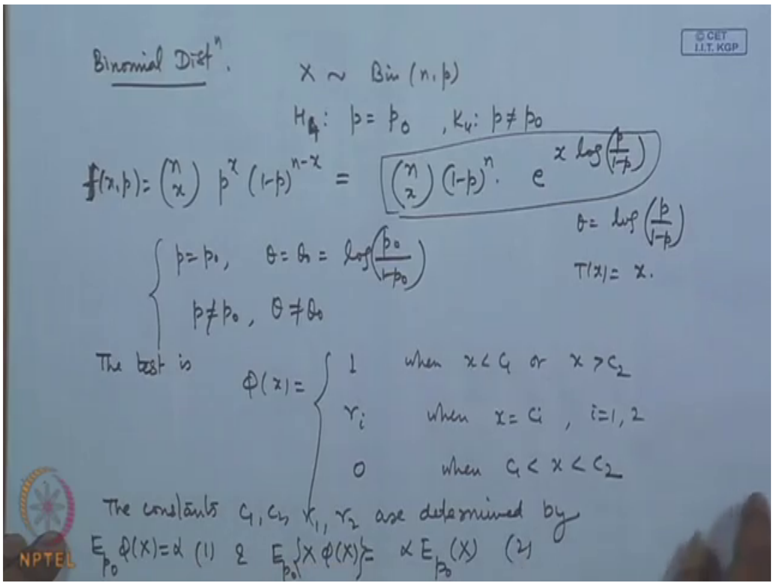

So, let us consider say X follows binomial n p, we are considering the testing problem for H 4. So, I will use this notation H 1 H 2 H 3 H 4 reference is clear here the alternative is p is not equal to p naught. So, here you have the distribution function and the form of the density function n C x p to the power x 1 minus p to the power n minus x. If we are writing in the form of the exponential family, we are writing it as 1 minus p to the power n, e to the power x log p by 1 minus p.

So, here this is x T x is equal to x and theta is equal to log p by 1 minus p and T x is equal to x. So, if I say p is equal to p naught then theta is equal to theta naught that is equal to log of p naught by 1 minus p naught and theta naught equal to theta naught then it is equivalent to p naught equal to p naught.

So, the hypothesis testing problem is restated in the form of this, where we are getting the one parameter exponential family. So, the test is phi x is equal to 1 when x is less than C 1, r x is greater than C 2 it is equal to gamma i when x is equal to C i for i is equal to 1 2. It is equal to 0 when C 1 is less than x is less than c. The constants C 1 C 2 gamma 1 gamma 2 are determined that is expectation of p naught phi x is equal to alpha and expectation of x phi x is equal to alpha expectation p naught x.

And of course, expectation of x in the binomial distribution will be n p. So, here it will become n p naught. So, let me call these conditions 1 and 2 here let me write down these conditions in detail.

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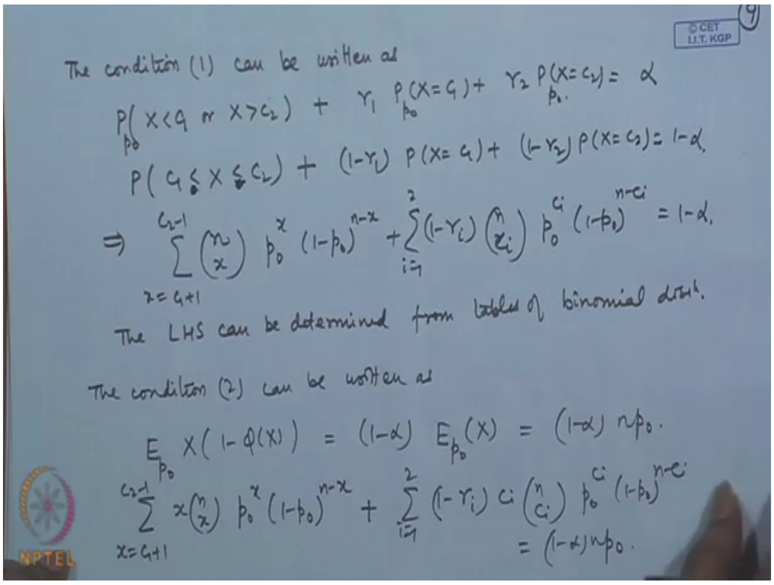

The condition 1 can be written as, this is actually probability of X less than C 1 or X greater than C 2 plus gamma 1 probability of X is equal to C 1 plus gamma 2 probability of X is equal to C 2 is equal to alpha.

I may consider the reverse of this also, I may consider probability of C 1 less than or equal to x less than or equal to C 2 under p naught of course, here. So, this I am removing from here and we can put it as 1 minus gamma 1 probability of x is equal to C 1 plus 1 minus gamma 2 probability of x equal to C 2 is equal to 1 minus alpha.

So, this condition is then sigma n x p naught to the power x, 1 minus p naught to the power n minus x where x is from C 1 plus 1 to C 2 minus 1 plus 1 minus gamma i, n C x n C c i p naught to the power C i 1 minus p naught to the power n minus C i, i is equal to 1 2 that is equal to 1 minus alpha. Now given the value of p naught and n one can do certain interpolation and determine. So, the left hand side can be determined from tables of binomial distribution.

Now, what is the second condition? The second condition; so again we consider one. So, X into 1 minus phi x that is equal to 1 minus alpha expectation of X. Of course, you may ask the question that why we are taking 1 minus this is for the simplicity otherwise here we have to write 2 regions, here we write only 1 region if I am taking 1 minus 1 therefore, it is convenient to consider 1 minus in both of these places.

Now, this is equal to 1 minus alpha into n p naught here. Now the left hand side if you look at once again this is becoming sigma x into n C x p naught to the power x, 1 minus p naught to the power n minus x for x is equal to C 1 plus 1 to C 2 minus 1 plus sigma 1 minus gamma i into n C i into C i, because we are considering multiplication of x. So, here will be C i n C i p naught to the power C i 1 minus p naught to the power n minus C i, i is equal to 1 2 that is equal to 1 minus alpha n p naught.

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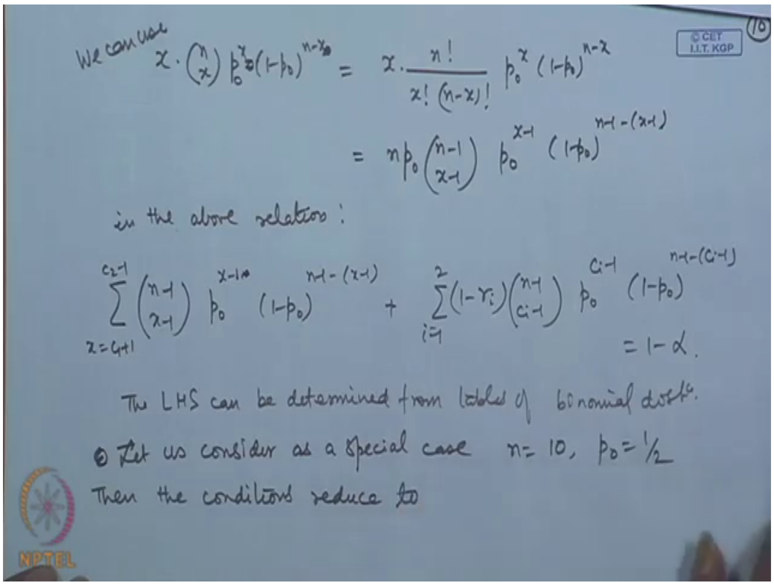

Now, here we do some sort of simplification. See this x into n C x that is equal to and of course, p naught to the power x into 1 minus p naught to the power n minus x naught minus x. So, this is x into n factorial divided by x factorial into n minus x factorial, p naught to the power x into 1 minus p naught to the power n minus x. This we can write as n into n minus 1 x minus 1 and then p naught here, p naught to the power x minus 1, 1 minus p naught to the power n minus 1 minus x minus 1 here.

So; obviously, we can use this in the above relation. So, here we have n C x into x into this term. So, n p naught will come out in the second term it is C i into n minus C i and again here we can use this thing n p naught can be written outside and if this will become n minus 1, c, C i minus 1 and this will become minus 1 here this will become minus 1 and n p naught n p naught will get cancelled out. So, this will give me simply n minus 1 C x minus 1, x is equal to C 1 plus 1 2 C 2 minus 1 p naught to the power x minus 1 and 1 minus p naught to the power n minus 1 minus x minus 1 plus 1 minus gamma i, n minus 1 C i minus 1 p naught to the power C i minus 1 1 minus p naught to the power n minus 1 minus C i minus 1 i is equal to 1 2 that is equal to 1 minus alpha.

And if you see this condition, there is slight difference from these conditions here you are having n C x here this is n minus 1 C x minus 1. So, once again you can see this left hand side can be determined from tables of binomial distribution. So, one can actually evaluate now, you will be given in a given problem n will be given to you p naught will be given to you. So, after substituting these values, this is reducing to looking at the values from the. So, for example, suppose I say p naught is equal to half, if I take p naught is equal to half then these conditions become extremely simple let me just show you it here.

If I take say p naught is equal to half, then this is becoming simply half to the power n let me just demonstrate. Let us consider as a special case say n is equal to say some value say 10 and p naught is equal to half then what it is reducing to? Then the conditions reduced to let me write it here one by one.

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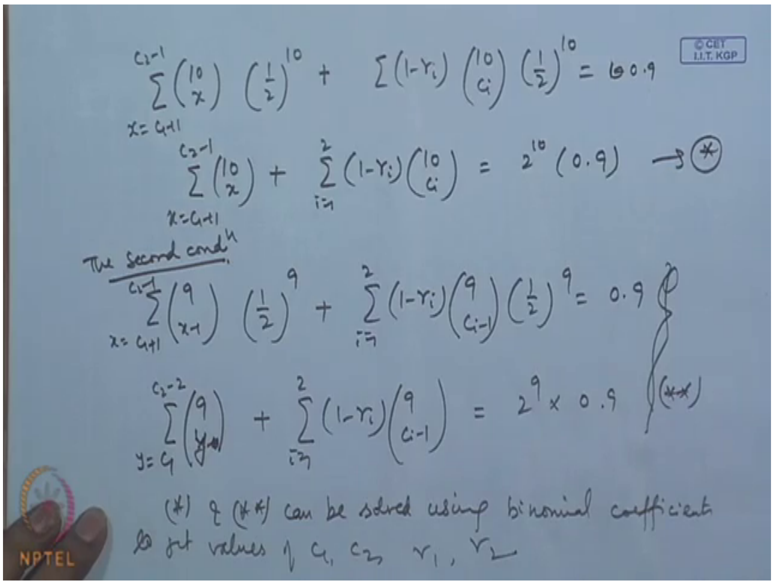

So, the first condition you can see here, it is n C x. So, this is becoming sigma 10 x x is equal to C 1 plus 1 to C 2 minus 1 this is becoming half to the power n that is half to the power 10 plus sigma 1 minus gamma i 10 C i and once again half to the power 10 is equal to 1 minus alpha suppose I say alpha is equal to 0.1. So, this is becoming 0.9 say suppose I am taking alpha is equal to 0.1

Then immediately you can see you can multiply by this 2 to the power 10 on this side. So, this is becoming 10 C x, x is equal to C 1 plus 1 to C 2 minus 1 plus sigma 1 minus gamma i i is equal to 1 to 2 10 C c i is equal to 2 to the power 10 into 0.9. And similarly if you look at the second condition, the second condition will become the second condition the second condition will become sigma 9 C x minus 1, x is equal to C 1 minus 1 C 1 plus 1 to C 2 minus 1 half to the power 9 plus sigma 1 minus gamma i, i is equal to 1 to 9 C c i half to the power nine is equal to 0.9. So, this will become equal to sigma 9 C x minus 1, we may put it in the terms of y say, y is equal to I am putting x minus 1 is equal to y.

So, this is from C 1 to C 2 minus 2 plus sigma i is equal 2 1 to 1 minus gamma I nine C and of course, this would have become minus 1 here. So, that is C i minus 1 that is equal to 2 to the power 9 into 0.9. So, these these conditions condition star and double star star, and double star can be solved using binomial coefficients to get values of C 1 C 2 gamma 1 and gamma 2.

So, in this problem I have demonstrated that a uniformly most powerful unbiased test exists. So, we have considered one parameter exponential family here. So, for the point null hypothesis and the alternative is actually two sided, we are able to exactly obtain the test function. Of course, the constants C 1 C 2 gamma and gamma 2 have to be further determined we have to actually carry out a little exercise here.

We have also demonstrated that for continuous distribution such as normal for testing for mean or testing for the variance, again we have shown that the two sided alternative UMP unbiased test will exist here. So, let us look at that case.

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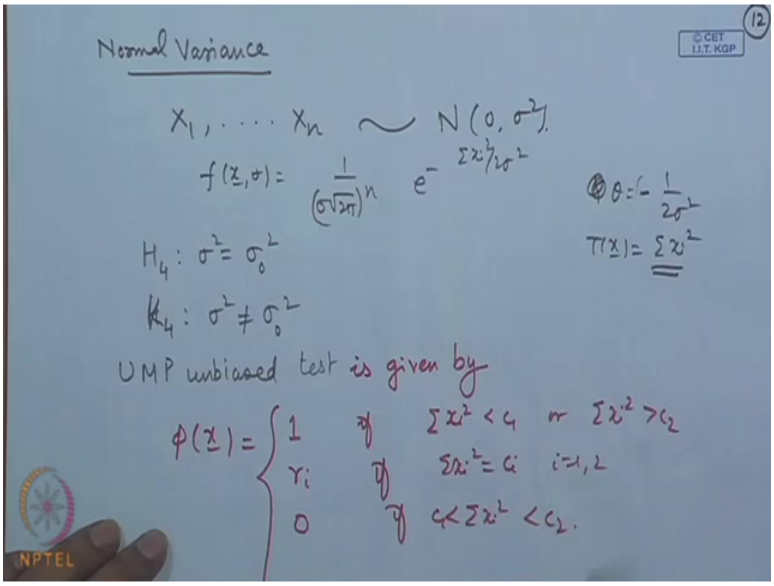

So, applications to testing for normal variance; we have a random sample from normal 0 sigma square distribution. So, we need to write down the joint distribution to look at the formulation here of T x, e to the power minus sigma x i square by 2 sigma square. So, here your parameter theta is equal to minus 1 by 2 sigma square, of course, you can see this is an increasing function here and T x is equal to sigma x i square.

So, if I consider the testing problem, sigma square is equal to sigma naught square against sigma square is not equal to sigma naught square, then the UMP unbiased test is given by phi x is equal to 1 if sigma x i square is less than C 1 or sigma x i square is greater than C 2, it is equal to gamma i if sigma x i square is equal to C i i is equal to 1 2 it is equal to 0 if sigma x i square lies between C 1 to C 2

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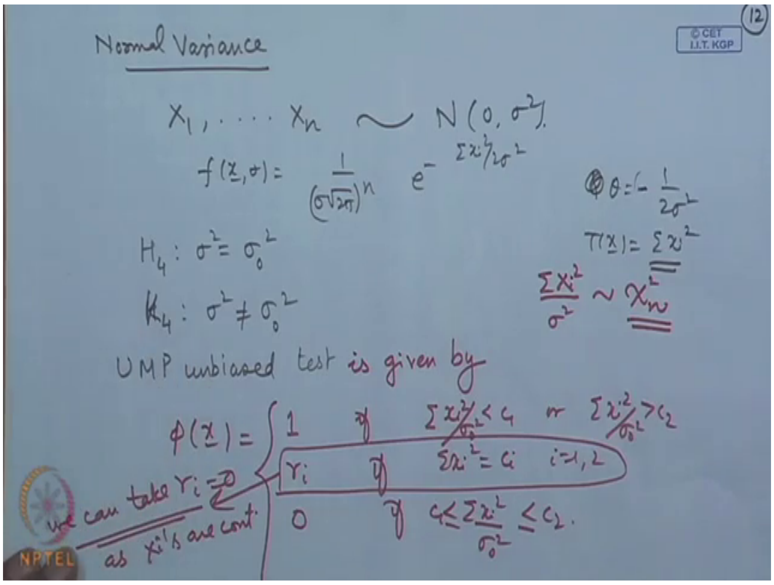

And now this part we can take, we can take gamma to be 0 as X i s continuous. So, this part can be then included here; also we can note here that sigma X i square by sigma square that will have chi square distribution on n degrees of freedom. So, if I am considering under the null hypothesis, I can consider division by sigma naught square here so, that the distributional properties will be easy to study then.

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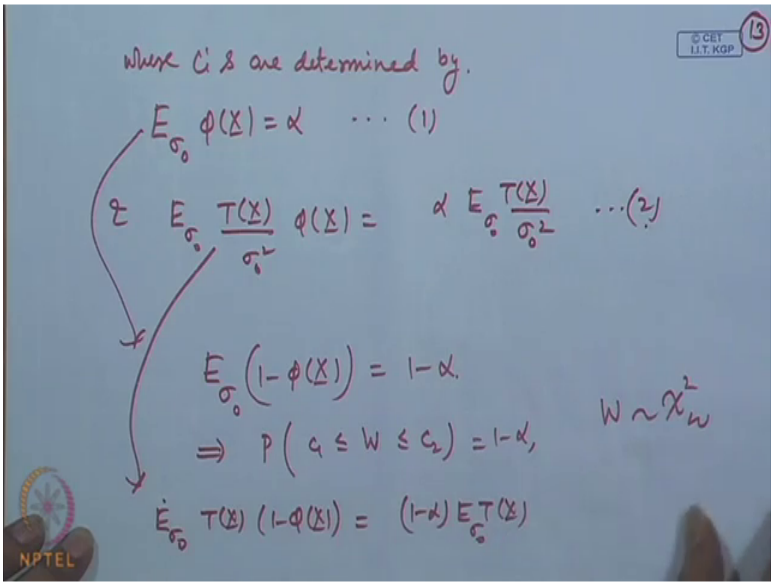

We will have say expectation of where C i s are determined by expectation of sigma naught phi x is equal to alpha and expectation of T X divided by sigma naught square, phi x is equal to alpha times expectation of T X by sigma naught square.

So, we will determine the constants C 1 and C 2 using both of these conditions. Since this condition can be simply written as expectation of sigma naught 1 minus phi x is equal to 1 minus alpha. So, 1 minus phi x means this is becoming probability of C 1 less than or equal to W less than or equal to C 2 is equal to 1 minus alpha where are W follows chi square n distribution.

So, this is simply to be determined from the tables of chi square distribution on n degrees of freedom. However, let us look at this condition 2 also what is this condition 2? In the condition 2 you are getting once again if I write 1 minus. So, this will become expectation of sigma naught T X 1 minus phi x, that is equal to 1 minus alpha times expectation of T X because expectation T X gets cancelled out on both the sides.

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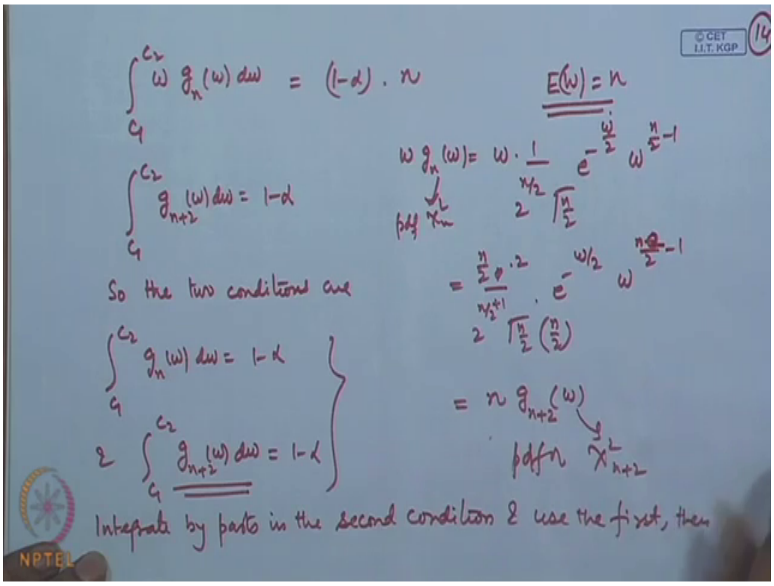

So, this is then reducing to integral W and of course, we are also considering division by sigma naught square; so this is W here.

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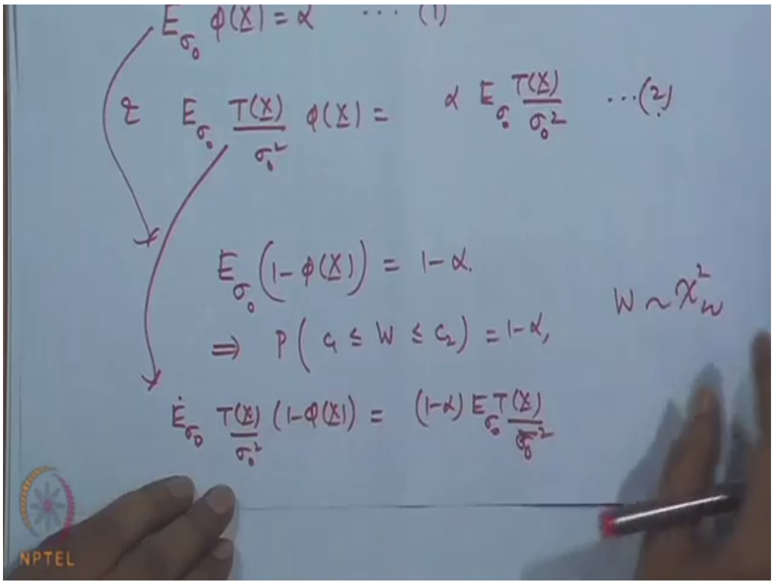

So, this is w into the density function of a chi square variable on n degrees of freedom from C 1 to C 2 this is equal to 1 minus alpha and this is n because expectation of w is n. So, if we consider the density of chi square on n degrees of freedom, that is 1 by 2 to the power n by 2 gamma n by 2, e to the power minus w by 2 w to the power n by 2 minus 1 then this is equal to 1 by 2 to the power n by 2 gamma n by 2, e to the power minus w by 2 w to the power now 1 power is getting added. So, this will become n plus 1 by 2 minus 1.

If that is so, then here I add the coefficient here, I write here gamma this is n by 2 here and then n by 2 here and then I add 1 power here and I add 1 power here. So, this is becoming n times g of n plus. So, this will become n plus 2 here rather than n plus. So, this is; that means, this is density of p d f of chi square n plus 2, if this is p d f of chi square n, then this is p d f of chi square n plus 2.

. So, this condition then can be written as C 1 to C 2 g n plus 2 w d w is equal to 1 minus alpha. So, the 2 conditions are then, C 1 to C 2 g n w d w is equal to 1 minus alpha and C 1 to C to g n plus 2 w d w is equal to 1 minus alpha.

Now if we consider integration by parts in this one, integrate by parts in the second condition and use the first.

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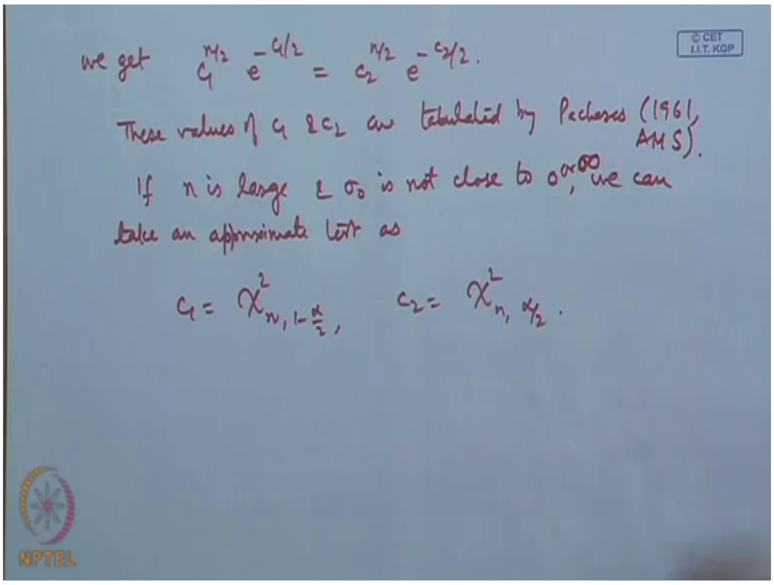

Then we get C 1 to the power n by 2, e to the power minus C 1 by 2 is equal to C 2 to the power n by 2 e to the power minus C 2 by 2.

Ah These tables of C 1 and C 2 are tabulated by Pechares in 1961 in a paper in Annals of Mathematical Statistics here. And another thing is that if n is large, then we can consider the 2 conditions to be exactly the same. If n is large and sigma naught is not close to 0 we can take 0 or infinity, we can take an approximate test as C 1 is equal to chi square n 1 minus alpha by 2 and C 2 is equal to chi square n alpha by 2.

So, today I have demonstrated that, applications of a 2 sided alternative hypothesis testing problem we have UMP unbiased test. And I have given applications to a binomial problem to a normal problem and of course, here you notice that the application is limited to one parameter exponential family. And in the one parameter exponential family I have taken a specific form theta into T x. So, for such a problem then we are able to derive the UMP unbiased test.

Now, in the next lecture I will be considering the completeness aspect also and we will introduce the test with the name and structure. Now using this structure we will show that we will be able to provide the solutions, when we are having the nuisance parameters. The simplest case for example, we have a normal distribution with the two parameters then what happens?

We will actually show that in all these cases UMP unbiased tests can be derived. So, as an application we will consider all the normal distribution problems for testing for the mean, testing for the variance in the presence of the other parameter, we will also look at certain applications to distribution such as exponential and others.

Here of course, there is a concept of the ancillary statistic that will be utilized and completeness and the Boscoes theorem will also be helpful. These things we did it in the first part of this course namely when we are discussing the point estimation, we have introduced these concepts. So, we need to look at those things again and we will be using this thing.

So, I will continue the concept of unbiasness in the testing problems in the following lecture.