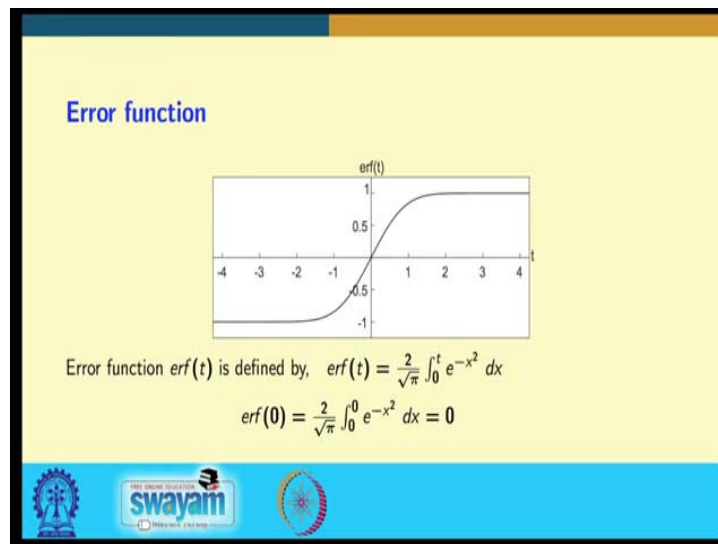


Transform Calculus and its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 09
Error Function, Dirac Delta Function and their Laplace Transform

In this lecture, initially, we will go through some useful well-known functions and we will evaluate their Laplace transform as well. The first one that we will cover is the Error function.

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The Error function denoted by $erf(t)$ is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

and the graph of this function is shown in the slide. Now clearly, at $t = 0$, the value will be equal to 0 i.e.,

$$erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-x^2} dx = 0.$$

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$$\begin{aligned} \operatorname{erf}(\infty) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} \cdot \frac{-\sqrt{u}}{2} du \quad \left(\begin{array}{l} u=x^2 \\ \rightarrow \\ \frac{1}{2} \end{array} \right) \\ &= \frac{2}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}\right) \\ &= 1. \end{aligned}$$

Next we see the value of $\operatorname{erf}(\infty)$. So, by definition, it will be

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx$$

We substitute $u = x^2$ so that $du = 2x dx$ i.e., $dx = \frac{du}{2\sqrt{u}}$ and the limits of the integration will remain unchanged. Then we have,

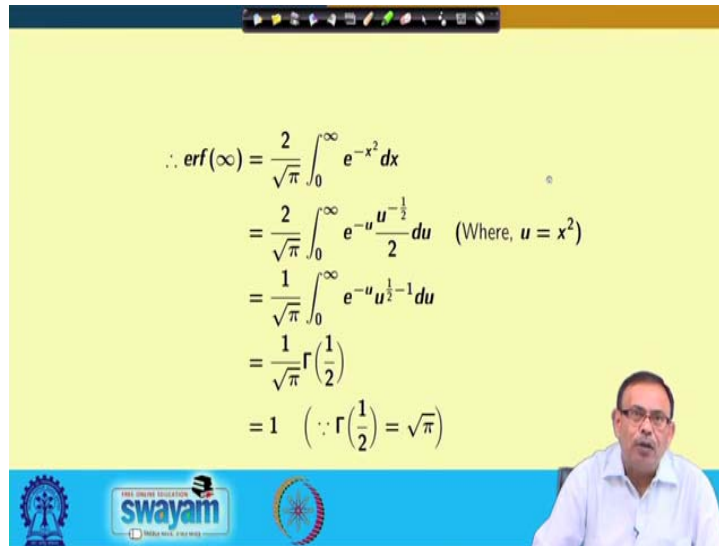
$$\operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du$$

We can express this as a Gamma function directly.

$$\therefore \operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \right)$$

So, the value of the Error function, as $t \rightarrow \infty$, is equal to 1.

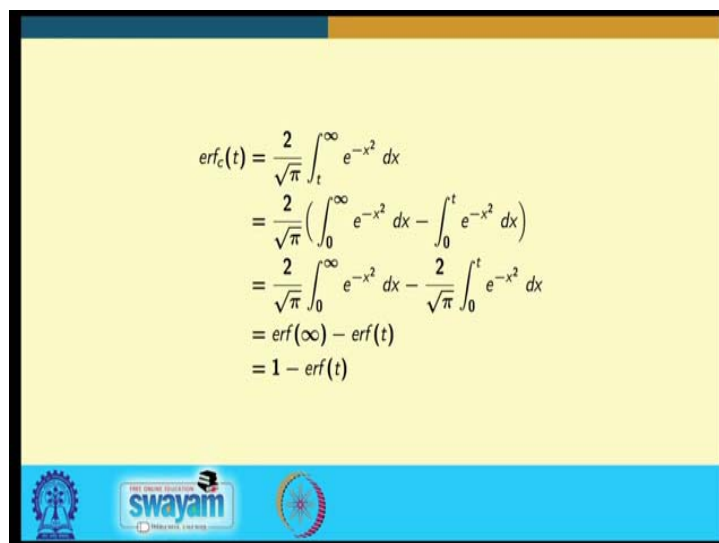
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$$\begin{aligned}\therefore \operatorname{erf}(\infty) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u} \frac{u^{-\frac{1}{2}}}{2} du \quad (\text{Where, } u = x^2) \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du \\ &= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \\ &= 1 \quad (\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi})\end{aligned}$$

There is another function which we call Complementary Error function denoted by $\operatorname{erf}_c(t)$ and it is defined as

$$\operatorname{erf}_c(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2} dx.$$

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$$\begin{aligned}\operatorname{erf}_c(t) &= \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2} dx \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-x^2} dx - \int_0^t e^{-x^2} dx \right) \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx - \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \\ &= \operatorname{erf}(\infty) - \operatorname{erf}(t) \\ &= 1 - \operatorname{erf}(t)\end{aligned}$$

So, we can break the integral into two parts as:

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-x^2} dx - \int_0^t e^{-x^2} dx \right)$$

This we can write down as

$$\begin{aligned} \operatorname{erfc}(t) &= \operatorname{erf}(\infty) - \operatorname{erf}(t) \\ &= 1 - \operatorname{erf}(t). \end{aligned}$$

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These functions are very useful for solving various kinds of engineering problems. Now we evaluate the Laplace transform of $\operatorname{erf}(t)$.

$$\begin{aligned} L\{\operatorname{erf}(t)\} &= L\left\{ \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \right\} \\ &= \frac{2}{\sqrt{\pi}} \frac{1}{s} L\{e^{-x^2}\}, \quad (\text{using Integral theorem}) \end{aligned}$$

From the basic definition of Laplace transform we get,

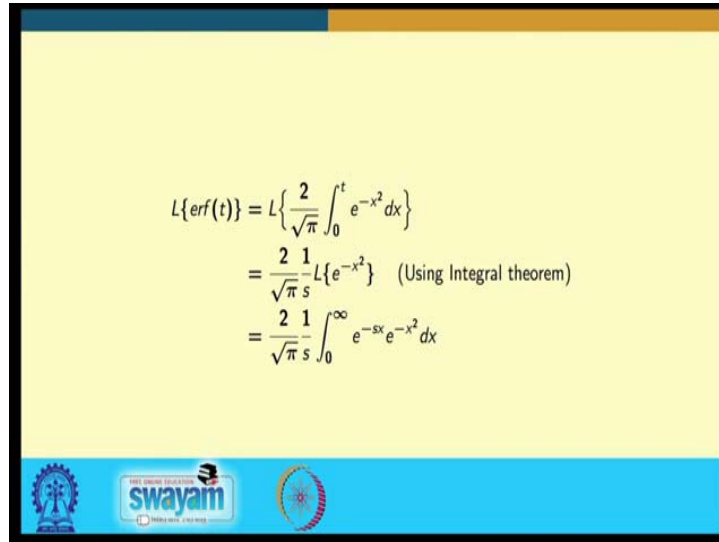
$$L\{\operatorname{erf}(t)\} = \frac{2}{\sqrt{\pi}} \frac{1}{s} e^{\frac{s^2}{4}} \int_0^{\infty} e^{-(x+\frac{s}{2})^2} dx$$

Refer the slide above for detailed steps. If we take $x + \frac{s}{2} = u$, in that case, this will be converted into

$$L\{\text{erf}(t)\} = \frac{2}{\sqrt{\pi}} \frac{1}{s} e^{\frac{s^2}{4}} \int_{\frac{s}{2}}^{\infty} e^{-u^2} du$$

$$= \frac{1}{s} e^{\frac{s^2}{4}} \text{erfc}\left(\frac{s}{2}\right).$$

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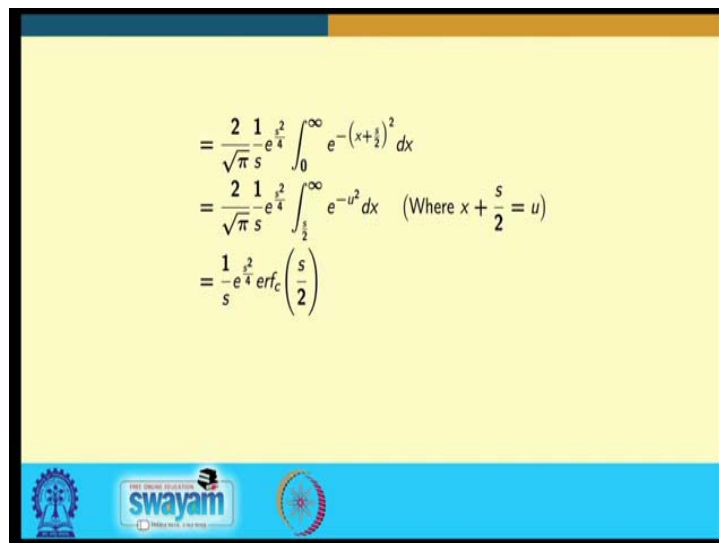
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$$L\{\text{erf}(t)\} = L\left\{\frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx\right\}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} L\{e^{-x^2}\} \quad (\text{Using Integral theorem})$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-sx} e^{-x^2} dx$$

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$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} e^{\frac{s^2}{4}} \int_0^{\infty} e^{-(x+\frac{s}{2})^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} e^{\frac{s^2}{4}} \int_{\frac{s}{2}}^{\infty} e^{-u^2} dx \quad (\text{Where } x + \frac{s}{2} = u)$$


$$= \frac{1}{s} e^{\frac{s^2}{4}} \text{erfc}\left(\frac{s}{2}\right)$$

Now, let us see what would be the Laplace transform of $\text{erf}(\sqrt{t})$.

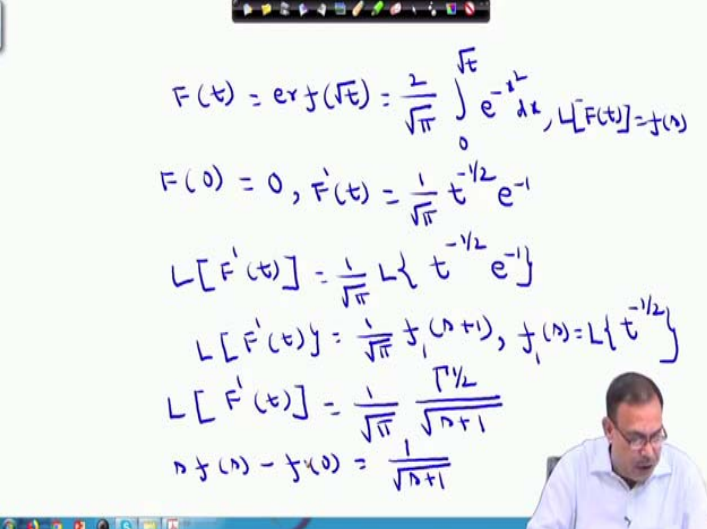
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Example
Find $L\{\text{erf}(\sqrt{t})\}$

Solution:

$$\text{Let, } F(t) = \text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx \text{ and } L\{F(t)\} = f(s)$$
$$\therefore F(0) = 0 \text{ and } F'(t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-t} \text{ (Using Leibniz Integral Rule)}$$
$$\therefore L\{F'(t)\} = \frac{1}{\sqrt{\pi}} L\{t^{-1/2} e^{-t}\}$$


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$$F(t) = \text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx, L\{F(t)\} = f(s)$$
$$F(0) = 0, F'(t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-t}$$
$$L\{F'(t)\} = \frac{1}{\sqrt{\pi}} L\{t^{-1/2} e^{-t}\}$$
$$L\{F'(t)\} = \frac{1}{\sqrt{\pi}} f_1(s), f_1(s) = L\{t^{-1/2}\}$$
$$L\{F'(t)\} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2)}{\sqrt{s+1}}$$
$$n f_1(s) - f_1(0) = \frac{1}{\sqrt{s+1}}$$

Let, $F(t) = \text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx$ and $L\{F(t)\} = f(s)$ so that from here, directly we can say $F(0) = 0$ and using Leibniz integral theorem,

$$F'(t) = \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-t}.$$

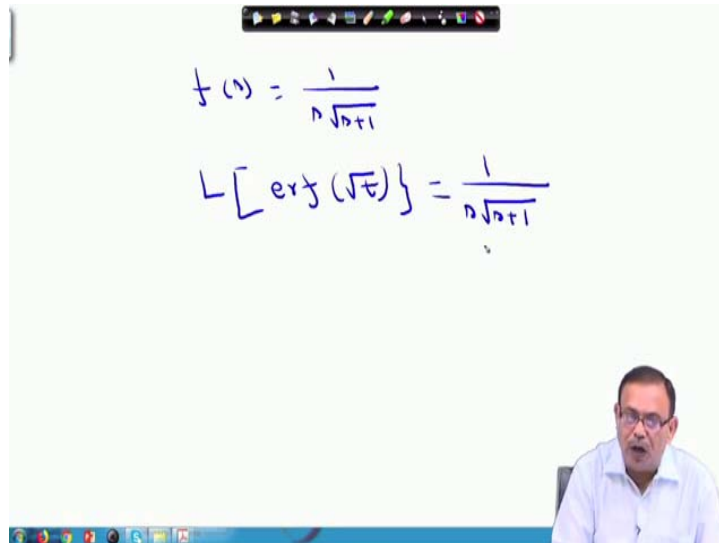
Therefore Laplace transform of $F'(t)$ is,

$$\begin{aligned}
 L\{F'(t)\} &= \frac{1}{\sqrt{\pi}} L\{t^{-1/2}e^{-t}\} = \frac{1}{\sqrt{\pi}} f_1(s+1) \quad \text{where, } f_1(s) = L\{t^{-1/2}\} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \\
 &= \frac{1}{\sqrt{s+1}} \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right).
 \end{aligned}$$

So, we can write down Laplace transform of $F'(t)$ as

$$\begin{aligned}
 sf(s) - F(0) &= \frac{1}{\sqrt{s+1}} \\
 \Rightarrow f(s) = L\{F(t)\} &= \frac{1}{s\sqrt{s+1}} \\
 \Rightarrow L\{\text{erf}(\sqrt{t})\} &= \frac{1}{s\sqrt{s+1}}
 \end{aligned}$$

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$$\begin{aligned} \Rightarrow L\{F'(t)\} &= \frac{1}{\sqrt{\pi}} f_1(s+1) \\ &\text{(Using First shifting Theorem, where } f_1(s) = L\{t^{-1/2}\}) \\ \Rightarrow L\{F'(t)\} &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2})}{\sqrt{s+1}} \quad (\because L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}) \\ \Rightarrow sf(s) - F(0) &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2})}{\sqrt{s+1}} \quad \text{(Using Differentiation Theorem)} \\ \Rightarrow f(s) &= \frac{1}{s\sqrt{s+1}} \quad (\because F(0) = 0 \text{ and } \Gamma(\frac{1}{2}) = \sqrt{\pi}) \\ \Rightarrow L\{\text{erf}(\sqrt{t})\} &= \frac{1}{s\sqrt{s+1}} \end{aligned}$$

Now, we want to find $L\{t \text{erf}(2\sqrt{t})\}$.

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Example
Find $L\{t \text{erf}(2\sqrt{t})\}$

Solution:

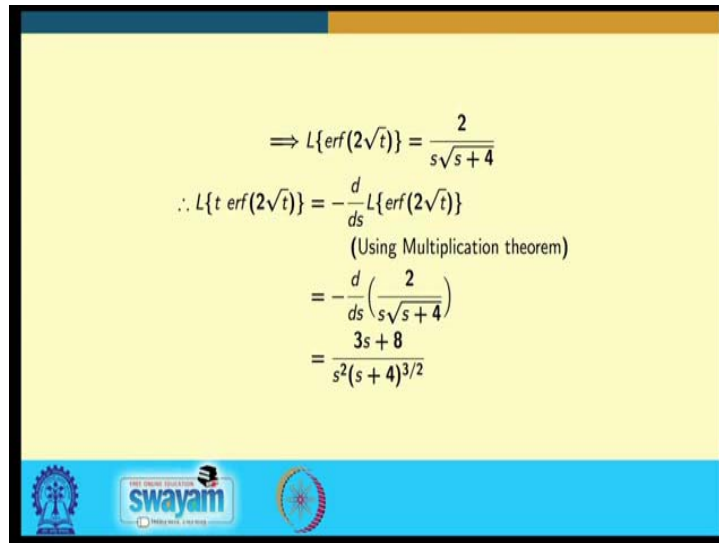
$$\begin{aligned} \text{We have, } L\{\text{erf}(\sqrt{t})\} &= \frac{1}{s\sqrt{s+1}} \\ \Rightarrow L\{\text{erf}(2\sqrt{t})\} &= L\{\text{erf}(\sqrt{4t})\} = \frac{1}{4} \frac{1}{\left(\frac{s}{4}\sqrt{\frac{s}{4}+1}\right)} \\ &\quad (\because L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)) \end{aligned}$$

We know that the Laplace transform of $\text{erf}(\sqrt{t})$ is $\frac{1}{s\sqrt{s+1}}$.

Using the change of scale property, now we can tell

$$L\{\text{erf}(2\sqrt{t})\} = L\{\text{erf}(\sqrt{4t})\} = \frac{1}{4} \frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}$$

(Refer Slide Time: 13:39)


$$\begin{aligned}\Rightarrow L\{\operatorname{erf}(2\sqrt{t})\} &= \frac{2}{s\sqrt{s+4}} \\ \therefore L\{t \operatorname{erf}(2\sqrt{t})\} &= -\frac{d}{ds}L\{\operatorname{erf}(2\sqrt{t})\} \\ &\quad \text{(Using Multiplication theorem)} \\ &= -\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right) \\ &= \frac{3s+8}{s^2(s+4)^{3/2}}\end{aligned}$$

Using the theorem on multiplication by power of t , we have

$$\begin{aligned}L\{t \operatorname{erf}(2\sqrt{t})\} &= -\frac{d}{ds}(L\{\operatorname{erf}(2\sqrt{t})\}) \\ &= -\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right) \\ &= \frac{3s+8}{s^2(s+4)^{3/2}}.\end{aligned}$$

This error function is used in various engineering problems as we always come across this function. And once we know its Laplace transform, it will be very useful for us to solve various problems.

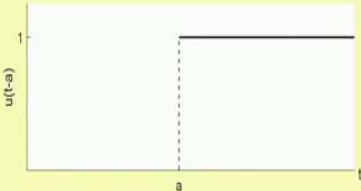
Next, we have the Unit step function or Heaviside's unit function defined as

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

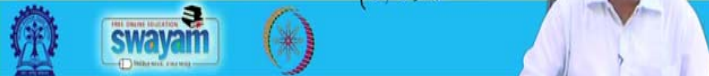
The graphical representation of the function is presented in the lecture slide.

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Unit step function or Heaviside's unit function




Unit step function or Heaviside's unit function $u(t - a)$ is defined by,

$$u(t - a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$


We will now evaluate its Laplace transform. Using the definition, we have,

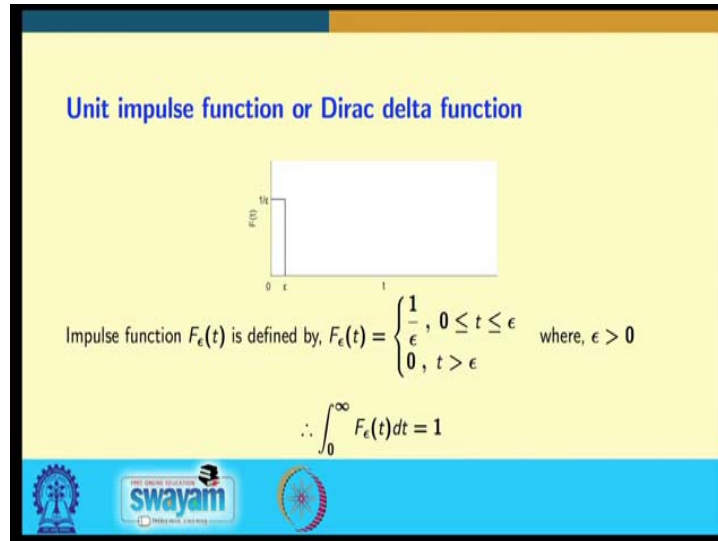
$$\begin{aligned} L\{u(t - a)\} &= \int_0^{\infty} e^{-st} u(t - a) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} \cdot 1 dt \\ &= \frac{e^{-as}}{s}. \end{aligned}$$

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$$\begin{aligned} L\{u(t - a)\} &= \int_0^{\infty} e^{-st} u(t - a) dt \\ &= \int_0^a e^{-st} \cdot 0 dt + \int_a^{\infty} e^{-st} \cdot 1 dt \\ &= \int_a^{\infty} e^{-st} dt \\ &= -\left[\frac{e^{-st}}{s}\right]_a^{\infty} \\ &= \frac{e^{-as}}{s} \end{aligned}$$


There is another important function which appears frequently in various engineering problems, which we call the Unit Impulse function or Dirac delta function denoted by $F_\epsilon(t)$.

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The Dirac delta function, whose graph is presented in the above slide, is defined by

$$F_\epsilon(t) = \begin{cases} 1/\epsilon, & 0 \leq t \leq \epsilon \\ 0, & t > \epsilon \end{cases}$$

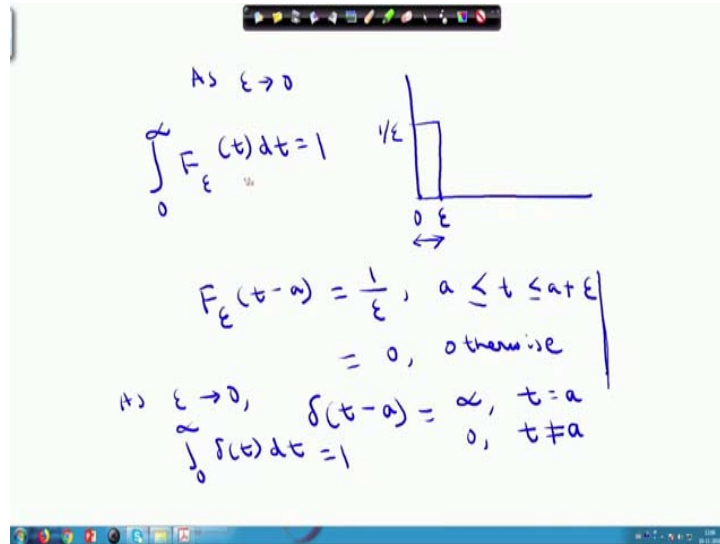
where $\epsilon > 0$. In other sense, we can write as

$$F_\epsilon(t - a) = \begin{cases} 1/\epsilon, & a \leq t \leq a + \epsilon \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Clearly, if we integrate the Dirac delta function from 0 to ∞ , we get the value as 1 i.e.,

$$\int_0^{\infty} F_\epsilon(t) dt = 1.$$

(Refer Slide Time: 17:42)



From the figure, we see that whenever ϵ approaches 0, the height approaches to ∞ in such a manner that the area of this rectangle always will remain 1. So as $\epsilon \rightarrow 0$, we write

$$F_{\epsilon}(t-a) = \delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}$$

and

$$\int_0^{\infty} \delta(t) dt = 1.$$

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Impulse function can be written as,

$$F_{\epsilon}(t-a) = \begin{cases} \frac{1}{\epsilon}, & a \leq t \leq a+\epsilon \\ 0, & \text{otherwise} \end{cases}$$

If $\epsilon \rightarrow 0$ then $F_{\epsilon}(t-a)$ is called Dirac's delta function and defined as,

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}$$
$$\therefore \int_0^{\infty} \delta(t) dt = 1$$

The slide has a yellow background. At the top, it says "Impulse function can be written as," followed by the piecewise definition of $F_{\epsilon}(t-a)$. Below that, it says "If $\epsilon \rightarrow 0$ then $F_{\epsilon}(t-a)$ is called Dirac's delta function and defined as," followed by the piecewise definition of $\delta(t-a)$ and the equation $\therefore \int_0^{\infty} \delta(t) dt = 1$. At the bottom right, there is a small video inset of a man in a white shirt. At the bottom left, there are logos for "THE ONLINE EDUCATION swayam" and "INDIAN INSTITUTE OF TECHNOLOGY KANPUR".

For any given function $f(t)$, we have,

$$\begin{aligned} \int_0^{\infty} f(t) F_{\epsilon}(t-a) dt &= \int_a^{a+\epsilon} f(t) \frac{1}{\epsilon} dt \\ &= (a + \epsilon - a) \cdot f(\eta) \cdot \frac{1}{\epsilon} \quad (\text{Using Mean Value Theorem}) \\ &= f(\eta) \end{aligned}$$

(Refer Slide Time: 21:16)

The slide displays the following mathematical derivation:

$$\begin{aligned} \int_0^{\infty} f(t) F_{\epsilon}(t-a) dt &= \int_a^{a+\epsilon} f(t) \frac{1}{\epsilon} dt \\ &= (a + \epsilon - a) \cdot f(\eta) \cdot \frac{1}{\epsilon} \\ &\quad (\text{Using Mean Value theorem where } a < \eta < a + \epsilon) \\ &= f(\eta) \end{aligned}$$

The slide also features the Swayam logo and a small video inset of the instructor in the bottom right corner.

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The slide shows the following handwritten mathematical expressions:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(t) F_{\epsilon}(t-a) dt &= \lim_{\epsilon \rightarrow 0} f(\eta) \\ \int_0^{\infty} f(t) \delta(t-a) dt &= f(a) \\ \int_0^{\infty} f(t) \delta(t) dt &= f(0) \end{aligned}$$

The slide includes a video inset of the instructor in the bottom right corner.

Now, taking limits on both sides as ϵ approaches 0, we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(t) F_{\epsilon}(t-a) dt &= \lim_{\epsilon \rightarrow 0} f(\eta) \\ \Rightarrow \int_0^{\infty} f(t) \delta(t-a) dt &= f(a) \quad (2) \\ \Rightarrow \int_0^{\infty} f(t) \delta(t) dt &= f(0) \quad \text{for } a = 0. \end{aligned}$$

If we put $f(t) = e^{-st}$ in (2), then we get

$$\begin{aligned} \int_0^{\infty} e^{-st} \delta(t-a) dt &= e^{-sa} \\ \Rightarrow L\{\delta(t-a)\} &= e^{-sa} \quad (\text{by definition}) \\ \Rightarrow L\{\delta(t)\} &= 1 \quad \text{for } a = 0. \end{aligned}$$

(Refer Slide Time: 23:34)

The image shows a whiteboard with handwritten mathematical derivations. The equations are:

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st} \delta(t-a) dt = e^{-sa}$$

$$L\{\delta(t)\} = 1$$

A person is visible in the bottom right corner of the frame, likely the presenter.

(Refer Slide Time: 24:50)

The slide displays the following mathematical derivations:

$$\begin{aligned} \therefore \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(t) F_{\epsilon}(t-a) dt &= \lim_{\epsilon \rightarrow 0} f(a) \\ \Rightarrow \int_0^{\infty} f(t) \delta(t-a) dt &= f(a) \\ \Rightarrow \int_0^{\infty} f(t) \delta(t) dt &= f(0) \end{aligned}$$
$$\begin{aligned} L\{\delta(t-a)\} &= \int_0^{\infty} e^{-st} \delta(t-a) dt \\ &= e^{-sa} \quad (\text{Using property of Dirac's delta function}) \end{aligned}$$
$$\therefore L\{\delta(t)\} = 1 \quad (\text{by putting } a = 0)$$

The slide also features the Swayam logo and a small video inset of a man in a white shirt.

Now, let us see some applications. Say, we want to find out the Laplace transform of

$$F(t) = \sin 2t \cdot \delta\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2t} \cdot \delta\left(t - \frac{\pi}{2}\right).$$

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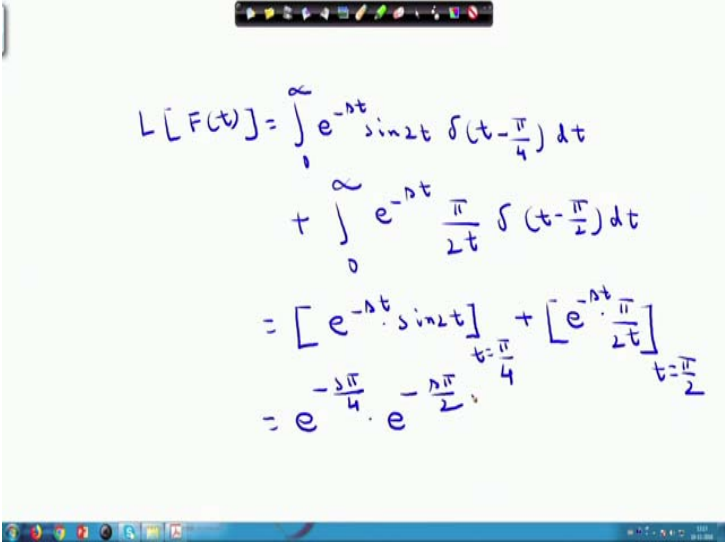
Example
Find the Laplace transformation of $F(t) = \sin 2t \delta\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2t} \delta\left(t - \frac{\pi}{2}\right)$

Solution:

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} \sin 2t \delta\left(t - \frac{\pi}{4}\right) dt + \int_0^{\infty} e^{-st} \frac{\pi}{2t} \delta\left(t - \frac{\pi}{2}\right) dt \\ &= \left[e^{-st} \sin 2t \right]_{t=\frac{\pi}{4}} + \left[e^{-st} \frac{\pi}{2t} \right]_{t=\frac{\pi}{2}} \\ &= e^{-\frac{s\pi}{4}} + e^{-\frac{s\pi}{2}} \end{aligned}$$

The slide also features the Swayam logo.

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The image shows a handwritten derivation of the Laplace transform of a function involving Dirac delta functions. The derivation is as follows:

$$\begin{aligned} L[F(t)] &= \int_0^{\infty} e^{-st} \sin 2t \delta\left(t - \frac{\pi}{4}\right) dt \\ &\quad + \int_0^{\infty} e^{-st} \frac{\pi}{2t} \delta\left(t - \frac{\pi}{2}\right) dt \\ &= \left[e^{-st} \sin 2t \right]_{t=\frac{\pi}{4}} + \left[e^{-st} \frac{\pi}{2t} \right]_{t=\frac{\pi}{2}} \\ &= e^{-\frac{s\pi}{4}} \cdot e^{-\frac{s\pi}{2}} \end{aligned}$$

Using the definition of Laplace transform, we have,

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} \left[\sin 2t \cdot \delta\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2t} \cdot \delta\left(t - \frac{\pi}{2}\right) \right] dt \\ &= \int_0^{\infty} e^{-st} \sin 2t \cdot \delta\left(t - \frac{\pi}{4}\right) dt + \frac{\pi}{2} \int_0^{\infty} e^{-st} \frac{1}{t} \cdot \delta\left(t - \frac{\pi}{2}\right) dt \\ &= \left[e^{-st} \sin 2t \right]_{t=\frac{\pi}{4}} + \frac{\pi}{2} \left[e^{-st} \frac{1}{t} \right]_{t=\frac{\pi}{2}} \\ &= e^{-\frac{s\pi}{4}} + e^{-\frac{s\pi}{2}}. \end{aligned}$$

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Dirac Delta Function (unit impulse function)

The idea of a very large force acting for a short time is of frequent occurrence in Mechanics. To deal with such and similar ideas, Dirac Delta function was introduced.

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The idea of a very large force acting for a very short duration is a frequent occurrence in mechanics. In order to deal with such and similar ideas, Dirac delta function was introduced and in mechanics, it is being used very frequently which marks its importance. Therefore, it becomes essential to have a knowledge about the Laplace transform of this function.

(Refer Slide Time: 28:59)

A rectangular pulse function $F(t)$ is defined by

$$F(t) = h[u_a(t - a) - u_b(t - b)],$$

where $0 < a < b$ and h is a constant.

$$\therefore L\{F(t)\} = \int_0^{\infty} e^{-st} h[u_a(t - a) - u_b(t - b)] dt$$
$$= \frac{h}{s} (e^{-sa} - e^{-sb})$$

Now if " a " is fixed, and " b " tends to " a " and $h \rightarrow \infty$ in such a way that

$$\lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b - a) = 1$$

The slide features a yellow background with a blue header and footer. The footer contains the Swayam logo and the text 'FREE ONLINE EDUCATION swayam'.

We define a rectangular pulse function as

$$F(t) = h[u_a(t - a) - u_b(t - b)]$$

where $0 < a < b$ and h is a constant. By definition, the Laplace transform of $F(t)$ is given by

$$\begin{aligned} L\{F(t)\} &= \int_0^{\infty} e^{-st} h[u_a(t-a) - u_b(t-b)] dt \\ &= \frac{h}{s} (e^{-sa} - e^{-sb}) \end{aligned}$$

Now, we keep a fixed and let $b \rightarrow a, h \rightarrow \infty$, such that

$$\lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b-a) = 1.$$

Then, we have,

$$\begin{aligned} \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} L\{F(t)\} &= \frac{1}{s} \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b-a) \left[\frac{e^{-sa} - e^{-sb}}{b-a} \right] \\ &= \frac{1}{s} \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} \left[\frac{e^{-sa} - e^{-sb}}{b-a} \right] \cdot \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b-a) \\ &= \frac{1}{s} s e^{-sa} \cdot 1 \quad (\text{using L'Hospital's Rule}) \\ &= e^{-sa}. \end{aligned}$$

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Then, $\lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} L\{F(t)\} = \frac{1}{s} \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b-a) \cdot \left[\frac{e^{-sa} - e^{-sb}}{b-a} \right]$

$$= \frac{1}{s} \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} \left[\frac{e^{-sa} - e^{-sb}}{b-a} \right] \cdot \lim_{\substack{b \rightarrow a \\ h \rightarrow \infty}} h(b-a)$$

$$= \frac{1}{s} \cdot s \cdot e^{-sa} \cdot 1 \quad [\text{Using L'Hospital's Rule}]$$

$$= e^{-sa}$$

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Thank you.