Transform Calculus and its Applications in Differential Equations
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Lecture – 09
Error Function, Dirac Delta Function and their Laplace Transform

In this lecture, initially, we will go through some useful well-known functions and we will evaluate their Laplace transform as well. The first one that we will cover is the Error function.

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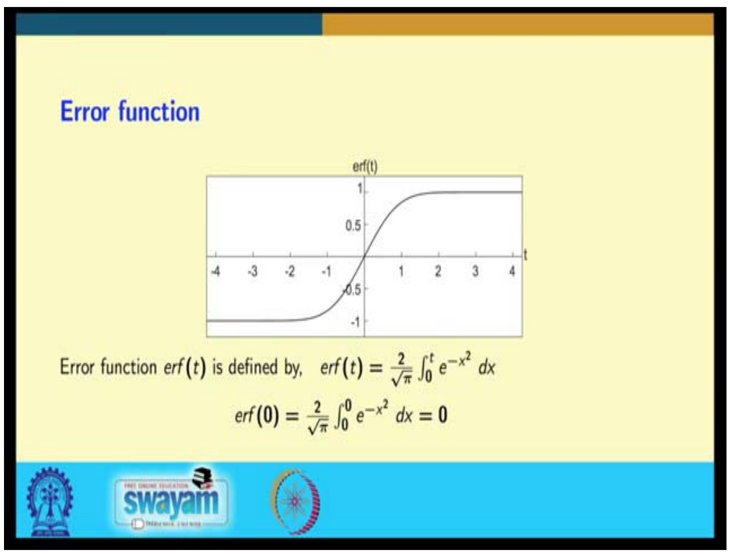

The Error function denoted by $\mathrm{erf}(t)$ is defined by

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$$

and the graph of this function is shown in the slide. Now clearly, at $t = 0$, the value will be equal to 0 i.e.,

$$\mathrm{erf}(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-x^2}\,dx = 0.$$

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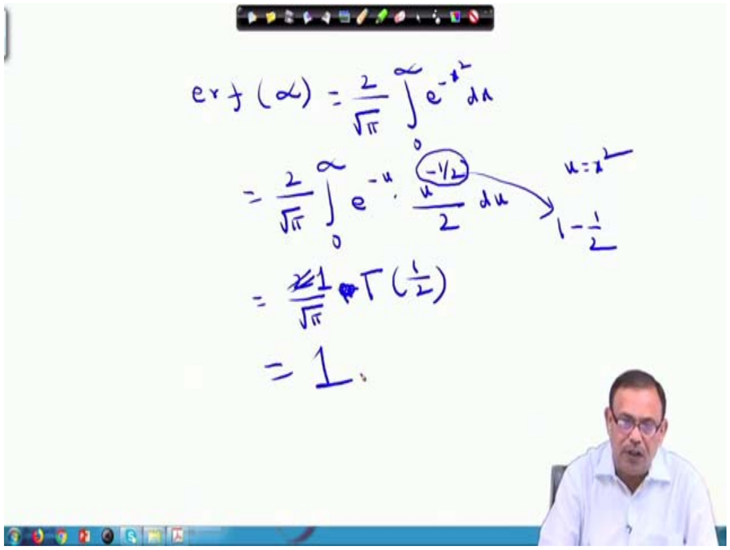

Next we see the value of erf(∞). So, by definition, it will be

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx$$

We substitute $u = x^2$ so that $du = 2xdx$ i.e., $dx = \frac{du}{2\sqrt{u}}$ and the limits of the integration will remain unchanged. Then we have,

$$\operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du$$

We can express this as a Gamma function directly.

$$\therefore \operatorname{erf}(\infty) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \quad \left(\because \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

So, the value of the Error function, as $t \to \infty$, is equal to 1.

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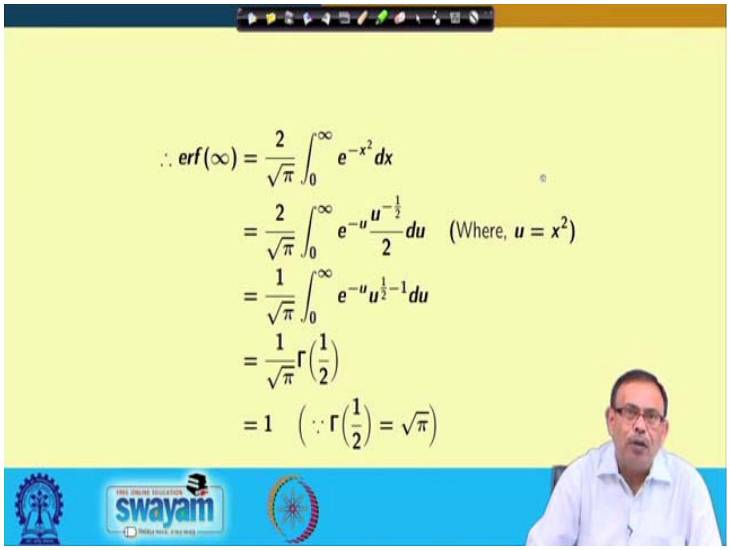

There is another function which we call Complementary Error function denoted by $\mathrm{erf}_c(t)$ and it is defined as

$$\mathrm{erf}_c(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx.$$

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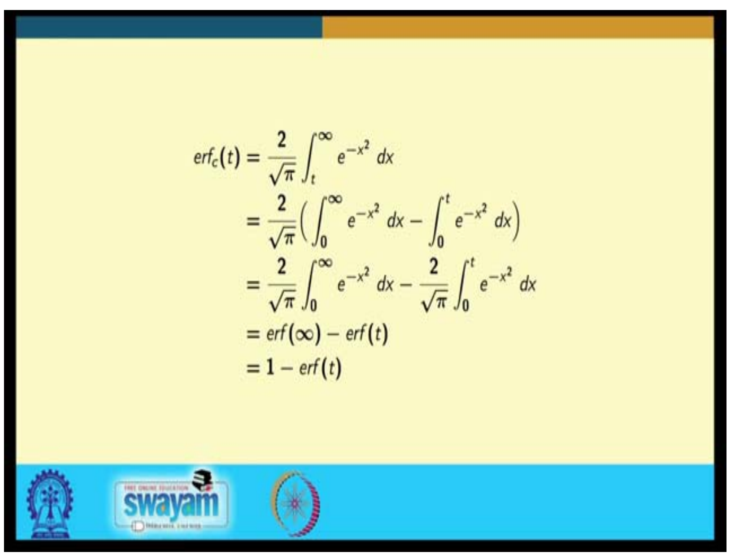

So, we can break the integral into two parts as:

$$\text{erf}_c(t) = \frac{2}{\sqrt{\pi}}\left(\int_0^\infty e^{-x^2}dx - \int_0^t e^{-x^2}dx\right)$$

This we can write down as

$$\text{erf}_c(t) = \text{erf}(\infty) - \text{erf}(t)$$
$$= 1 - \text{erf}(t).$$

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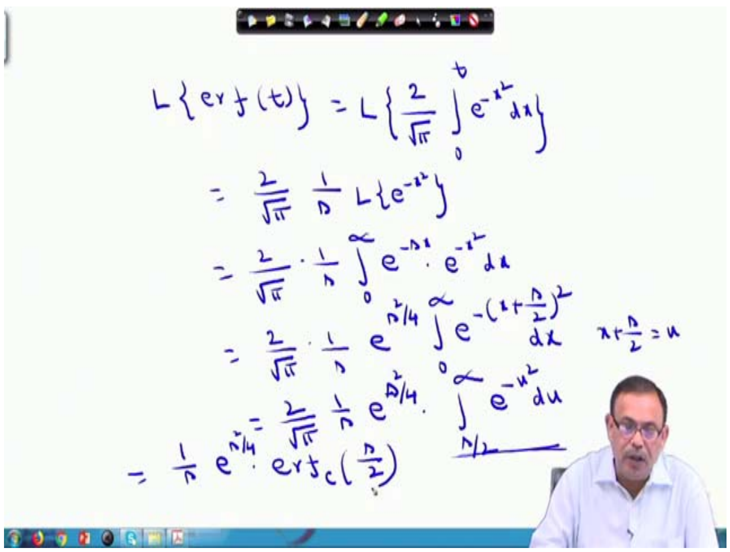

These functions are very useful for solving various kinds of engineering problems. Now we evaluate the Laplace transform of $\text{erf}(t)$.

$$L\{\text{erf}(t)\} = L\left\{\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx\right\}$$
$$= \frac{2}{\sqrt{\pi}}\frac{1}{s}L\{e^{-x^2}\}, \qquad \text{(using Integral theorem)}$$

From the basic definition of Laplace transform we get,

$$L\{\text{erf}(t)\} = \frac{2}{\sqrt{\pi}}\frac{1}{s}e^{\frac{s^2}{4}}\int_0^\infty e^{-\left(x+\frac{s}{2}\right)^2}dx$$

Refer the slide above for detailed steps. If we take $x + \frac{s}{2} = u$, in that case, this will be converted into

$$L\{\mathrm{erf}(t)\} = \frac{2}{\sqrt{\pi}}\frac{1}{s}e^{\frac{s^2}{4}}\int_{\frac{s}{2}}^{\infty} e^{-u^2}du$$

$$= \frac{1}{s}e^{\frac{s^2}{4}}\,\mathrm{erf_c}\left(\frac{s}{2}\right).$$

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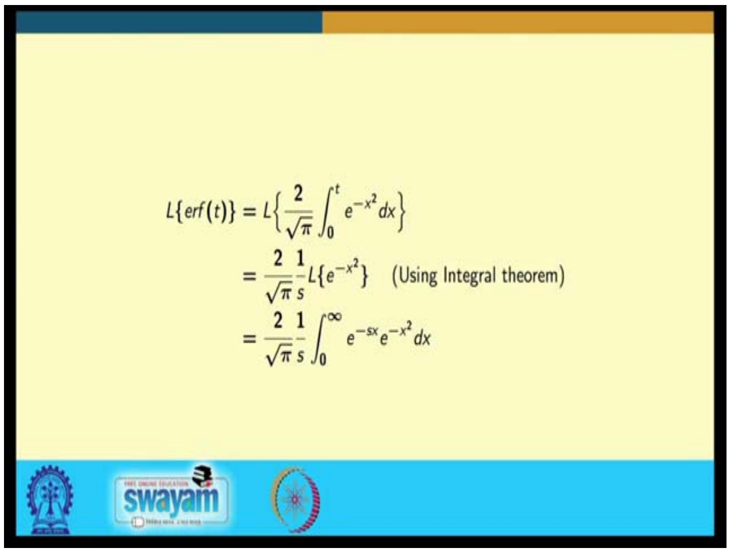

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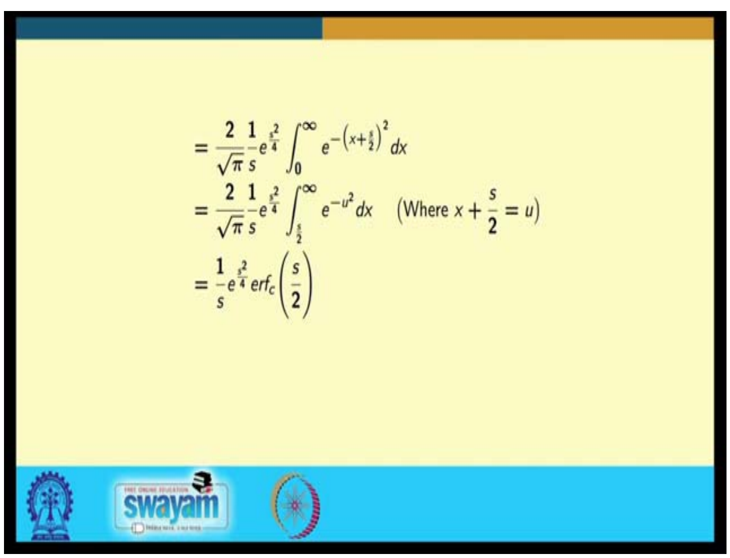

Now, let us see what would be the Laplace transform of $\mathrm{erf}(\sqrt{t})$.

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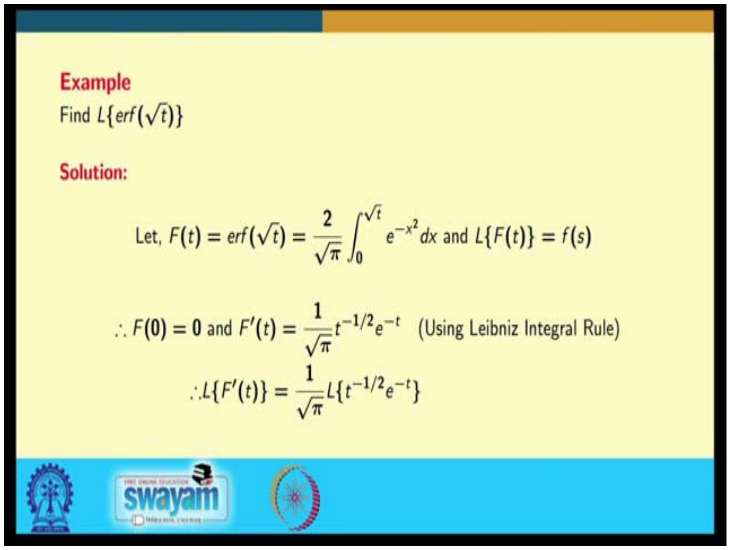

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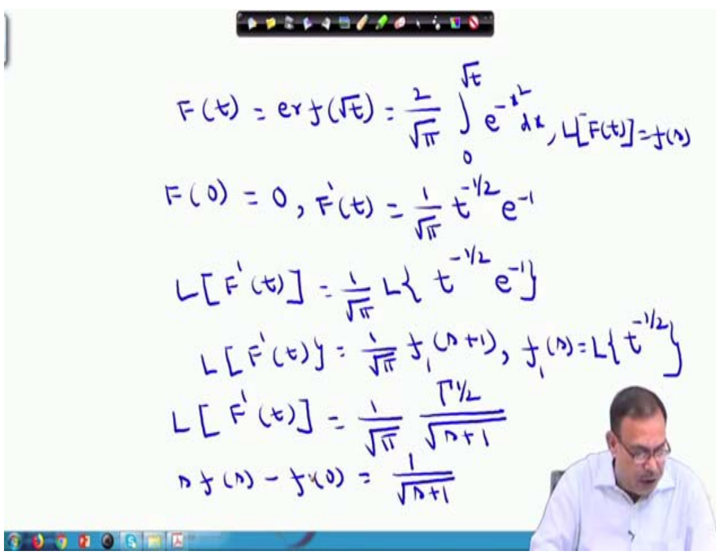

Let, $F(t) = \text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx$ and $L\{F(t)\} = f(s)$ so that from here, directly we can say $F(0) = 0$ and using Leibniz integral theorem,

$$F'(t) = \frac{1}{\sqrt{\pi}}\, t^{-1/2}e^{-t}.$$

Therefore Laplace transform of $F'(t)$ is,

$$L\{F'(t)\} = \frac{1}{\sqrt{\pi}}\, L\{t^{-1/2}e^{-t}\} = \frac{1}{\sqrt{\pi}}\, f_1(s+1) \quad \text{where,} \quad f_1(s) = L\{t^{-1/2}\} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}$$

$$= \frac{1}{\sqrt{s+1}} \quad \left(\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right).$$

So, we can write down Laplace transform of $F'(t)$ as

$$sf(s) - F(0) = \frac{1}{\sqrt{s+1}}$$

$$\Rightarrow f(s) = L\{F(t)\} = \frac{1}{s\sqrt{s+1}}$$

$$\Rightarrow L\{erf(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}}$$

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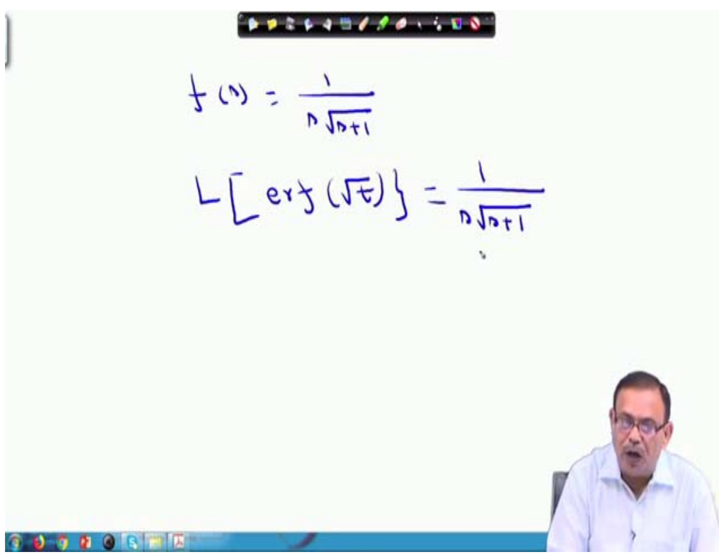

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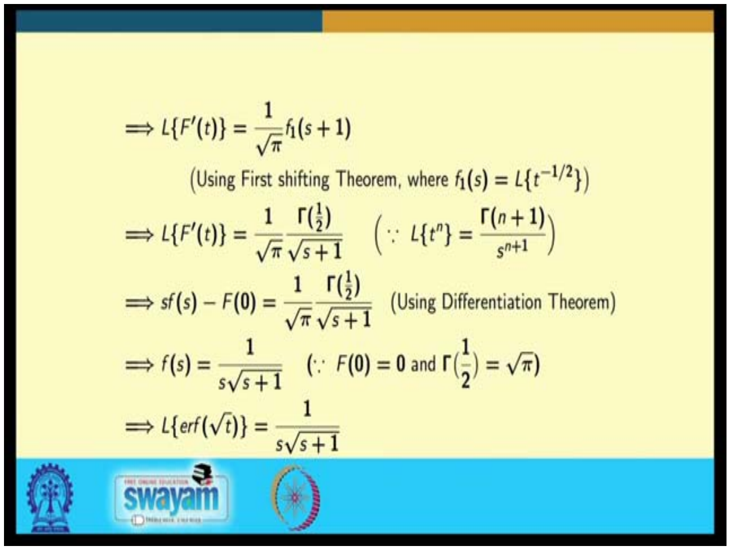

Now, we want to find $L\{t\,\mathrm{erf}(2\sqrt{t})\}$.

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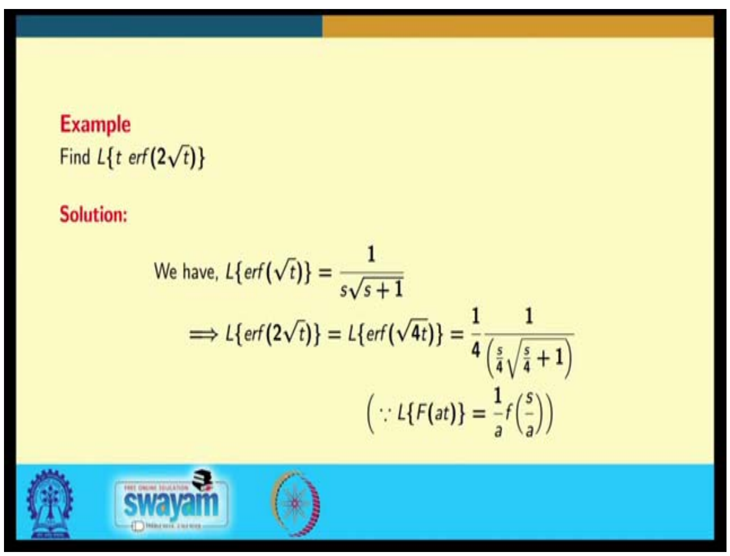

We know that the Laplace transform of $\mathrm{erf}(\sqrt{t})$ is $\frac{1}{s\sqrt{s+1}}$.

Using the change of scale property, now we can tell

$$L\{\mathrm{erf}(2\sqrt{t})\} = L\{\mathrm{erf}(\sqrt{4t})\} = \frac{1}{4} \frac{1}{\frac{s}{4}\sqrt{\frac{s}{4}+1}} = \frac{2}{s\sqrt{s+4}}.$$

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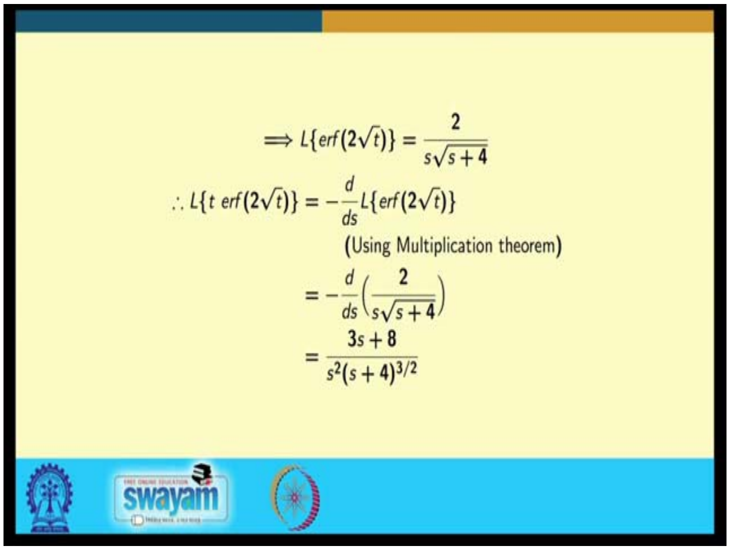

Using the theorem on multiplication by power of t, we have

$$\begin{aligned} L\{t\,\mathrm{erf}(2\sqrt{t})\} &= -\frac{d}{ds}\left(L\{\mathrm{erf}(2\sqrt{t})\}\right) \\ &= -\frac{d}{ds}\left(\frac{2}{s\sqrt{s+4}}\right) \\ &= \frac{3s+8}{s^2(s+4)^{3/2}}. \end{aligned}$$

This error function is used in various engineering problems as we always come across this function. And once we know its Laplace transform, it will be very useful for us to solve various problems.

Next, we have the Unit step function or Heaviside’s unit function defined as

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

The graphical representation of the function is presented in the lecture slide.

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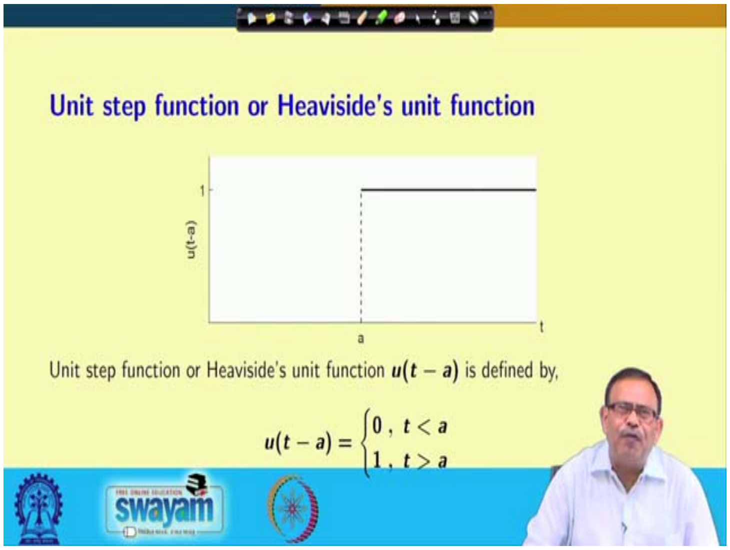

We will now evaluate its Laplace transform. Using the definition, we have,

$$
\begin{aligned}
L\{u(t-a)\} &= \int_0^\infty e^{-st}\, u(t-a)dt \\
&= \int_0^a e^{-st}\,.\,0\; dt + \int_a^\infty e^{-st}\,.\,1\; dt \\
&= \frac{e^{-as}}{s}.
\end{aligned}
$$

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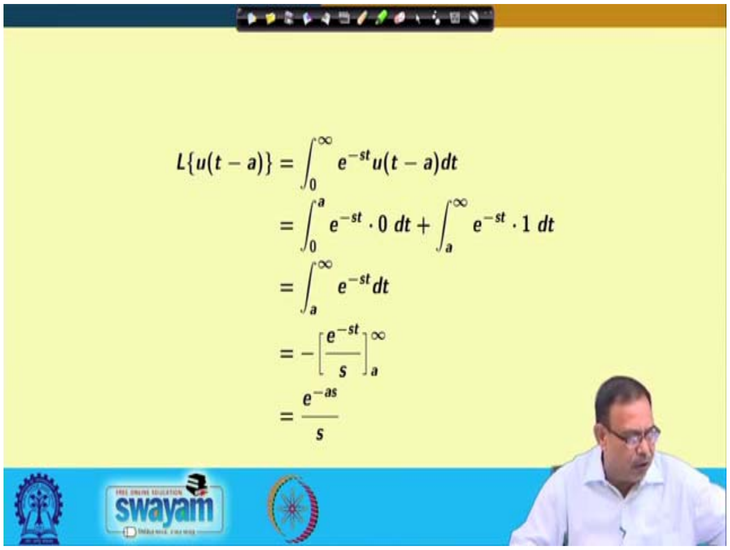

There is another important function which appears frequently in various engineering problems, which we call the Unit Impulse function or Dirac delta function denoted by $F_\epsilon(t)$.

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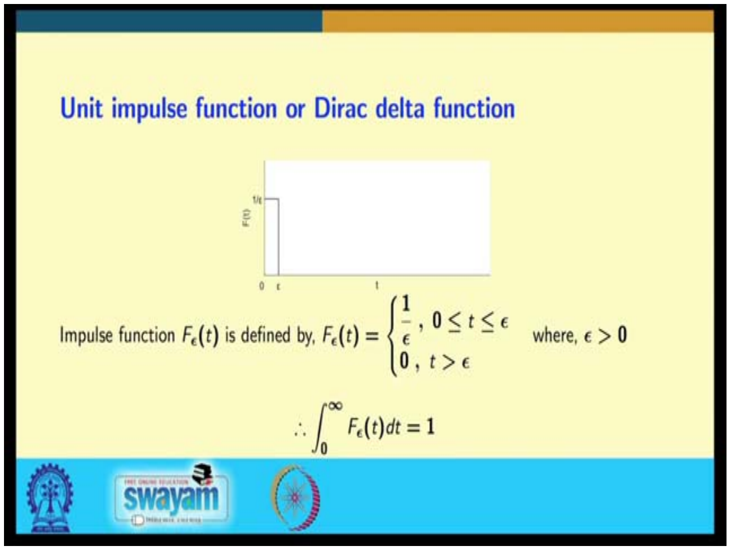

The Dirac delta function, whose graph is presented in the above slide, is defined by

$$F_\epsilon(t) = \begin{cases} 1/\epsilon, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$$

where $\epsilon > 0$. In other sense, we can write as

$$F_\epsilon(t-a) = \begin{cases} 1/\epsilon, & a \le t \le a+\epsilon \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Clearly, if we integrate the Dirac delta function from 0 to ∞, we get the value as 1 i.e.,

$$\int_0^\infty F_\epsilon(t)\, dt = 1.$$

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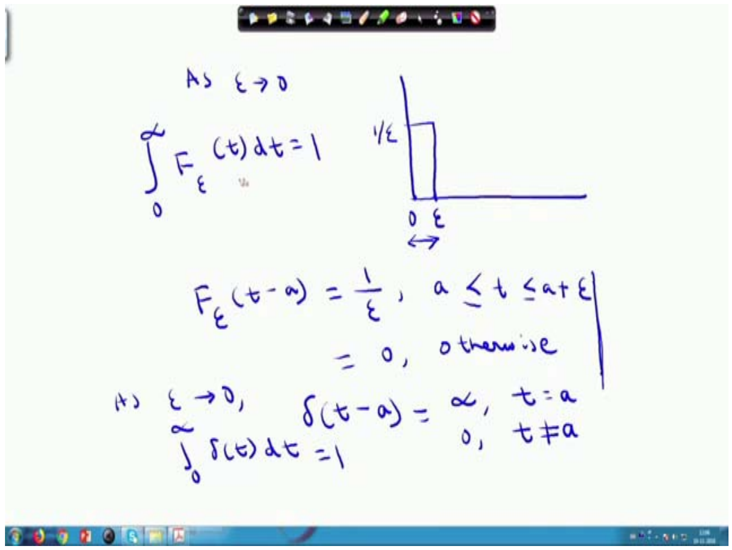

From the figure, we see that whenever ϵ approaches 0, the height approaches to ∞ in such a manner that the area of this rectangle always will remain 1. So as $\epsilon \to 0$, we write

$$F_\epsilon(t-a) = \delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$$

and

$$\int_0^\infty \delta(t)\, dt = 1.$$

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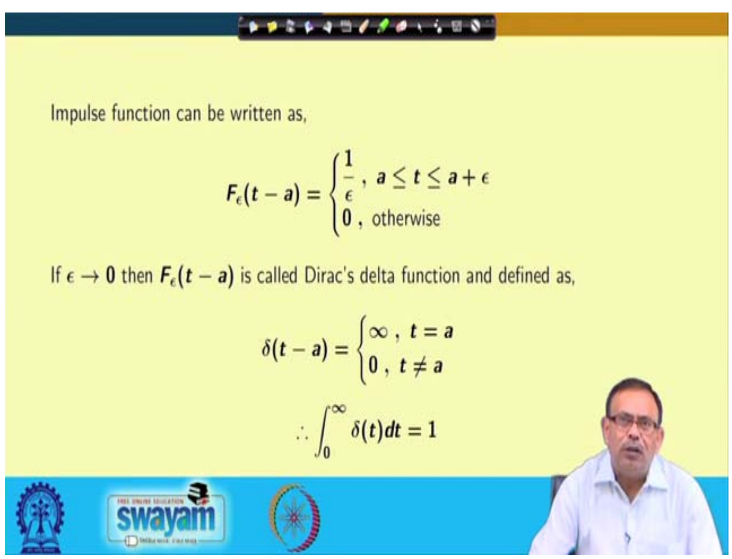

For any given function $f(t)$, we have,

$$\int_0^\infty f(t)\, F_\epsilon(t-a)dt = \int_a^{a+\epsilon} f(t)\ \frac{1}{\epsilon}\ dt$$

$$= (a+\epsilon-a).\, f(\eta).\frac{1}{\epsilon} \quad \text{(Using Mean Value Theorem)}$$

$$= f(\eta)$$

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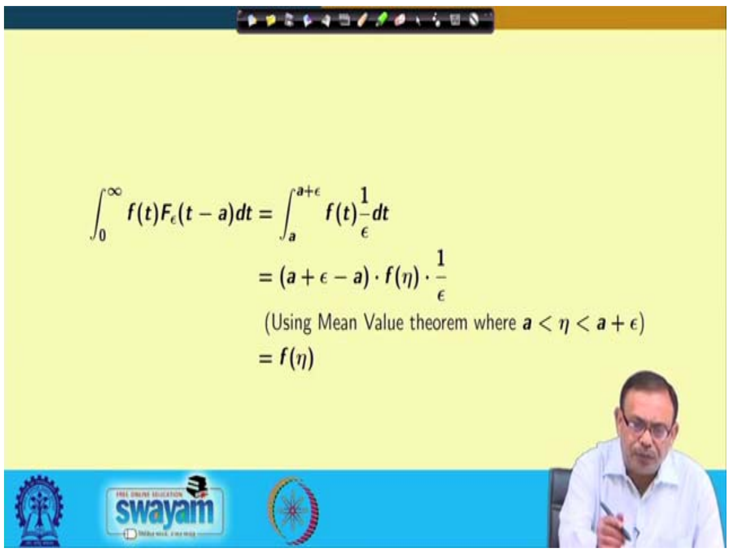

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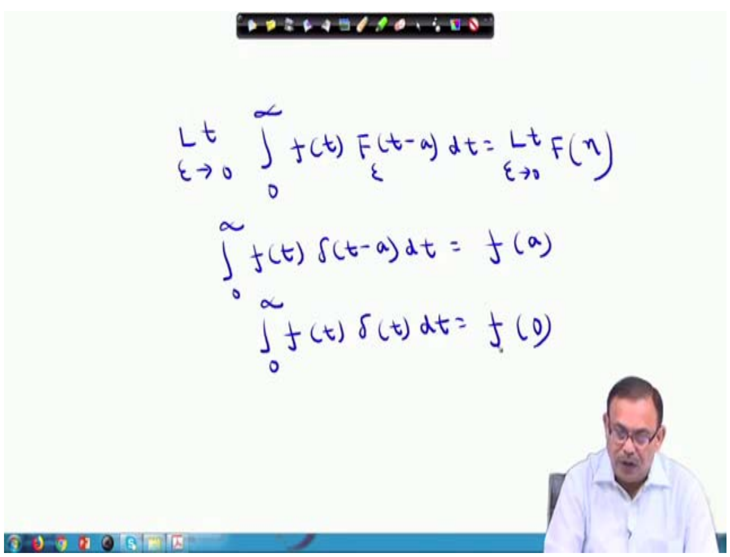

Now, taking limits on both sides as ϵ approaches 0, we have

$$\lim_{\epsilon \to 0} \int_0^\infty f(t)\, F_\epsilon(t-a)dt = \lim_{\epsilon \to 0} f(\eta)$$

$$\Rightarrow \int_0^\infty f(t)\, \delta(t-a)dt = f(a) \qquad (2)$$

$$\Rightarrow \int_0^\infty f(t)\, \delta(t)dt = f(0) \quad \text{for } a = 0.$$

If we put $f(t) = e^{-st}$ in (2), then we get

$$\int_0^\infty e^{-st}\, \delta(t-a)dt = e^{-sa}$$

$$\Rightarrow L\{\delta(t-a)\} = e^{-sa} \qquad \text{(by definition)}$$

$$\Rightarrow L\{\delta(t)\} = 1 \quad \text{for } a = 0.$$

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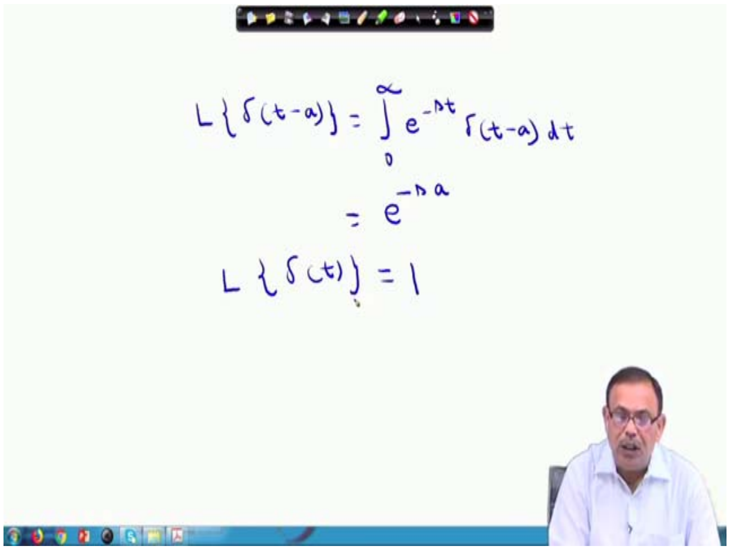

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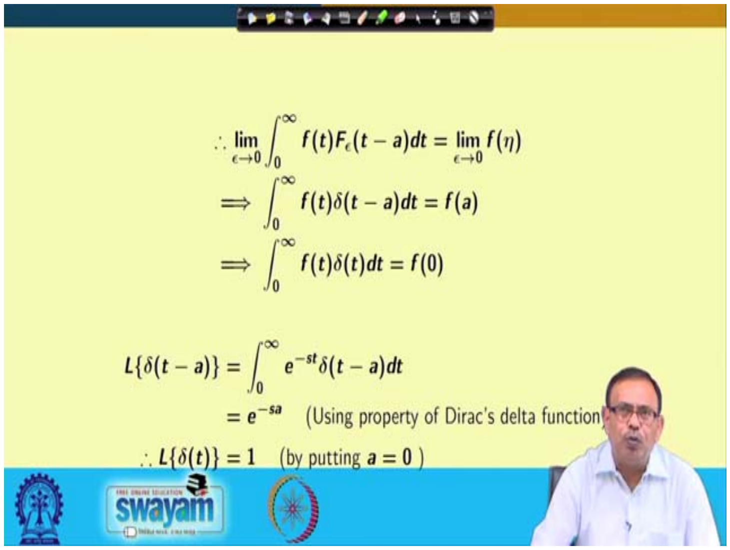

Now, let us see some applications. Say, we want to find out the Laplace transform of

$$F(t) = \sin 2t\,.\,\delta\left(t-\frac{\pi}{4}\right) + \frac{\pi}{2t}.\,\delta\left(t-\frac{\pi}{2}\right).$$

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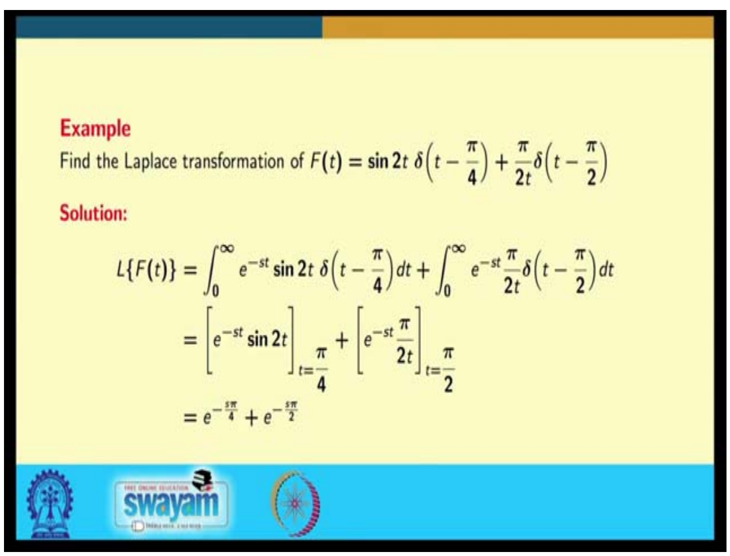

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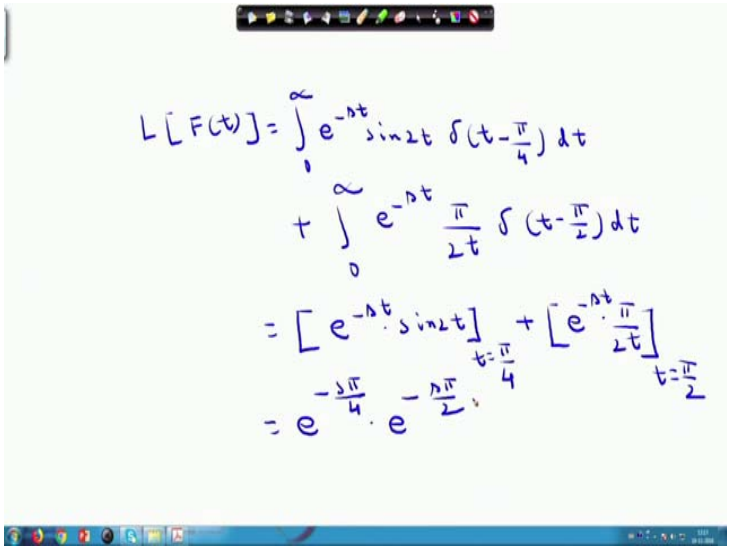

Using the definition of Laplace transform, we have,

$$
\begin{aligned}
L\{F(t)\} &= \int_0^\infty e^{-st}\left[\sin 2t\,.\,\delta\left(t-\frac{\pi}{4}\right)+\frac{\pi}{2t}.\,\delta\left(t-\frac{\pi}{2}\right)\right]dt \\
&= \int_0^\infty e^{-st}\sin 2t\,.\,\delta\left(t-\frac{\pi}{4}\right)dt+\frac{\pi}{2}\int_0^\infty e^{-st}\frac{1}{t}.\,\delta\left(t-\frac{\pi}{2}\right)dt \\
&= \left[e^{-st}\sin 2t\right]_{t=\frac{\pi}{4}}+\frac{\pi}{2}\left[e^{-st}\frac{1}{t}\right]_{t=\frac{\pi}{2}} \\
&= e^{-\frac{\pi s}{4}}+e^{-\frac{\pi s}{2}}.
\end{aligned}
$$

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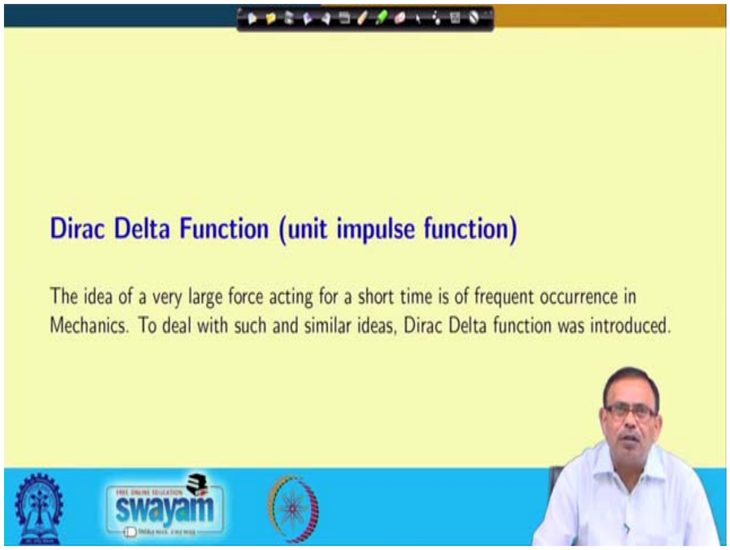

The idea of a very large force acting for a very short duration is a frequent occurrence in mechanics. In order to deal with such and similar ideas, Dirac delta function was introduced and in mechanics, it is being used very frequently which marks its importance. Therefore, it becomes essential to have a knowledge about the Laplace transform of this function.

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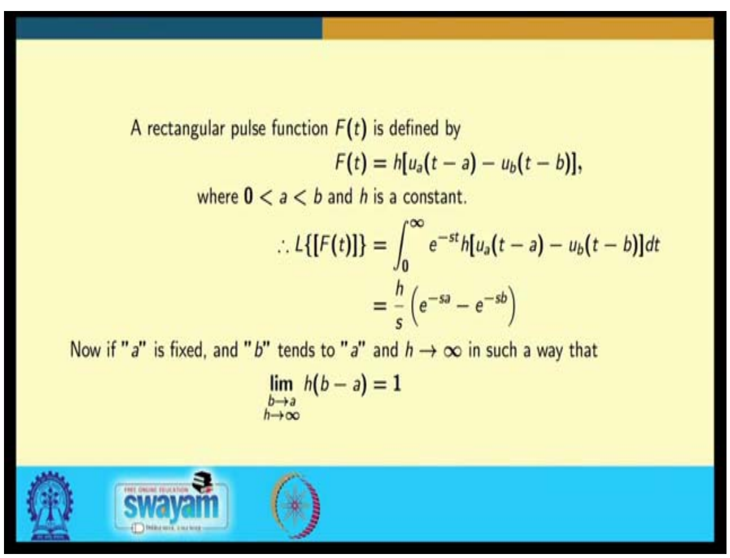

We define a rectangular pulse function as

$$F(t) = h[u_a(t-a) - u_b(t-b)]$$

where $0 < a < b$ and h is a constant. By definition, the Laplace transform of $F(t)$ is given by

$$L\{F(t)\} = \int_0^\infty e^{-st}\, h[u_a(t-a) - u_b(t-b)]dt$$
$$= \frac{h}{s}(e^{-sa} - e^{-sb})$$

Now, we keep a fixed and let $b \to a, h \to \infty$, such that

$$\lim_{\substack{b\to a \\ h\to\infty}} h(b-a) = 1.$$

Then, we have,

$$\lim_{\substack{b\to a \\ h\to\infty}} L\{F(t)\} = \frac{1}{s} \lim_{\substack{b\to a \\ h\to\infty}} h(b-a)\left[\frac{e^{-sa} - e^{-sb}}{b-a}\right]$$
$$= \frac{1}{s} \lim_{b\to a}\left[\frac{e^{-sa} - e^{-sb}}{b-a}\right] . \lim_{\substack{b\to a \\ h\to\infty}} h(b-a)$$
$$= \frac{1}{s} s e^{-sa}. 1 \qquad \text{(using L'Hospital's Rule)}$$
$$= e^{-sa}.$$

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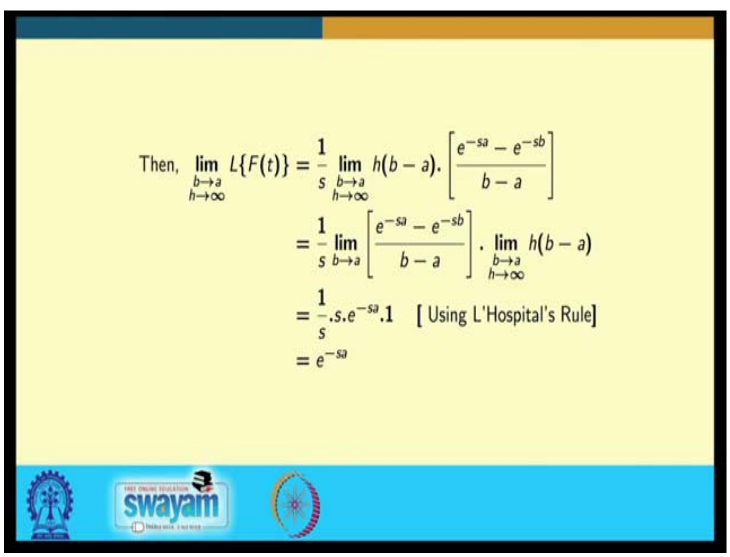

Thank you.