

**Transform Calculus and its Applications in Differential Equations**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture – 08**  
**Laplace Transform of some special Functions**

Welcome again. In the last lecture, we have discussed the Laplace transform of a periodic function.

(Refer Slide Time: 00:31)

$$L\{\sin t\} = \frac{1}{1 - e^{-s\pi}} \int_0^{\pi} e^{-st} \sin t \, dt$$

$$= \frac{1}{1 - e^{-s\pi}} \times I(\text{say})$$

$$\therefore I = \int_0^{\pi} e^{-st} \sin t \, dt$$

$$= -[\cos te^{-st}]_0^{\pi} - s \int_0^{\pi} e^{-st} \cos t \, dt$$

$$= -[\cos te^{-st}]_0^{\pi} - s[\sin te^{-st}]_0^{\pi} - s^2 \int_0^{\pi} e^{-st} \sin t \, dt$$

The slide also features a video inset of Prof. Adrijit Goswami in the bottom right corner and logos for IIT Kharagpur and SWAYAM in the bottom left corner.

(Refer Slide Time: 00:40)

$$L[F(t)] = \frac{\int_0^T e^{-st} F(t) \, dt}{1 - e^{-sT}}$$

The slide also features a video inset of Prof. Adrijit Goswami in the bottom right corner.

(Refer Slide Time: 02:02)

**Example**  
Find the Laplace transformation of  $F(t) = |\sin t|$ .

**Solution:**

$F(t) = |\sin t|$  is a periodic function with period  $\pi$  defined by,

$F(t) = \sin t, 0 \leq t < \pi$   
 $F(t + \pi) = F(t)$

Let us take the example where we need to find the Laplace transform of  $F(t) = |\sin t|$ . We know that  $F(t) = |\sin t|$  is a periodic function with period  $\pi$ , which is defined by

$$F(t) = \sin t, \quad 0 \leq t \leq \pi \quad \text{and} \quad F(t + \pi) = F(t).$$

And, on the right side of the slide, we can see the graph for this periodic function.

So, our aim is to find out  $L\{|\sin t|\}$ , where  $|\sin t|$  is a periodic function of period  $\pi$ . So, let us see the solution process for this.

(Refer Slide Time: 03:02)

$$L[|\sin t|] = \frac{1}{1 - e^{-s\pi}} \int_0^{\pi} e^{-st} \sin t dt$$

$$= \frac{1}{1 - e^{-s\pi}} \times I$$

$$I = \int_0^{\pi} e^{-st} \sin t dt =$$

$$= - \left[ \cos t e^{-st} \right]_0^{\pi} - \int_0^{\pi} e^{-st} \cos t dt$$

$$= - \left[ \cos t e^{-st} \right]_0^{\pi} - \int_0^{\pi} \sin t e^{-st} dt$$

$$= - \left[ \cos t e^{-st} \right]_0^{\pi} - \int_0^{\pi} e^{-st} \sin t dt$$

(I)

We can write Laplace transform of periodic function using the derived formula as:

$$L\{|\sin t|\} = \frac{1}{1 - e^{-s\pi}} \int_0^{\pi} e^{-st} \sin t \, dt$$

Now, this integral can be evaluated easily using the formula  $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$ . Therefore,

$$L\{|\sin t|\} = \frac{1}{1 - e^{-s\pi}} I \quad \text{where, } I = \int_0^{\pi} e^{-st} \sin t \, dt.$$

And, we can evaluate this integral to obtain

$$I = \frac{e^{-s\pi} + 1}{1 + s^2}.$$

(Refer Slide Time: 05:29)

$$\begin{aligned} (1+s^2)I &= -[(\cos t + s \sin t)e^{-st}]_0^{\pi} \\ &= e^{-s\pi} + 1 \\ I &= \frac{1 + e^{-s\pi}}{1 + s^2} \\ L[|\sin t|] &= \frac{1}{1 + s^2} \cdot \frac{1 + e^{-s\pi}}{1 - e^{-s\pi}} \end{aligned}$$

So, that Laplace transform of  $|\sin t|$  is given by

$$L\{|\sin t|\} = \frac{1}{1 + s^2} \frac{1 + e^{-s\pi}}{1 - e^{-s\pi}}$$

(Refer Slide Time: 06:59)

The slide displays the following mathematical derivation:

$$\begin{aligned} \therefore (1 + s^2)I &= -[(\cos t + s \sin t)e^{-st}]_0^\pi \\ \Rightarrow (1 + s^2)I &= e^{-s\pi} + 1 \\ \Rightarrow I &= \frac{e^{-s\pi} + 1}{1 + s^2} \end{aligned}$$
$$\therefore L\{\sin t\} = \frac{1}{1 + s^2} \frac{1 + e^{-s\pi}}{1 - e^{-s\pi}}$$

The slide also features the Swamyam logo and a video feed of the presenter in the bottom right corner.

Now, let us see another function, which we call Sine Integral function defined by

$$Si(t) = \int_0^t \frac{\sin x}{x} dx.$$

(Refer Slide Time: 07:10)

The slide is titled "Sine Integral Function" and contains a graph of the function  $Si(t)$  versus  $t$ . The x-axis ranges from -14 to 14 with major ticks every 2 units. The y-axis ranges from -2 to 2 with major ticks every 1 unit. The graph shows an odd function passing through the origin (0,0). For  $t > 0$ , the function increases from 0, reaching a local maximum of approximately 1.8 at  $t \approx 2.5$ , then oscillates with decreasing amplitude, crossing the x-axis at  $t \approx 7.7$  and  $t \approx 14.5$ . For  $t < 0$ , the function decreases from 0, reaching a local minimum of approximately -1.8 at  $t \approx -2.5$ , and oscillates with decreasing amplitude.

Below the graph, the text states: "Sine Integral Function  $Si(t)$  is defined by,  $Si(t) = \int_0^t \frac{\sin x}{x} dx$ ".

The slide also features the Swamyam logo and a video feed of the presenter in the bottom right corner.

And the graph of Sine Integral function is also given in the slide.

Now, we will try to solve this problem using various methods to show that one particular problem can be solved using various techniques and we will get the same result.

First, using Integral Theorem, we will solve the problem.

(Refer Slide Time: 08:08)

The image shows a handwritten derivation on a green background. It starts with the Laplace transform of an integral:  $L\left\{\int_0^t F(x) dx\right\} = \frac{f(s)}{s}$ . To the right, it defines  $Si(t) = \int_0^t \frac{\sin x}{x} dx$ . Below this, it shows the Laplace transform of  $Si(t)$ :  $L\{Si(t)\} = \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}$ . This is then evaluated as  $\frac{1}{s} \int_0^\infty \frac{1}{1+x^2} dx = \frac{1}{s} [\tan^{-1} x]_0^\infty$ , which simplifies to  $\frac{1}{s} \left[\frac{\pi}{2} - \tan^{-1} s\right] = \frac{1}{s} \tan^{-1} \frac{1}{s}$ .

We know from the Laplace transform of integral,

$$L\left\{\int_0^t F(x) dx\right\} = \frac{f(s)}{s}, \text{ where, } f(s) = L\{F(t)\}.$$

Therefore,

$$L\{Si(t)\} = L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}.$$

So, only thing we have to do is to evaluate the Laplace transform of  $\frac{\sin t}{t}$  and this we can write down using division theorem as,


$$\begin{aligned} L\{Si(t)\} &= \frac{1}{s} \int_s^\infty \frac{1}{x^2 + 1} dx \quad \left[ \because L\{\sin t\} = \frac{1}{s^2 + 1} \right] \\ &= \frac{1}{s} [\tan^{-1} x]_s^\infty \\ &= \frac{1}{s} \left[ \frac{\pi}{2} - \tan^{-1} s \right] \\ &= \frac{1}{s} \tan^{-1} \frac{1}{s}. \end{aligned}$$

So, this is the simplest solution using the integral theorem.

(Refer Slide Time: 11:22)

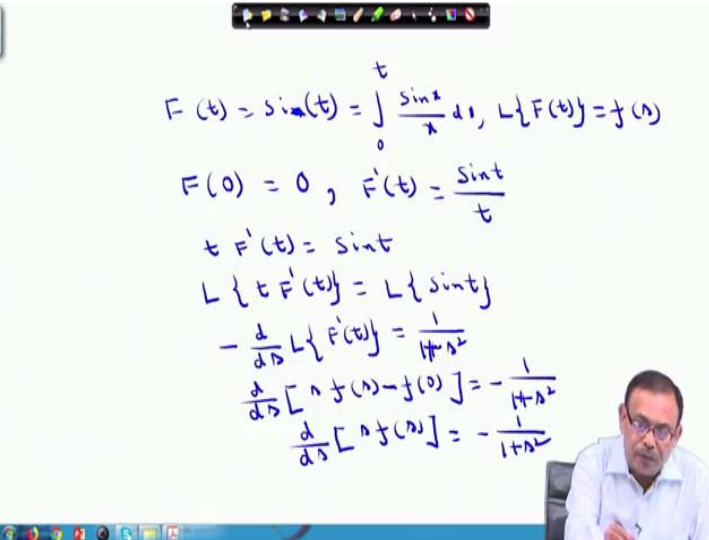
**Method 1 : Using integral theorem**

We know that  $L\left\{\int_0^t F(x)dx\right\} = \frac{f(s)}{s}$

$$\begin{aligned}\therefore L\{Si(t)\} &= \frac{1}{s} L\left\{\frac{\sin t}{t}\right\} \\ &= \frac{1}{s} \int_s^\infty \frac{1}{x^2+1} dx \quad (\text{Using division theorem}) \\ &= \frac{1}{s} [\tan^{-1} x]_s^\infty = \frac{1}{s} \tan^{-1} \frac{1}{s}\end{aligned}$$


The next method is using Differential, Multiplication and Initial Value Theorem. In the last lecture, we have discussed the Initial and Final Value Theorems.

(Refer Slide Time: 12:01)


$$\begin{aligned}F(t) = Si(t) &= \int_0^t \frac{\sin x}{x} dx, \quad L\{F(t)\} = f(s) \\ F(0) &= 0, \quad F'(t) = \frac{\sin t}{t} \\ t F'(t) &= \sin t \\ L\{t F'(t)\} &= L\{\sin t\} \\ -\frac{d}{ds} L\{F'(t)\} &= \frac{1}{1+s^2} \\ \frac{d}{ds} [n f(s) - f(0)] &= -\frac{1}{1+s^2} \\ \frac{d}{ds} [n f(s)] &= -\frac{1}{1+s^2}\end{aligned}$$

So, here let

$$F(t) = Si(t) = \int_0^t \frac{\sin x}{x} dx \quad \text{and} \quad f(s) = L\{F(t)\}.$$

From here, clearly  $F(0) = 0$ .

Using the Leibniz integral rule, we can always say,  $F'(t) = \frac{d}{dt} \left( \int_0^t \frac{\sin x}{x} dx \right) = \frac{\sin t}{t}$ .

$$\Rightarrow tF'(t) = \sin t.$$

So, once we are obtaining  $F(0)$  and  $F'(t)$ , we can write down

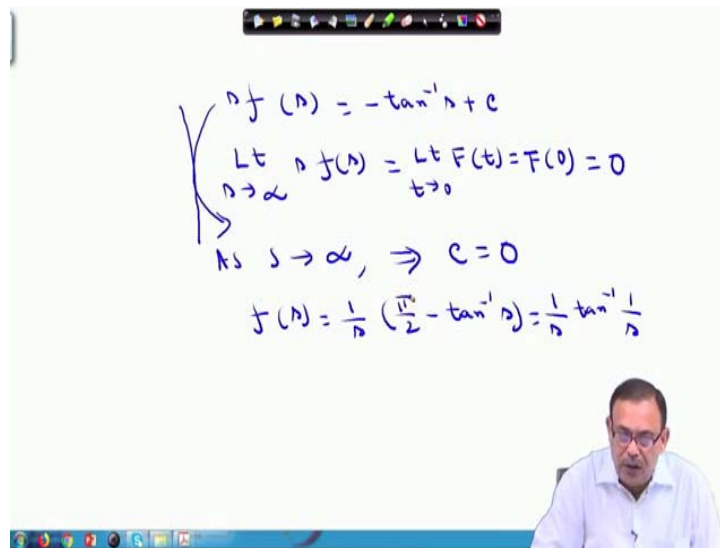
$$\begin{aligned} L\{tF'(t)\} &= L\{\sin t\} \\ \Rightarrow -\frac{d}{ds} [s f(s) - F(0)] &= \frac{1}{s^2 + 1} \end{aligned} \quad (1)$$

since  $L\{F'(t)\} = s f(s) - F(0)$  using Laplace Transform of derivative of a function and  $L\{tF'(t)\} = -\frac{d}{ds} [L\{F'(t)\}]$  using the property of multiplication by  $t$ .

Now integrating both sides of (1), we have,

$$s f(s) = -\tan^{-1} s + c, \quad \text{where, } c \text{ is constant of integration.}$$

(Refer Slide Time: 14:54)



Now, we have to find out the value of  $c$ . Here comes the role of the Initial Value Theorem.

From the results of the Initial Value Theorem, we have,

$$\begin{aligned} \lim_{s \rightarrow \infty} s f(s) &= \lim_{t \rightarrow 0} F(t) = F(0) = 0 \\ \Rightarrow \lim_{s \rightarrow \infty} s f(s) &= 0 \end{aligned}$$

$$\Rightarrow 0 = -\tan^{-1} \infty + c$$

$$\Rightarrow 0 = -\frac{\pi}{2} + c$$

$$\Rightarrow c = \frac{\pi}{2}$$

$$\therefore f(s) = L\{F(t)\} = \frac{1}{s} \left( \frac{\pi}{2} - \tan^{-1} s \right) = \frac{1}{s} \tan^{-1} \frac{1}{s}.$$

So, here actually we are using the concept of Initial Value Theorem to find out the value of the arbitrary constant  $c$  and we are getting the same result as earlier.


(Refer Slide Time: 16:38)

**Method 2 : Using Differential, multiplication and initial value theorem**

Let,  $F(t) = Si(t) = \int_0^t \frac{\sin x}{x} dx$  and  $L\{F(t)\} = f(s)$

$\therefore F(0) = 0$  and  $F'(t) = \frac{\sin t}{t}$  (Using Leibniz Integral Rule)

**Leibniz Integral Rule**

$$\frac{d}{dx} \int_{Q(x)}^{P(x)} F(x, t) dt = \int_{Q(x)}^{P(x)} \frac{\delta F(x, t)}{\delta x} dt + F(x, Q(x)) \frac{\delta Q(x)}{\delta x} - F(x, P(x)) \frac{\delta P(x)}{\delta x}$$




(Refer Slide Time: 16:56)

$\therefore tF'(t) = \sin t$   
 $\Rightarrow L\{tF'(t)\} = L\{\sin t\}$   
 $\Rightarrow -\frac{d}{ds}L\{F'(t)\} = \frac{1}{1+s^2}$  (Using Multiplication Theorem)  
 $\Rightarrow \frac{d}{ds}(sf(s) - F(0)) = -\frac{1}{1+s^2}$  (Using Differentiation Theorem)  
 $\Rightarrow \frac{d}{ds}(sf(s)) = -\frac{1}{1+s^2}$   
 $\Rightarrow sf(s) = -\tan^{-1}s + c$  (integrating both sides)

The slide also features a video inset of a man in a white shirt and glasses, and logos for Swamyam and other educational institutions at the bottom.

(Refer Slide Time: 17:20)

By Initial value theorem,  
 $\lim_{s \rightarrow \infty} sf(s) = \lim_{t \rightarrow 0} F(t) = F(0) = 0$   
 $\therefore c = \frac{\pi}{2}$   
 $\therefore f(s) = \frac{1}{s} \left( \frac{\pi}{2} - \tan^{-1}s \right) = \frac{1}{s} \tan^{-1} \frac{1}{s}$

The slide also features logos for Swamyam and other educational institutions at the bottom.

(Refer Slide Time: 17:57)

**Method 3 : Using infinite series**

$$\begin{aligned}\int_0^t \frac{\sin x}{x} dx &= \int_0^t \frac{1}{x} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) dx \\ &= \int_0^t \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right) dx \\ &= t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \frac{t^7}{7 \cdot 7!} + \dots\end{aligned}$$

Now, the third is by using infinite series expansion. We know that  $\sin x$  can be expressed in terms of infinite series as:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (2)$$

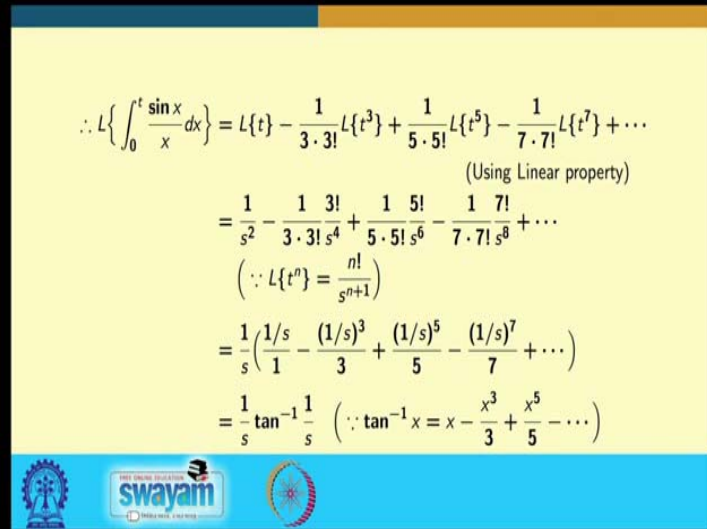
Since our given function is  $\frac{\sin x}{x}$ , so dividing (2) by  $x$ , we obtain

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

And, we can integrate it very easily within the limits 0 to  $t$  to obtain:

$$\int_0^t \frac{\sin x}{x} dx = t - \frac{t^3}{3 \cdot 3!} + \frac{t^5}{5 \cdot 5!} - \frac{t^7}{7 \cdot 7!} + \dots$$

(Refer Slide Time: 18:37)



$$\begin{aligned} \therefore L\left\{\int_0^t \frac{\sin x}{x} dx\right\} &= L\{t\} - \frac{1}{3 \cdot 3!} L\{t^3\} + \frac{1}{5 \cdot 5!} L\{t^5\} - \frac{1}{7 \cdot 7!} L\{t^7\} + \dots \\ &\quad \text{(Using Linear property)} \\ &= \frac{1}{s^2} - \frac{1 \cdot 3!}{3 \cdot 3! s^4} + \frac{1 \cdot 5!}{5 \cdot 5! s^6} - \frac{1 \cdot 7!}{7 \cdot 7! s^8} + \dots \\ &\quad \left(\because L\{t^n\} = \frac{n!}{s^{n+1}}\right) \\ &= \frac{1}{s} \left(\frac{1}{s} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \dots\right) \\ &= \frac{1}{s} \tan^{-1} \frac{1}{s} \quad \left(\because \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) \end{aligned}$$

So, taking Laplace transform on both side, and using linearity property, we get

$$\begin{aligned} L\left\{\int_0^t \frac{\sin x}{x} dx\right\} &= L\{t\} - \frac{1}{3 \cdot 3!} L\{t^3\} + \frac{1}{5 \cdot 5!} L\{t^5\} - \frac{1}{7 \cdot 7!} L\{t^7\} + \dots \\ &= \frac{1}{s^2} - \frac{1 \cdot 3!}{3 \cdot 3! s^4} + \frac{1 \cdot 5!}{5 \cdot 5! s^6} - \frac{1 \cdot 7!}{7 \cdot 7! s^8} + \dots \quad \left(\because L\{t^n\} = \frac{n!}{s^{n+1}}\right) \\ &= \frac{1}{s} \left[\frac{1}{s} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \dots\right] \\ &= \frac{1}{s} \tan^{-1} \frac{1}{s} \quad \left(\because \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) \end{aligned}$$

So, we can solve the same problem using various techniques.

The fourth method, which we will show is by using substitution.

(Refer Slide Time: 19:57)

**Method 4 : Using substitution**

Let  $x = tv$

$$\therefore \int_0^t \frac{\sin x}{x} dx = \int_0^1 \frac{\sin tv}{v} dv$$

$$\therefore L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = L\left\{\int_0^1 \frac{\sin tv}{v} dv\right\}$$

$$= \int_0^\infty e^{-st} \left(\int_0^1 \frac{\sin tv}{v} dv\right) dt \quad (\text{Using definition})$$

Let us substitute  $x = tv$  in  $\int_0^t \frac{\sin x}{x} dx$  so that  $dx = tv$  and the limits of integration will be changed from  $[0, t]$  to  $[0, 1]$ . Therefore,

$$\int_0^t \frac{\sin x}{x} dx = \int_0^1 \frac{\sin tv}{v} dv$$

Taking Laplace transform on both sides, we can write

$$L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = L\int_0^1 \frac{\sin tv}{v} dv$$

$$= \int_0^\infty e^{-st} \left(\int_0^1 \frac{\sin tv}{v} dv\right) dt \quad (\text{using definition})$$

We now change the order of integration to obtain

$$L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = \int_0^1 \frac{1}{v} \left(\int_0^\infty e^{-st} \sin vt dt\right) dv$$

$$= \int_0^1 \frac{1}{v} L\{\sin vt\} dv$$

$$= \int_0^1 \frac{1}{v} \left(\frac{v}{s^2 + v^2}\right) dv$$

$$\Rightarrow L \left\{ \int_0^t \frac{\sin x}{x} dx \right\} = \left[ \frac{1}{s} \tan^{-1} \frac{v}{s} \right]_0^1$$

$$= \frac{1}{s} \tan^{-1} \frac{1}{s}.$$

(Refer Slide Time: 20:31)

$$= \int_0^1 \frac{1}{v} \left( \int_0^\infty e^{-st} \sin tv dt \right) dv \quad (\text{By changing of order})$$

$$= \int_0^1 \frac{L\{\sin tv\}}{v} dv$$

$$= \int_0^1 \frac{dv}{s^2 + v^2}$$

$$= \left[ \frac{1}{s} \tan^{-1} \frac{v}{s} \right]_0^1 = \frac{1}{s} \tan^{-1} \frac{1}{s}$$

(Refer Slide Time: 21:12)

$$\int_0^1 \frac{L\{\sin tv\}}{v} dv$$

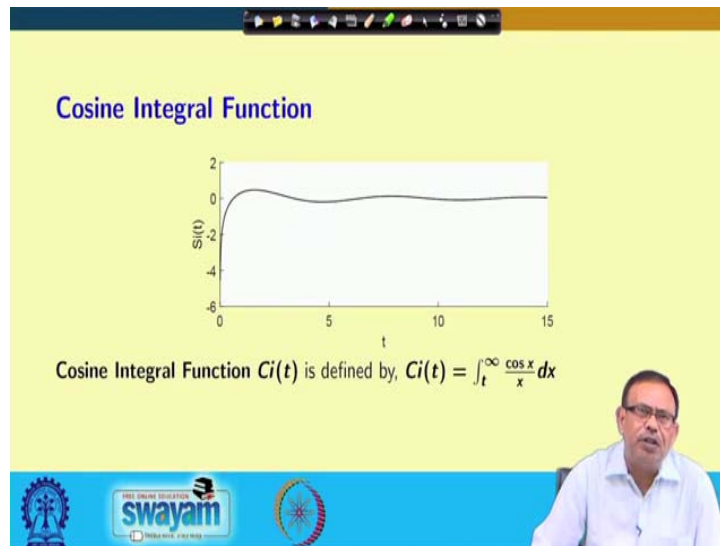
$$= \int_0^1 \frac{dv}{s^2 + v^2} = \left[ \frac{1}{s} \tan^{-1} \frac{v}{s} \right]_0^1$$

$$= \frac{1}{s} \tan^{-1} \frac{1}{s}$$

Therefore, the same problem can be solved in 4 different ways.

Now, let us consider another function, the Cosine Integral Function.

(Refer Slide Time: 22:28)



The graphical representation can be viewed in the above slide. The function is defined by

$$Ci(t) = \int_t^\infty \frac{\cos x}{x} dx.$$

(Refer Slide Time: 22:52)

The slide shows the derivation of the derivative of the Cosine Integral Function. It starts with the definition: "Let,  $F(t) = Ci(t) = \int_t^\infty \frac{\cos x}{x} dx$  and  $L\{F(t)\} = f(s)$ ". Then, it states: " $\therefore F'(t) = -\frac{\cos t}{t}$  (Using Leibniz Integral Rule)". The slide has a yellow background and a blue footer with logos for "swayam" and other educational institutions.

Now, the solution procedure for finding the Laplace transform remains similar. We assume  $F(t) = Ci(t) = \int_t^\infty \frac{\cos x}{x} dx$  and  $L\{F(t)\} = f(s)$ . Clearly, using Leibniz rule of integration, we have  $F'(t) = -\frac{\cos t}{t}$ .

$$\Rightarrow tF'(t) = -\cos t.$$

Now taking the Laplace transform on both sides and using the multiplication theorem and Laplace transform of derivative (refer to the attached lecture slide for step by step details), we can easily obtain the result as

$$sf(s) = \frac{1}{2} \log(s^2 + 1) + c. \quad (3)$$

(Refer Slide Time: 23:14)

$\therefore tF'(t) = -\cos t$   
 $\Rightarrow L\{tF'(t)\} = -L\{\cos t\}$   
 $\Rightarrow -\frac{d}{ds} L\{F'(t)\} = -\frac{s}{1+s^2}$  (Using Multiplication Theorem)  
 $\Rightarrow \frac{d}{ds} (sf(s) - F(0)) = \frac{s}{1+s^2}$  (Using Differentiation Theorem)  
 $\Rightarrow \frac{d}{ds} (sf(s)) = \frac{s}{1+s^2}$   
 $\Rightarrow sf(s) = \frac{1}{2} \log(s^2 + 1) + c$  (integrating both sides)

Now, we have to find out the value of the constant of integration  $c$ . For that we will use the Final Value Theorem which states

$$\lim_{s \rightarrow 0} sf(s) = \lim_{t \rightarrow \infty} F(t).$$

Clearly, by the definition of  $F(t)$ ,  $\lim_{t \rightarrow \infty} F(t) = 0$  so that  $\lim_{s \rightarrow 0} sf(s)$  also equals 0. Therefore,

(3) implies

$$c = 0$$

so that  $sf(s) = \frac{1}{2}\log(s^2 + 1)$ . Therefore,

$$f(s) = L\{Ci(t)\} = \frac{1}{2s}\log(s^2 + 1).$$

(Refer Slide Time: 24:25)

By Final value theorem,

$$\lim_{s \rightarrow 0} sf(s) = \lim_{t \rightarrow \infty} F(t) = \int_0^{\infty} \frac{\cos x}{x} dx = 0$$

$\therefore c = 0$

$$\therefore f(s) = \frac{1}{2s}\log(s^2 + 1)$$

The slide features a yellow background with a blue header and footer. The footer contains the Swayam logo and the text 'swayam' and 'Digital Work, Digital Study'.

Now, we come to the Exponential Integral Function whose graphical representation can be viewed in the slide attached below:

(Refer Slide Time: 25:09)

**Exponential Integral Function**

The graph shows the Exponential Integral Function  $E(t)$  plotted against  $t$ . The vertical axis  $E(t)$  ranges from 0 to 6, and the horizontal axis  $t$  ranges from 0 to 15. The curve starts at a high value for small  $t$  and decays rapidly towards zero as  $t$  increases.

Exponential Integral Function  $E(t)$  is defined by,  $E(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx$

Let,  $F(t) = E(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx$  and  $L\{F(t)\} = f(s)$

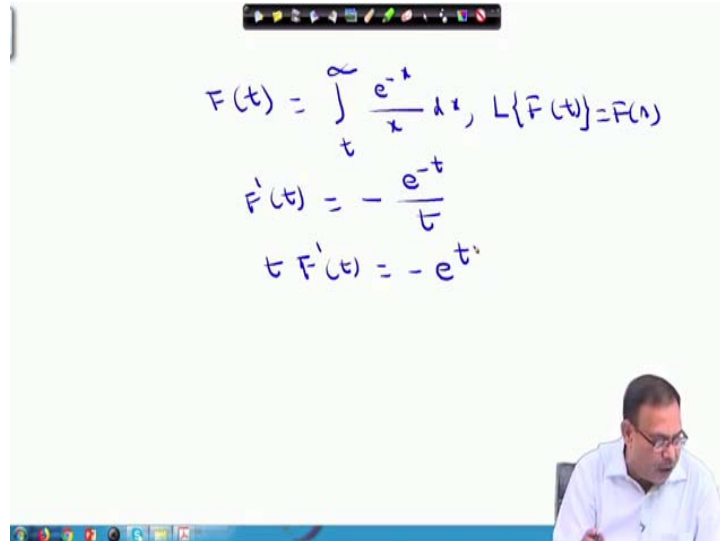
The slide features a yellow background with a blue header and footer. The footer contains the Swayam logo and the text 'swayam' and 'Digital Work, Digital Study'.

The Exponential Integral function is defined as



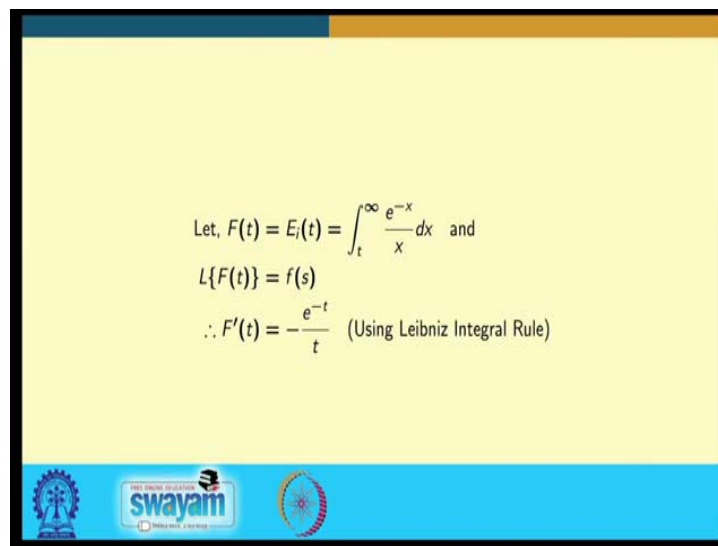
$$E_i(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx.$$

(Refer Slide Time: 25:34)



We assume  $F(t) = E_i(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx$  and  $L\{F(t)\} = f(s)$  so that using Leibniz integral rule,  $F'(t) = -\frac{e^{-t}}{t}$ .

(Refer Slide Time: 26:23)



Now, we will follow the similar process as earlier. All the steps are clearly presented in the attached slide. As we can see, we obtain the final result after integration as

$$sf(s) = \log(s + 1) + c \quad (4)$$

where  $c$  is the constant of integration whose value we need to evaluate.

(Refer Slide Time: 26:25)

$$\begin{aligned} \therefore tF'(t) &= -e^{-t} \\ \Rightarrow L\{tF'(t)\} &= -L\{e^{-t}\} \\ \Rightarrow -\frac{d}{ds}L\{F'(t)\} &= -\frac{1}{s+1} \quad (\text{Using Multiplication Theorem}) \\ \Rightarrow \frac{d}{ds}(sf(s) - F(0)) &= \frac{1}{s+1} \quad (\text{Using Differentiation Theorem}) \\ \Rightarrow \frac{d}{ds}(sf(s)) &= \frac{1}{s+1} \\ \Rightarrow sf(s) &= \log(s+1) + c \quad (\text{integrating both sides}) \end{aligned}$$

So, we use the Final Value theorem as earlier to evaluate  $c$ .

(Refer Slide Time: 27:03)

By Final value theorem,

$$\lim_{s \rightarrow 0} sf(s) = \lim_{t \rightarrow \infty} F(t) = \int_0^{\infty} \frac{e^{-x}}{x} dx = 0$$

$$\therefore c = 0$$

$$\therefore f(s) = \frac{\log(s+1)}{s}$$

The Final Value Theorem states

$$\lim_{s \rightarrow 0} sf(s) = \lim_{t \rightarrow \infty} F(t).$$

Clearly, by the definition of  $F(t)$ ,  $\lim_{t \rightarrow \infty} F(t) = 0$  so that  $\lim_{s \rightarrow 0} sf(s)$  also equals 0. Therefore, (4) implies

$$c = 0$$

so that  $sf(s) = \log(s + 1)$ . Therefore,

$$f(s) = L\{E_i(t)\} = \frac{1}{s} \log(s + 1).$$

So, these are some special functions, whose Laplace transforms have been derived. In the next lecture also, we will initially start with some more special functions, which are very useful in various engineering problems, statistics and many more. So, we will try to evaluate the Laplace transforms of those useful and frequently used functions in the next lecture.