Transform Calculus and its Applications in Differential Equations
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Lecture – 08
Laplace Transform of some special Functions

Welcome again. In the last lecture, we have discussed the Laplace transform of a periodic function.

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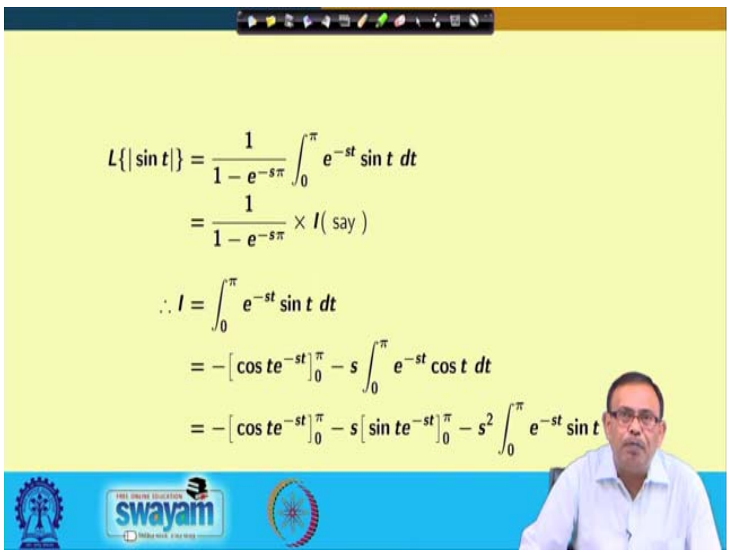

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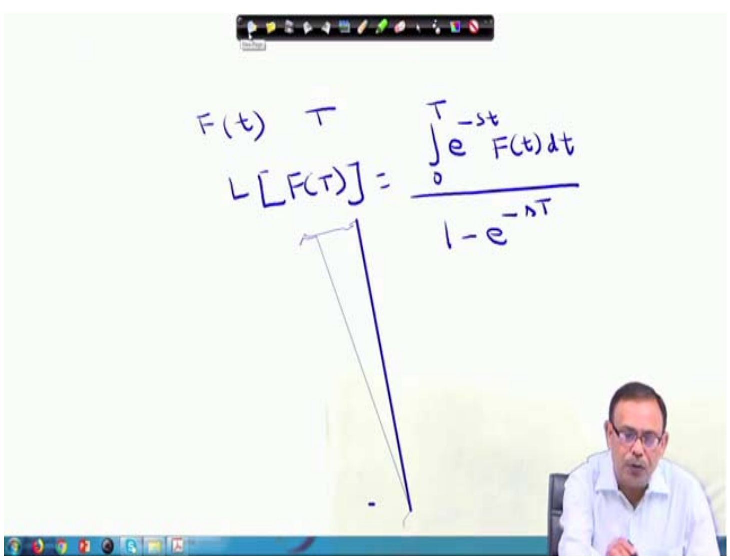

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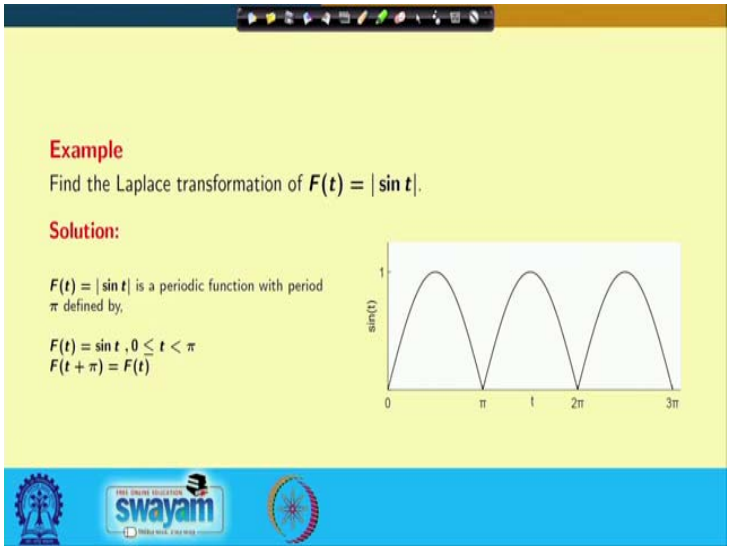

Let us take the example where we need to find the Laplace transform of $F(t) = |\sin t|$. We know that $F(t) = |\sin t|$ is a periodic function with period π, which is defined by

$$F(t) = \sin t \ , \ \ 0 \leq t \leq \pi \quad \text{and} \quad F(t+\pi) = F(t).$$

And, on the right side of the slide, we can see the graph for this periodic function.

So, our aim is to find out $L\{|\sin t|\}$, where $|\sin t|$ is a periodic function of period π. So, let us see the solution process for this.

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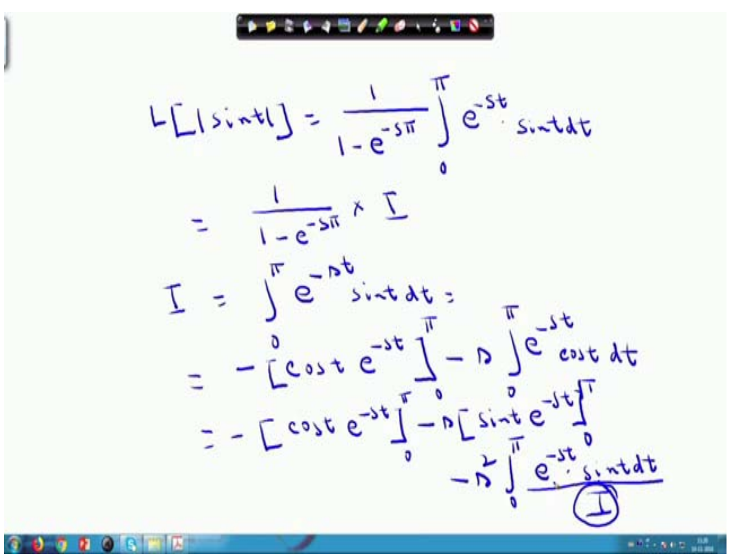

We can write Laplace transform of periodic function using the derived formula as:

$$L\{|\sin t|\} = \frac{1}{1 - e^{-s\pi}} \int_0^{\pi} e^{-st} \sin t \ dt$$

Now, this integral can be evaluated easily using the formula $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$. Therefore,

$$L\{|\sin t|\} = \frac{1}{1 - e^{-s\pi}} I \qquad \text{where, } \ I = \int_0^{\pi} e^{-st} \sin t \ dt.$$

And, we can evaluate this integral to obtain

$$I = \frac{e^{-s\pi} + 1}{1 + s^2}.$$

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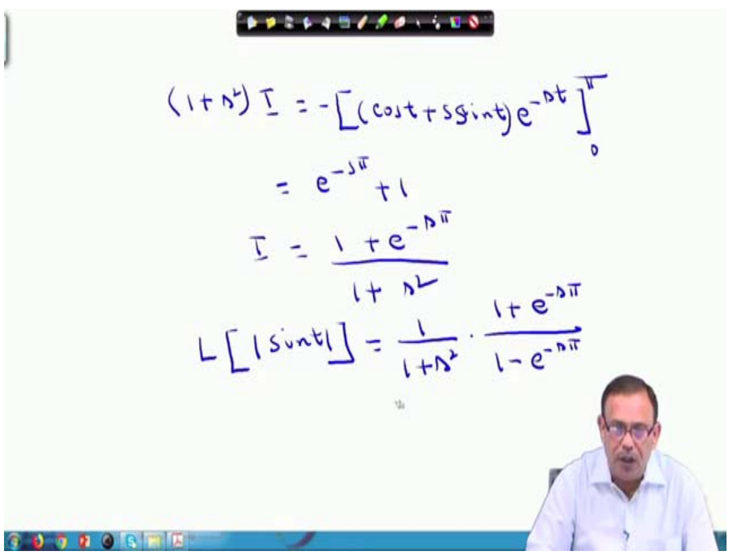

So, that Laplace transform of $|\sin t|$ is given by

$$L\{|\sin t|\} = \frac{1}{1 + s^2} \frac{1 + e^{-s\pi}}{1 - e^{-s\pi}}.$$

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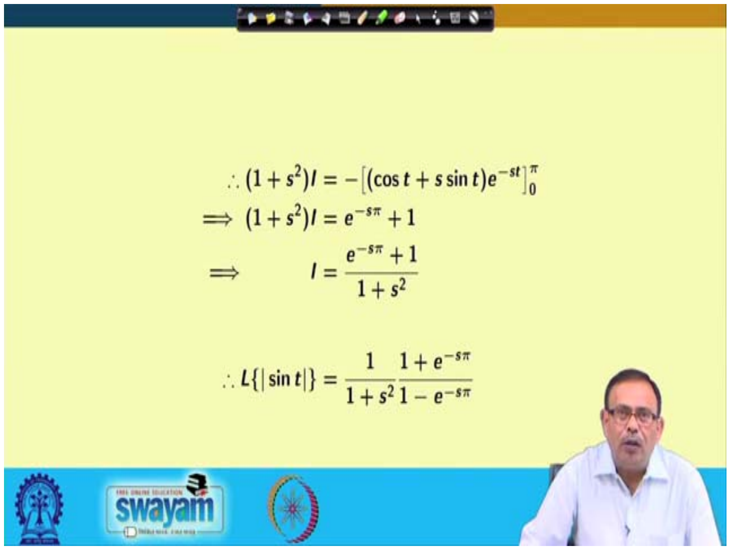

Now, let us see another function, which we call Sine Integral function defined by

$$Si(t) = \int_0^t \frac{\sin x}{x}\, dx.$$

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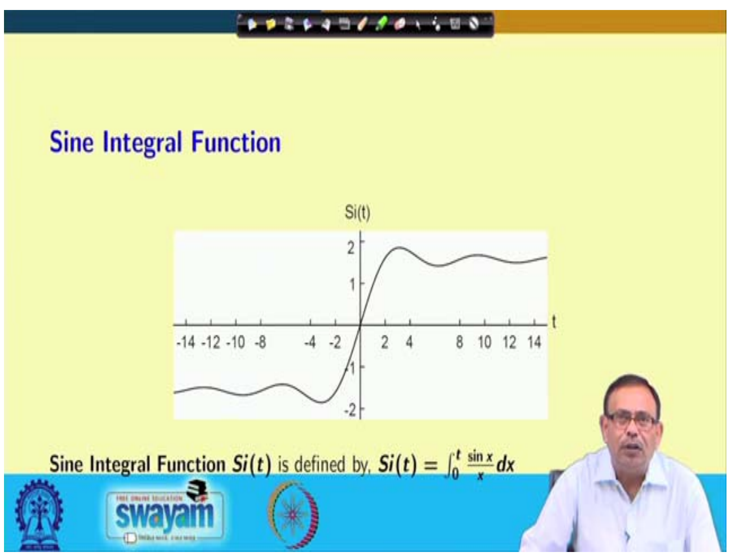

And the graph of Sine Integral function is also given in the slide.

Now, we will try to solve this problem using various methods to show that one particular problem can be solved using various techniques and we will get the same result.

First, using Integral Theorem, we will solve the problem.

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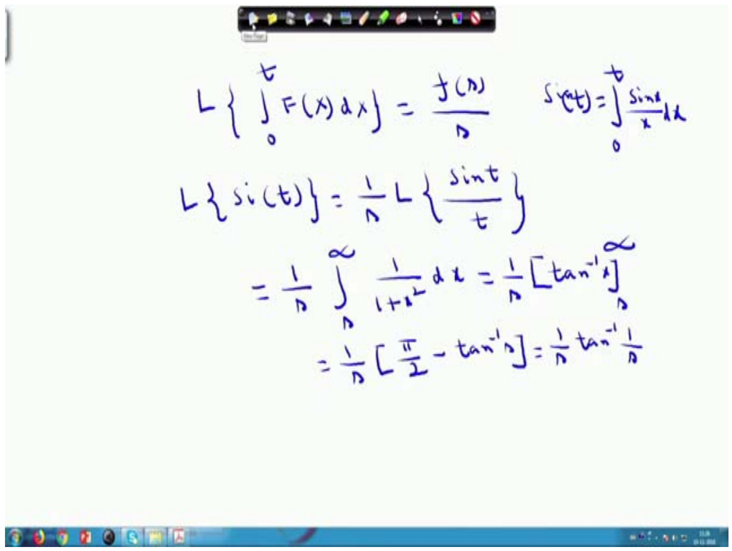

We know from the Laplace transform of integral,

$$L\left\{\int_0^t F(x)dx\right\} = \frac{f(s)}{s} \quad , \text{ where, } \; f(s) = L\{F(t)\}.$$

Therefore,

$$L\{Si(t)\} = L\left\{\int_0^t \frac{\sin x}{x}dx\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\}.$$

So, only thing we have to do is to evaluate the Laplace transform of $\frac{\sin t}{t}$ and this we can write down using division theorem as,

$$L\{Si(t)\} = \frac{1}{s}\int_s^\infty \frac{1}{x^2+1}dx \quad \left[\because L\{\sin t\} = \frac{1}{s^2+1}\right]$$

$$= \frac{1}{s}[\tan^{-1}x]_s^\infty$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}s\right]$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

So, this is the simplest solution using the integral theorem.

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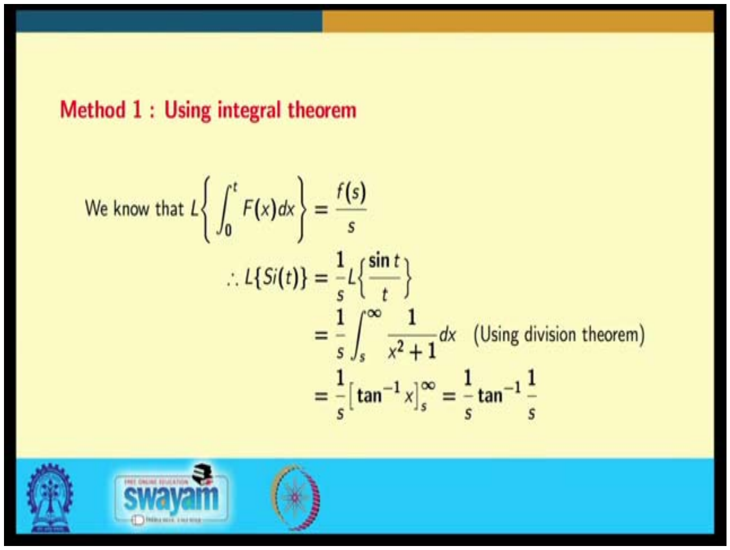

The next method is using Differential, Multiplication and Initial Value Theorem. In the last lecture, we have discussed the Initial and Final Value Theorems.

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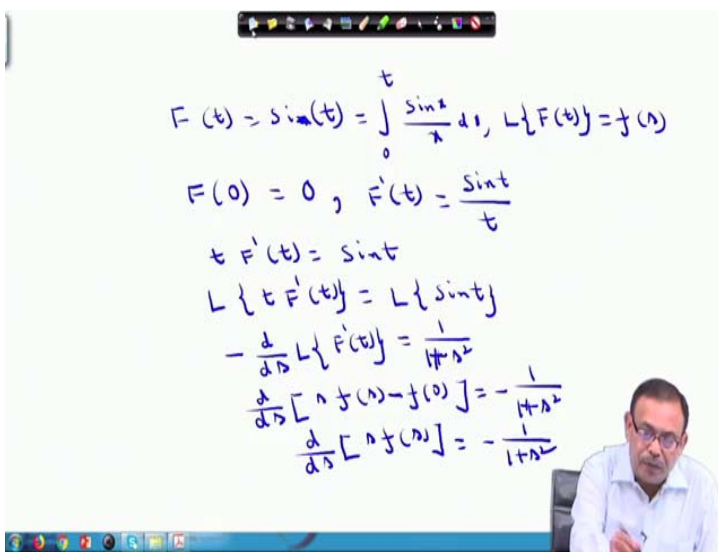

So, here let

$$F(t) = Si(t) = \int_0^t \frac{\sin x}{x}dx \quad \text{and} \quad f(s) = L\{F(t)\}.$$

From here, clearly $F(0) = 0$.

Using the Leibniz integral rule, we can always say, $F'(t) = \frac{d}{dt}\left(\int_0^t \frac{\sin x}{x} dx\right) = \frac{\sin t}{t}$.

$$\Rightarrow tF'(t) = \sin t.$$

So, once we are obtaining $F(0)$ and $F'(t)$, we can write down

$$L\{tF'(t)\} = L\{\sin t\}$$

$$\Rightarrow -\frac{d}{ds}[s\, f(s) - F(0)] = \frac{1}{s^2+1} \qquad (1)$$

since $L\{F'(t)\} = s\, f(s) - F(0)$ using Laplace Transform of derivative of a function and $L\{tF'(t)\} = -\frac{d}{ds}[L\{F'(t)\}]$ using the property of multiplication by t.

Now integrating both sides of (1), we have,

$$s\, f(s) = -\tan^{-1} s + c, \quad \text{where, c is constant of integration.}$$

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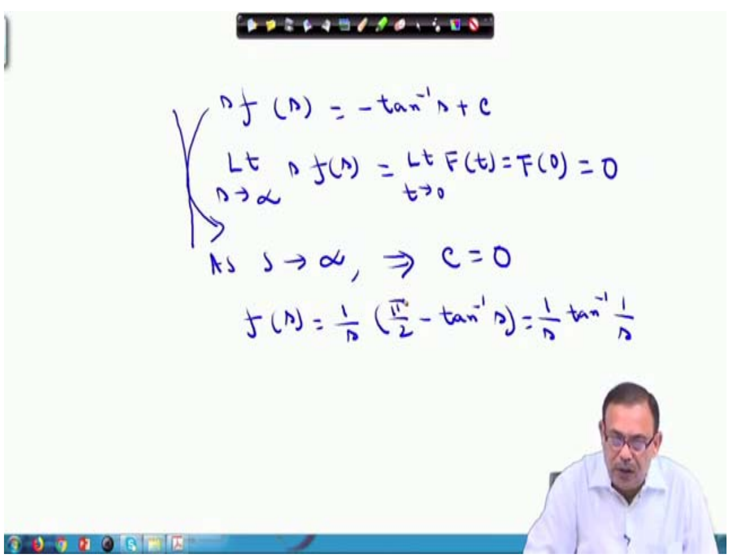

Now, we have to find out the value of c. Here comes the role of the Initial Value Theorem. From the results of the Initial Value Theorem, we have,

$$\lim_{s\to\infty} sf(s) = \lim_{t\to 0} F(t) = F(0) = 0$$

$$\Rightarrow \lim_{s\to\infty} sf(s) = 0$$

$$\Rightarrow 0 = -\tan^{-1}\infty + c$$

$$\Rightarrow 0 = -\frac{\pi}{2} + \mathrm{c}$$

$$\Rightarrow \mathrm{c} = \frac{\pi}{2}$$

$$\therefore f(s) = L\{F(t)\} = \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right) = \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

So, here actually we are using the concept of Initial Value Theorem to find out the value of the arbitrary constant c and we are getting the same result as earlier.

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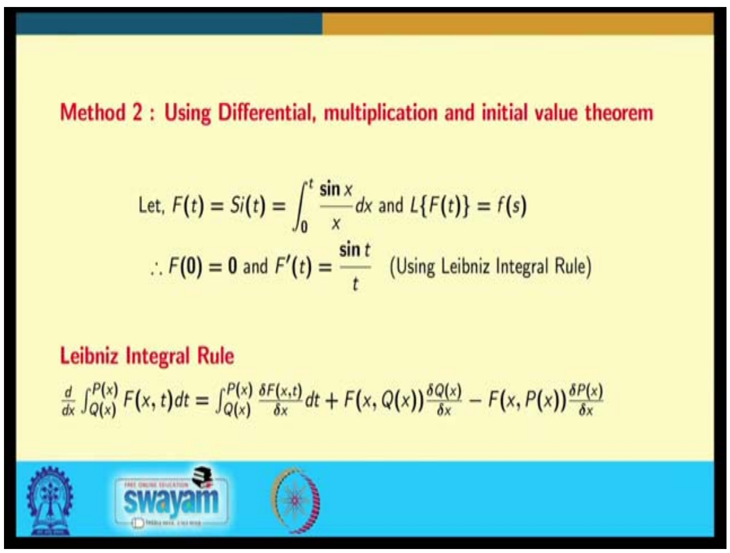

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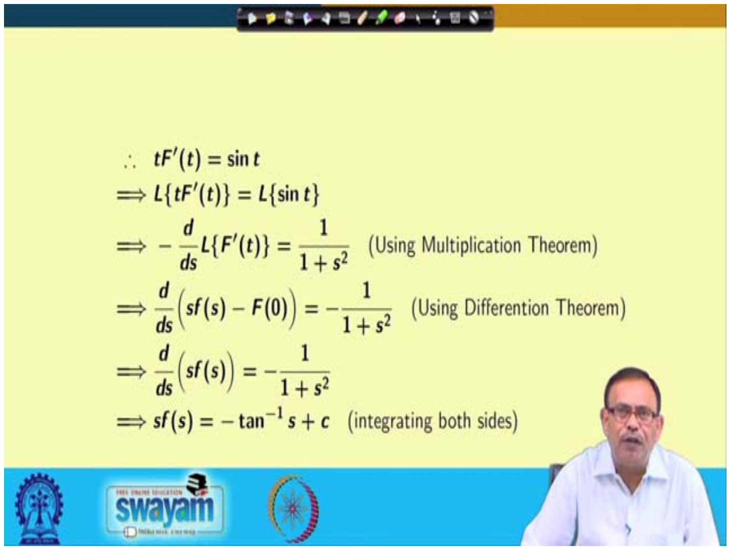

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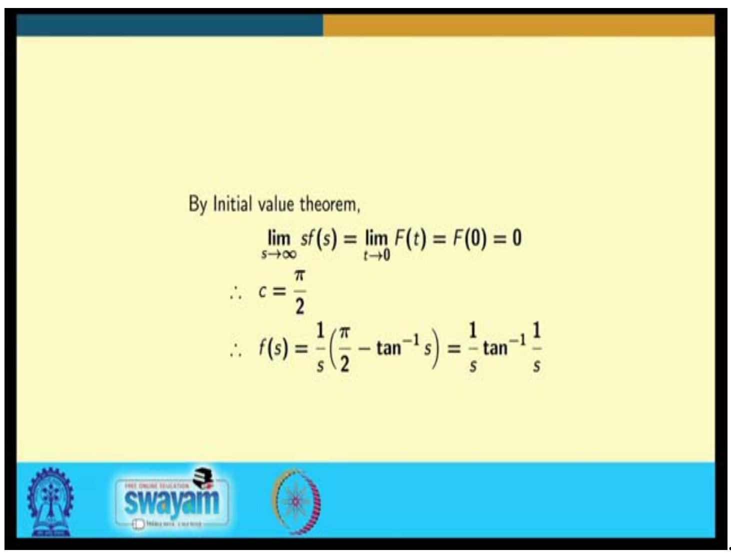

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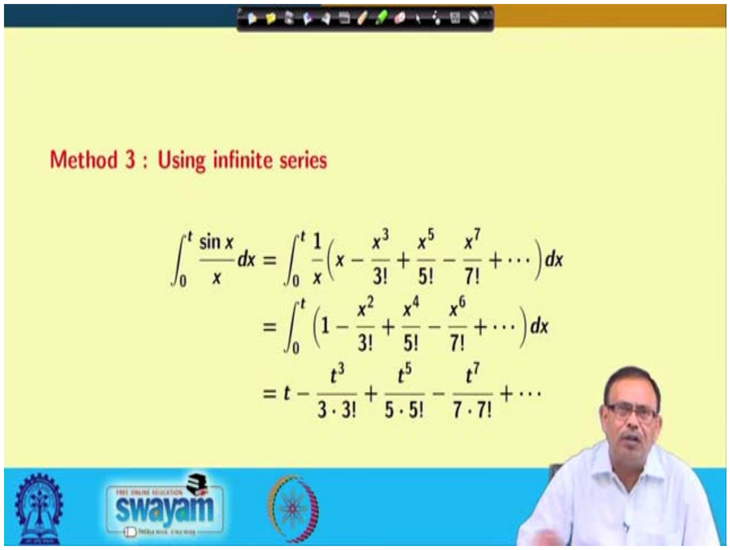

Now, the third is by using infinite series expansion. We know that $\sin x$ can be expressed in terms of infinite series as:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (2)$$

Since our given function is $\frac{\sin x}{x}$, so dividing (2) by x, we obtain

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

And, we can integrate it very easily within the limits 0 to t to obtain:

$$\int_0^t \frac{\sin x}{x} dx = t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \cdots$$

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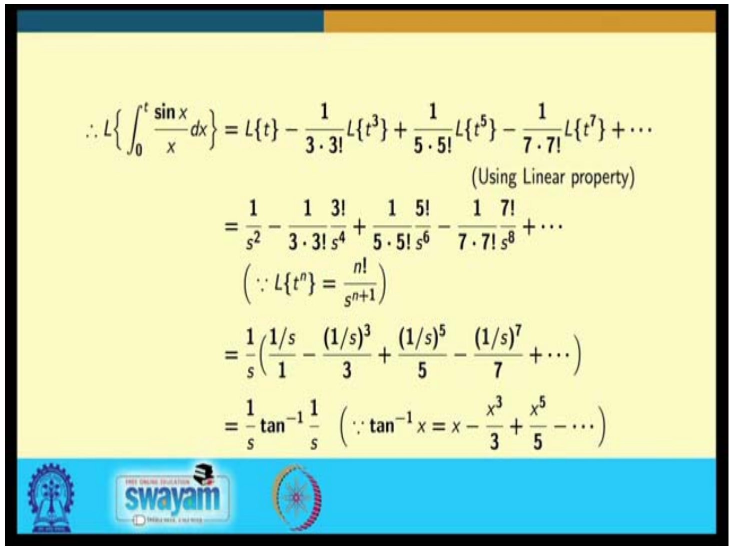

So, taking Laplace transform on both side, and using linearity property, we get

$$
\begin{aligned}
L\left\{\int_0^t \frac{\sin x}{x}dx\right\} &= L\{t\} - \frac{1}{3.3!}L\{t^3\} + \frac{1}{5.5!}L\{t^5\} - \frac{1}{7.7!}L\{t^7\} + \cdots \\
&= \frac{1}{s^2} - \frac{1}{3.3!}\frac{3!}{s^4} + \frac{1}{5.5!}\frac{5!}{s^6} - \frac{1}{7.7!}\frac{7!}{s^8} + \cdots \qquad \left(\because L\{t^n\} = \frac{n!}{s^{n+1}}\right) \\
&= \frac{1}{s}\left[\frac{1/s}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \cdots\right] \\
&= \frac{1}{s}\tan^{-1}\frac{1}{s} \qquad \left(\because \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)
\end{aligned}
$$

So, we can solve the same problem using various techniques.

The fourth method, which we will show is by using substitution.

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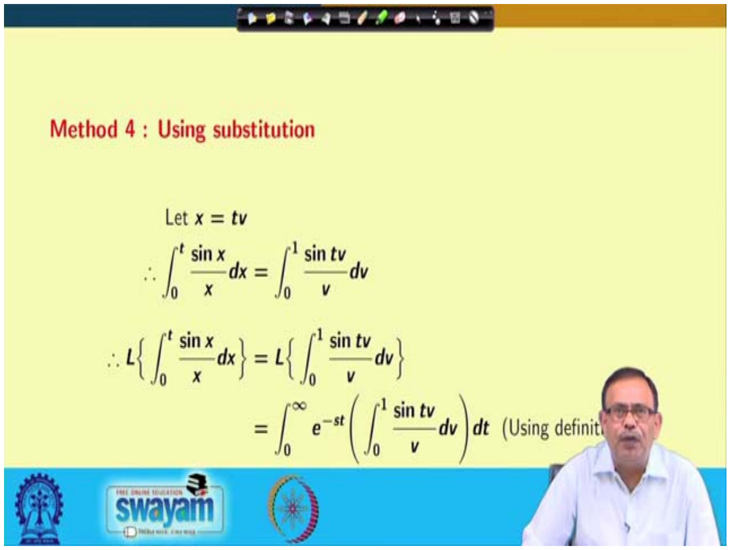

Let us substitute $x = tv$ in $\int_0^t \frac{\sin x}{x} dx$ so that $dx = tdv$ and the limits of integration will be changed from $[0, t]$ to $[0,1]$. Therefore,

$$\int_0^t \frac{\sin x}{x} dx = \int_0^1 \frac{\sin tv}{v} dv$$

Taking Laplace transform on both sides, we can write

$$L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = L \int_0^1 \frac{\sin tv}{v} dv$$

$$= \int_0^\infty e^{-st} \left(\int_0^1 \frac{\sin tv}{v} dv\right) dt \qquad \text{(using definition)}$$

We now change the order of integration to obtain

$$L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = \int_0^1 \frac{1}{v} \left(\int_0^\infty e^{-st} \sin vt \, dt\right) dv$$

$$= \int_0^1 \frac{1}{v} L\{\sin vt\} dv$$

$$= \int_0^1 \frac{1}{v} \left(\frac{v}{s^2 + v^2}\right) dv$$

$$\Rightarrow L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\} = \left[\frac{1}{s}\tan^{-1}\frac{v}{s}\right]_0^1$$
$$= \frac{1}{s}\tan^{-1}\frac{1}{s}.$$

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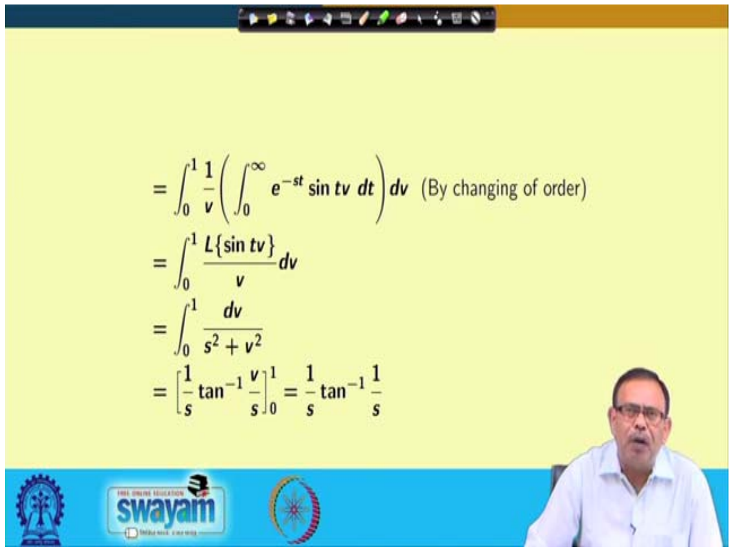

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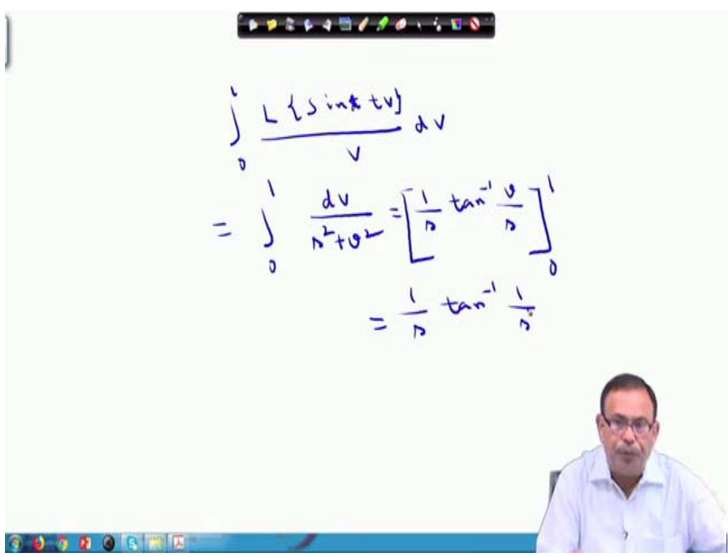

Therefore, the same problem can be solved in 4 different ways.

Now, let us consider another function, the Cosine Integral Function.

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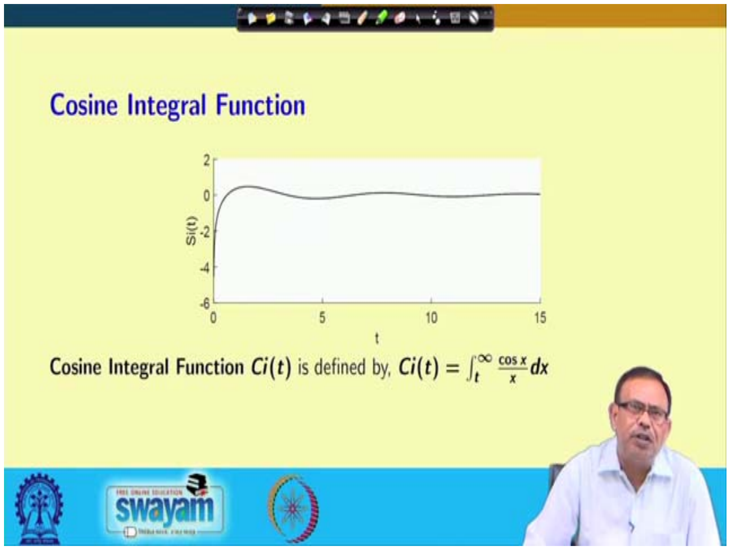

The graphical representation can be viewed in the above slide. The function is defined by

$$Ci(t) = \int_t^\infty \frac{\cos x}{x} dx.$$

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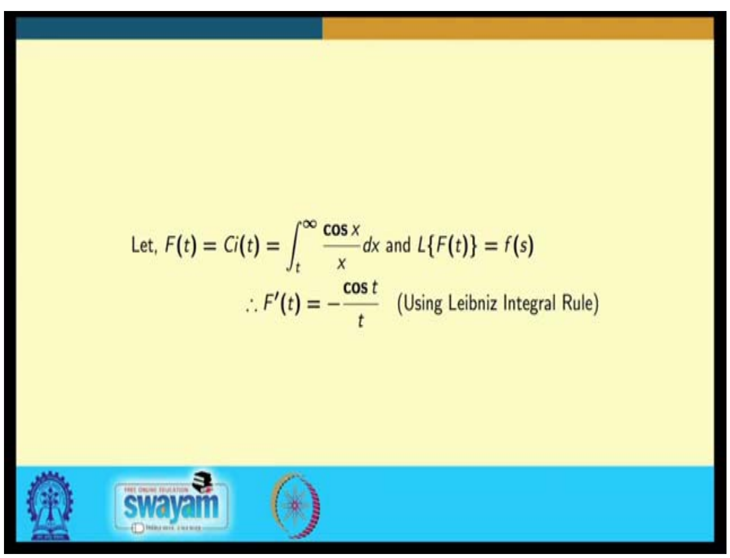

Now, the solution procedure for finding the Laplace transform remains similar. We assume $F(t) = Ci(t) = \int_t^\infty \frac{\cos x}{x} dx$ and $L\{F(t)\} = f(s)$. Clearly, using Leibniz rule of integration, we have $F'(t) = -\frac{\cos t}{t}$.

$$\Rightarrow tF'(t) = -\cos t.$$

Now taking the Laplace transform on both sides and using the multiplication theorem and Laplace transform of derivative (refer to the attached lecture slide for step by step details), we can easily obtain the result as

$$sf(s) = \frac{1}{2}\log(s^2+1) + c. \qquad (3)$$

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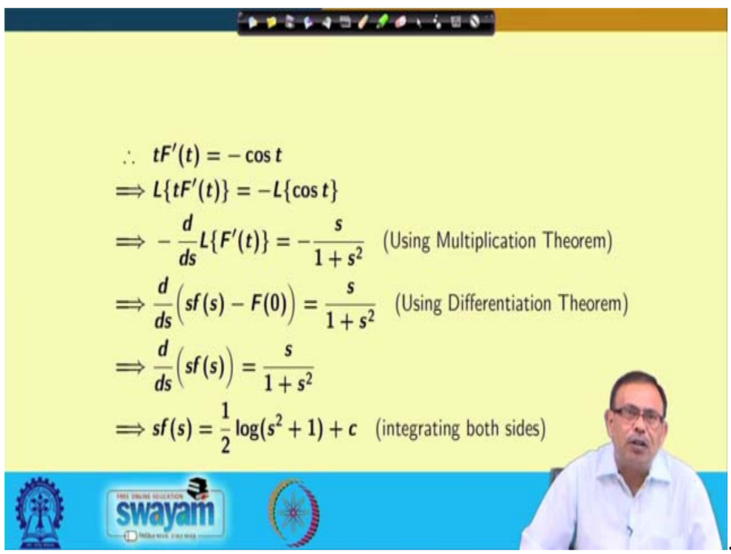

Now, we have to find out the value of the constant of integration c. For that we will use the Final Value Theorem which states

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t).$$

Clearly, by the definition of $F(t)$, $\lim_{t\to\infty} F(t) = 0$ so that $\lim_{s\to 0} sf(s)$ also equals 0. Therefore, (3) implies

$$c = 0$$

so that $sf(s)=\frac{1}{2}\log(s^2+1)$. Therefore,

$$f(s)=L\{Ci(t)\}=\frac{1}{2s}\log(s^2+1).$$

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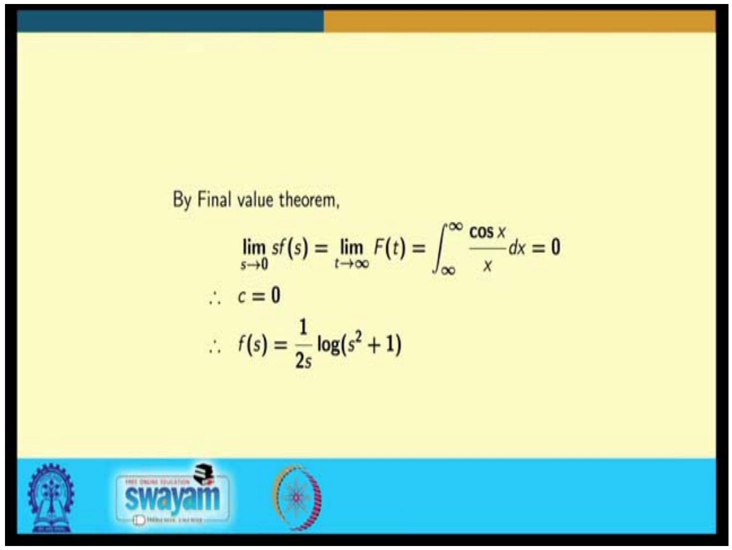

Now, we come to the Exponential Integral Function whose graphical representation can be viewed in the slide attached below:

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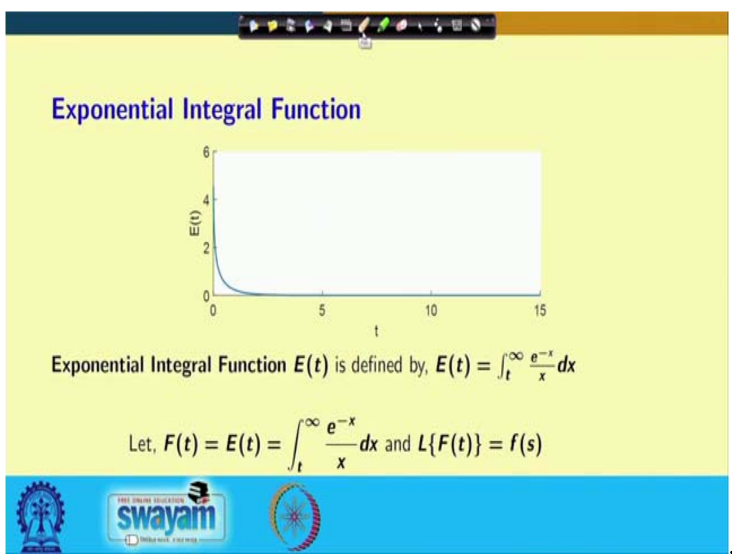

The Exponential Integral function is defined as

$$E_i(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx.$$

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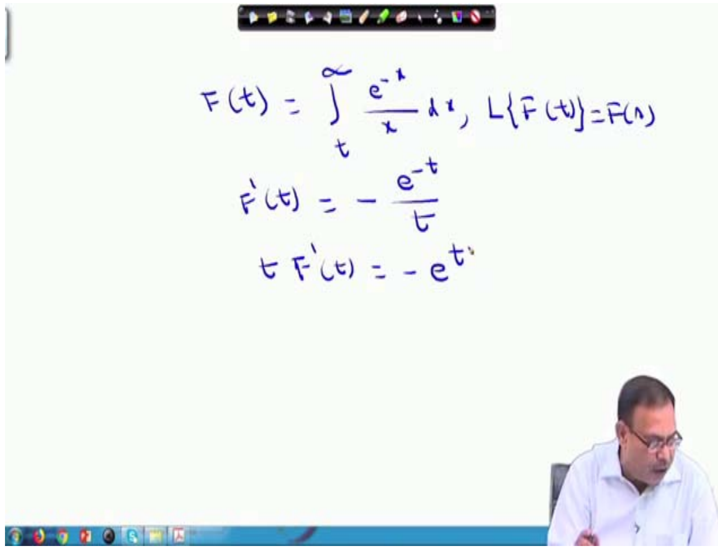

We assume $F(t) = E_i(t) = \int_t^{\infty} \frac{e^{-x}}{x} dx$ and $L\{F(t)\} = f(s)$ so that using Leibniz integral rule, $F'(t) = -\frac{e^{-t}}{t}$.

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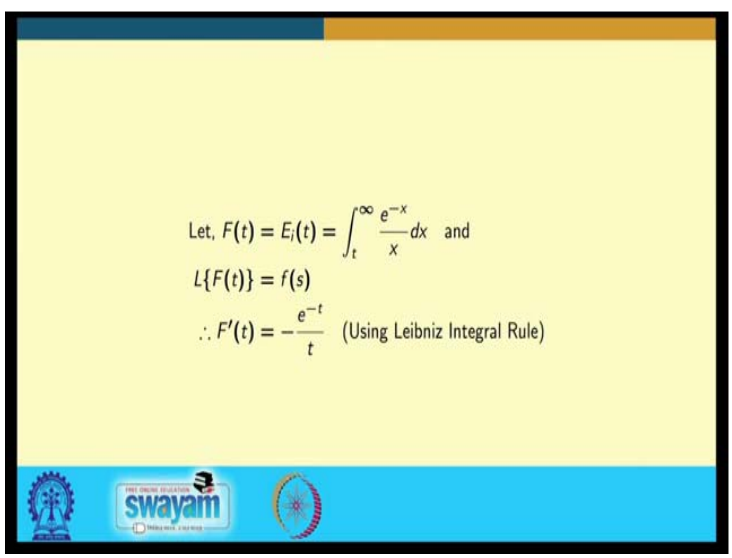

Now, we will follow the similar process as earlier. All the steps are clearly presented in the attached slide. As we can see, we obtain the final result after integration as

$$sf(s) = \log(s+1) + c \qquad (4)$$

where c is the constant of integration whose value we need to evaluate.

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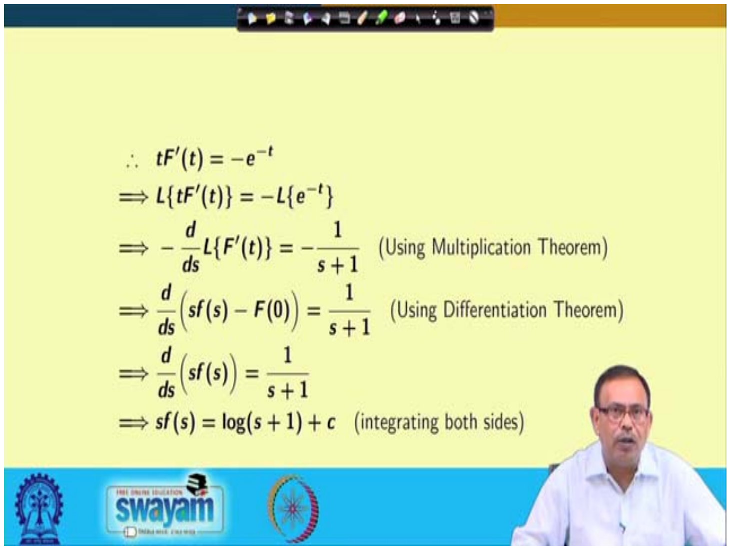

So, we use the Final Value theorem as earlier to evaluate c.

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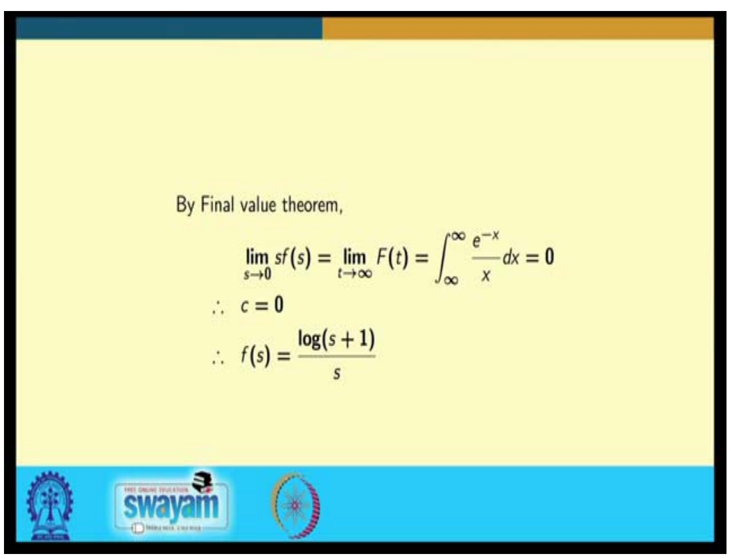

The Final Value Theorem states

$$\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t).$$

Clearly, by the definition of $F(t)$, $\lim_{t\to\infty} F(t) = 0$ so that $\lim_{s\to 0} sf(s)$ also equals 0. Therefore, (4) implies

$$c = 0$$

so that $sf(s) = \log(s+1)$. Therefore,

$$f(s) = L\{E_i(t)\} = \frac{1}{s}\log(s+1).$$

So, these are some special functions, whose Laplace transforms have been derived. In the next lecture also, we will initially start with some more special functions, which are very useful in various engineering problems, statistics and many more. So, we will try to evaluate the Laplace transforms of those useful and frequently used functions in the next lecture.