Transform Calculus and its Applications in Differential Equations
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Lecture – 07
Laplace Transform of Periodic Function

In this lecture, initially we will discuss two theorems: one is the Initial Value Theorem, another one is the Final Value Theorem. And, after that we will see what would be the Laplace Transform of Periodic Functions and their properties.

So we start with the Initial Value Theorem.

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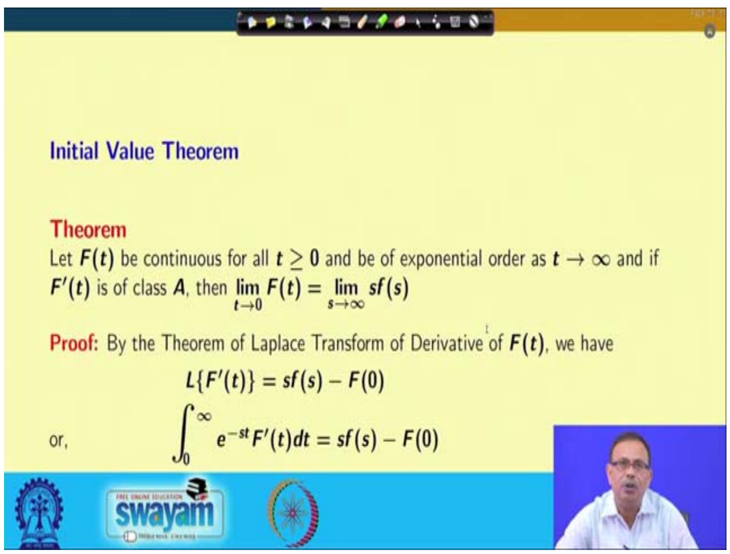

The Initial Value Theorem states that if $F(t)$ is a continuous function for all $t \geq 0$ and is of exponential order as $t \to \infty$ and if $F'(t)$ is of class A (that is $F'(t)$ is piecewise continuous and is of exponential order as $t \to \infty$), then, $\lim_{t\to 0} F(t)$ is equal to $\lim_{s\to\infty} sf(s)$. So, basically whenever t is approaching 0, the value of the function $F(t)$ should be equal to the limiting value of $sf(s)$ whenever s approaches ∞.

Let us see the proof first and afterwards we will see what is the use of these particular theorems.

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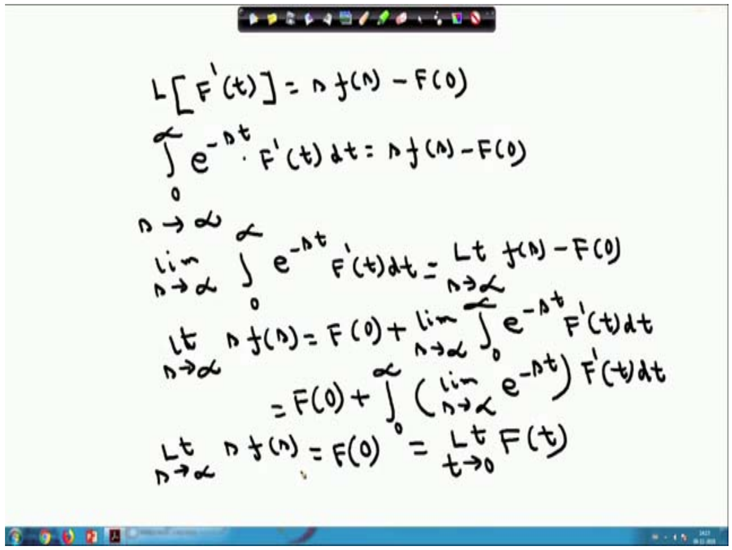

From the theorem of Laplace transform of derivatives, we know

$$L\{F'(t)\} = sf(s) - F(0)$$

where $f(s) = L\{F(t)\}$. Then on the LHS, we use the definition of Laplace Transform to get

$$\int_0^\infty e^{-st} F'(t)\, dt = sf(s) - F(0)$$

Now, if we make $s \to \infty$ on both sides of the above equation, then we have,

$$\lim_{s\to\infty} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s\to\infty} sf(s) - F(0)$$

$$\begin{aligned}
\Rightarrow \lim_{s\to\infty} sf(s) &= F(0) + \lim_{s\to\infty} \int_0^\infty e^{-st} F'(t)\, dt \\
&= F(0) + \int_0^\infty \left(\lim_{s\to\infty} e^{-st}\right) F'(t)\, dt \\
&= F(0) \\
&= \lim_{t\to 0} F(t).
\end{aligned}$$

So, our proof for the Initial Value Theorem is complete.

So, please note that if we consider a function $F(t)$, its limiting value as t approaches 0 always will be equal to the limiting value of the function $sf(s)$ whenever s approaches ∞.

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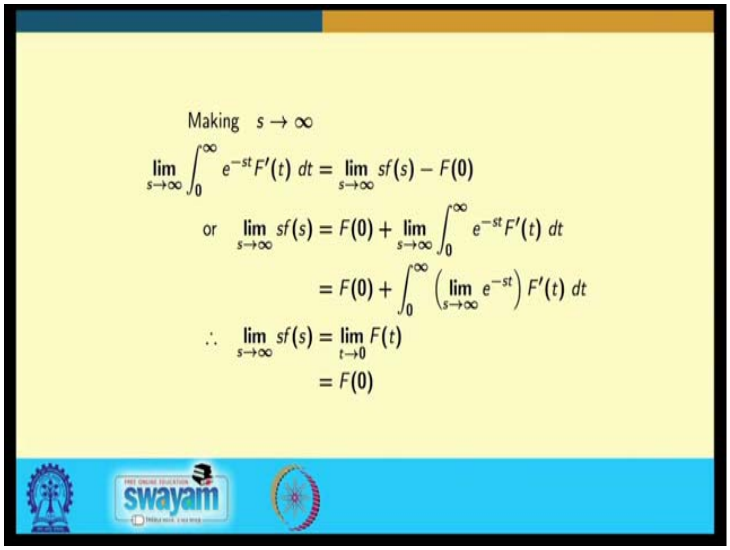

Now we come to the Final Value Theorem.

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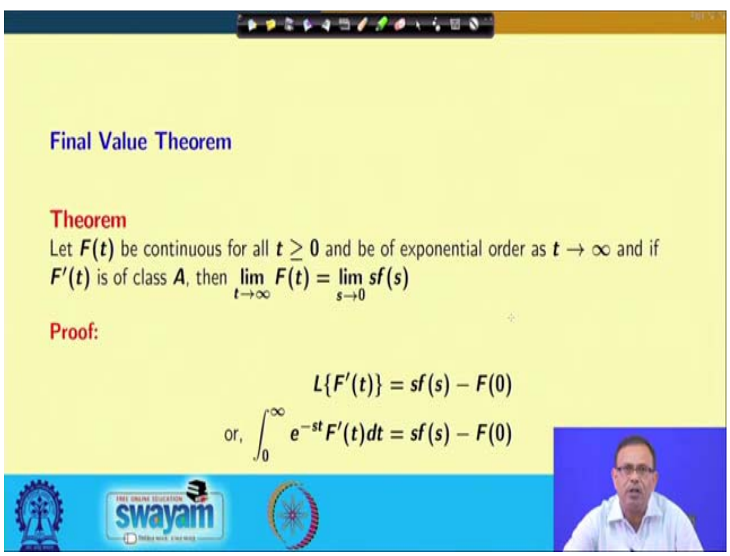

Let $F(t)$ be a continuous function for all $t \geq 0$ and is of exponential order as t approaches ∞ and if $F'(t)$ is of class A, then $\lim_{t\to\infty} F(t)$ equals $\lim_{s\to 0} sf(s)$.

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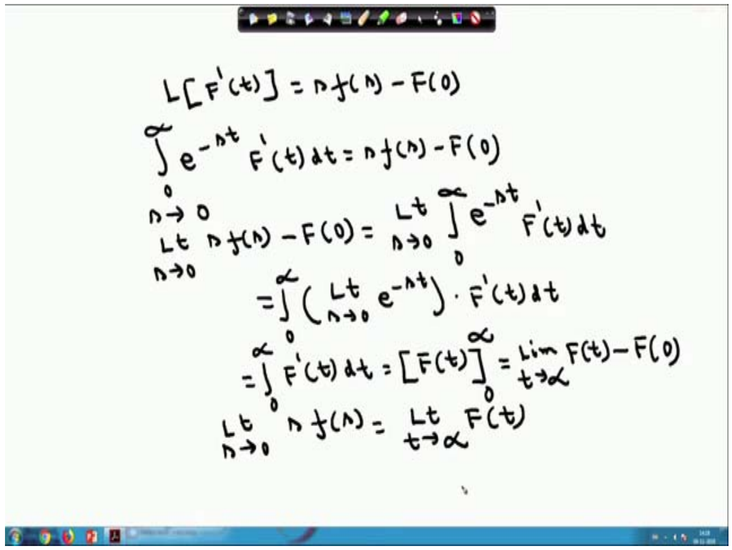

So, let us see the proof for this. From the theorem of Laplace transform of derivatives, we know $L\{F'(t)\} = sf(s) - F(0)$ where $f(s) = L\{F(t)\}$. Here on the LHS, we use the definition of Laplace Transform to get

$$\int_0^\infty e^{-st} F'(t)\, dt = sf(s) - F(0)$$

Now, if we make $s \to 0$ on both sides of the above equation, then we have,

$$\lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s\to 0} sf(s) - F(0)$$

$$\begin{aligned}
\Rightarrow \lim_{s\to 0} sf(s) &= F(0) + \lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\, dt \\
&= F(0) + \int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) F'(t)\, dt \\
&= F(0) + \int_0^\infty F'(t)\, dt \\
&= F(0) + [F(t)]_{t=0}^{\infty} \\
&= F(0) + \lim_{t\to\infty} F(t) - F(0) \\
&= \lim_{t\to\infty} F(t).
\end{aligned}$$

So, our proof for the Final Value Theorem is complete.

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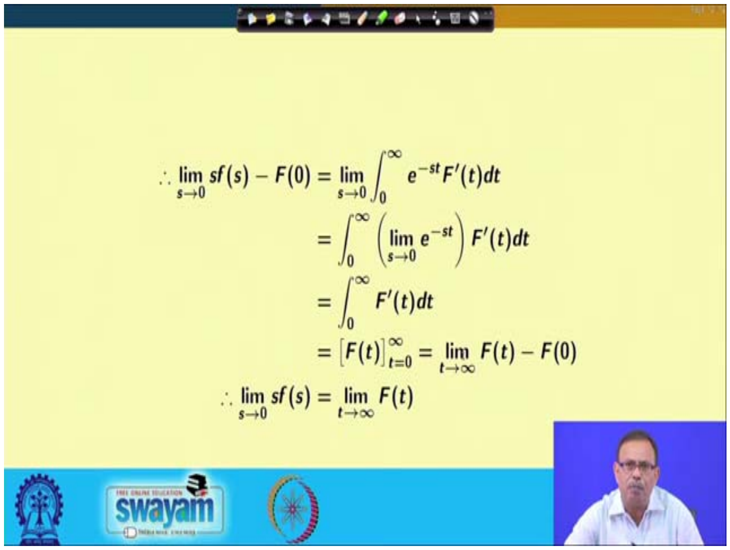

Now we have the Fundamental Theorem for Periodic Functions.

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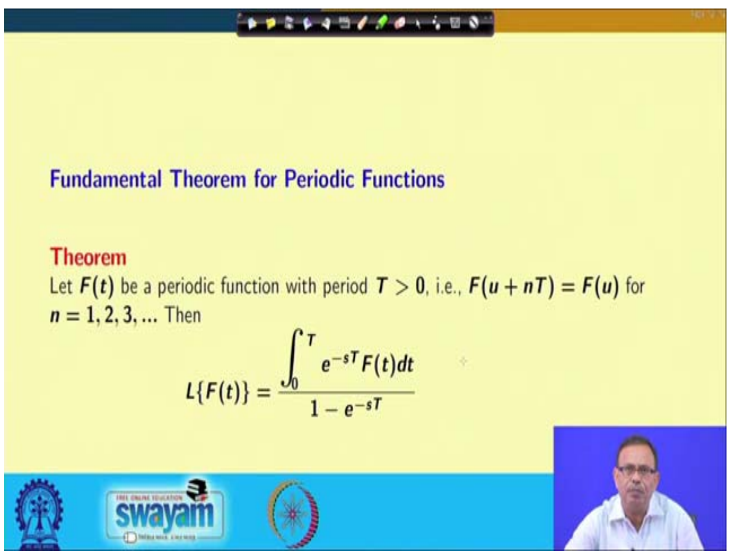

Suppose $F(t)$ is a periodic function with period T>0 so that $F(u + nT) = F(u)$, where n can take values $1, 2, 3$, ... In that case, the Laplace transform of the periodic function $F(t)$ is given by

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}.$$

So, using this formula directly, depending upon the periodic function $F(t)$ and the value of the period T, we can easily evaluate the Laplace transform of a periodic function.

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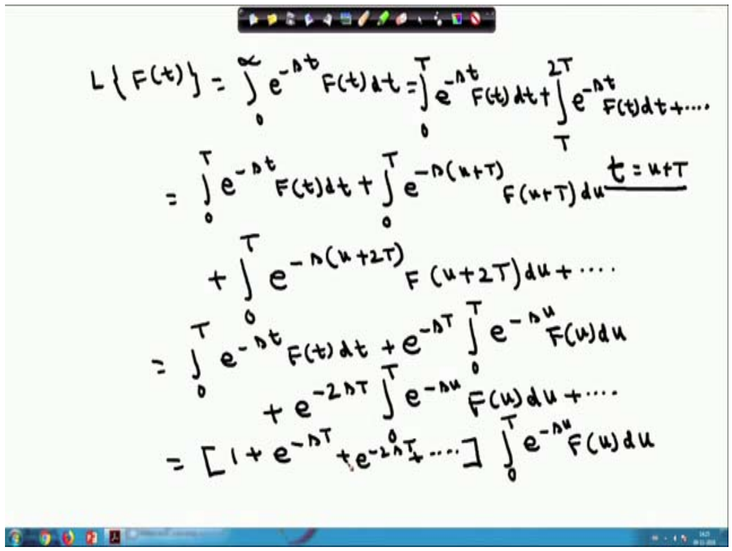

First let us go through the proof of this. We know from the definition of Laplace transform,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$$

Since the function $F(t)$ is of period T, so we break the interval $[0, \infty)$ into a number of sub-intervals as follows:

$$L\{F(t)\} = \int_0^T e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t)\, dt + \int_{2T}^{3T} e^{-st} F(t)\, dt + \cdots$$

Now we make a substitution in each of the integrals starting from the second. We put $t = u + T$ in the second integral, $t = u + 2T$ in the third integral and so on. In such a situation, every integral will have the limits from 0 to T. So, $L\{F(t)\}$ becomes

$$\int_0^T e^{-su} F(u)\, du + \int_0^T e^{-s(u+T)} F(u+T)\, du + \int_0^T e^{-s(u+2T)} F(u+2T)\, du + \cdots$$

[Please note that we have just changed the parameter from t to u in the case of the first integral]

Since $F(t)$ is a periodic function of period T, so $F(u + nT) = F(u)$ holds. So we have,

$$L\{F(t)\} = \int_0^T e^{-su} F(u)\, du + \int_0^T e^{-s(u+T)} F(u)\, du + \int_0^T e^{-s(u+2T)} F(u)\, du + \cdots$$

Now, clearly, $\int_0^T e^{-su} F(u)\, du$ is present in every term of the above expression so that we can take it in common.

$$L\{F(t)\} = [1 + e^{-sT} + e^{-2sT} + \cdots] \int_0^T e^{-su} F(u)\, du$$

$$= \left[\frac{1}{1 - e^{-sT}}\right] \int_0^T e^{-su} F(u)\, du$$

$$= \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}.$$

This completes the proof.

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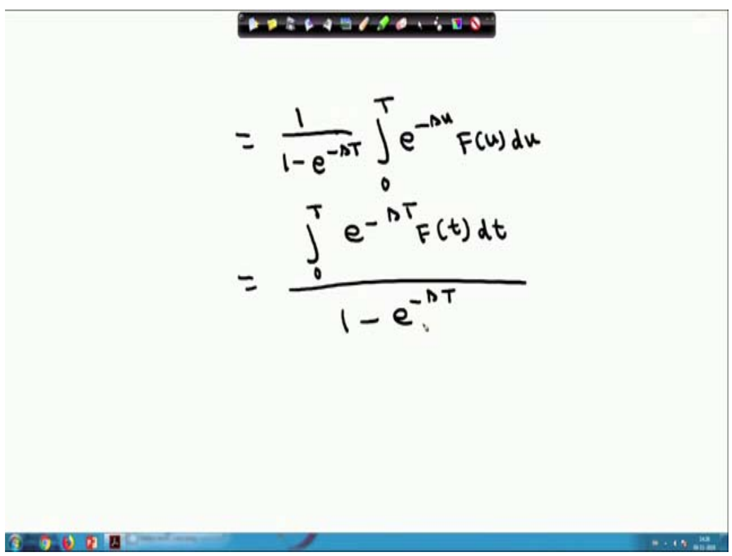

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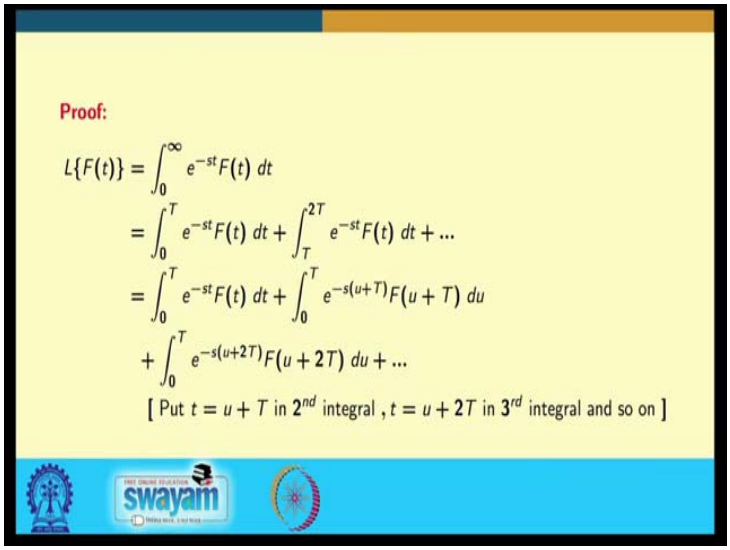

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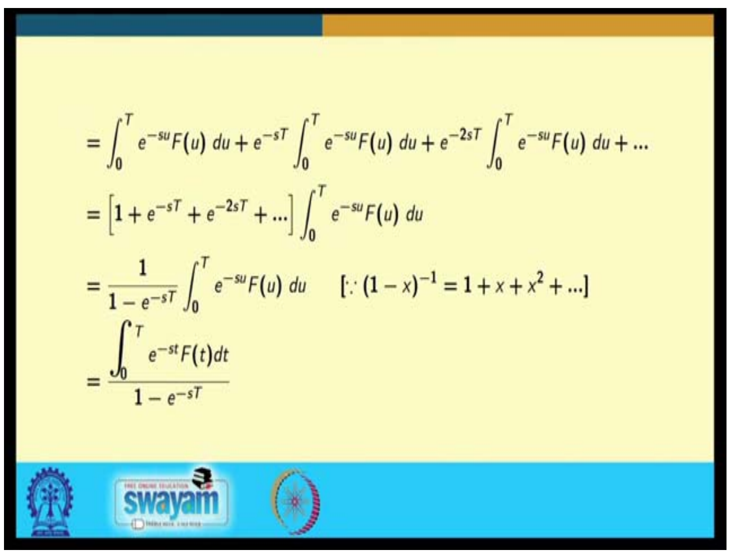

Here is presented a list of the various properties and theorems discussed so far in a tabular form for a better recapitulation.

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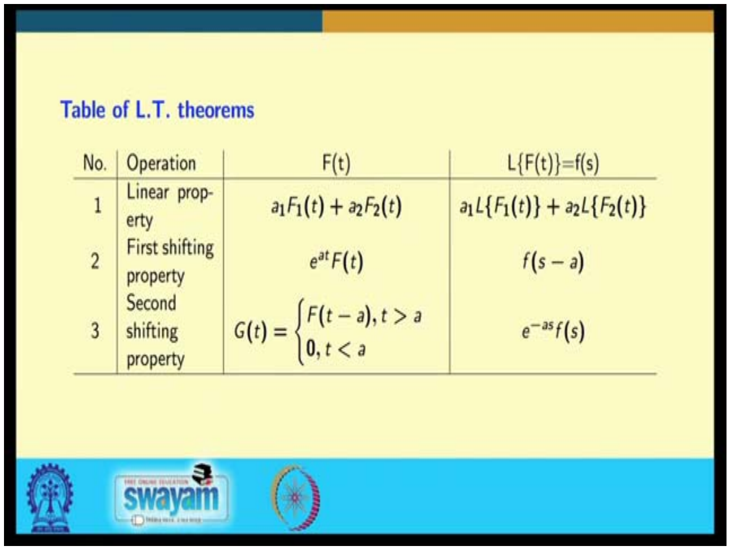

Table of L.T. theorems

No.	Operation	F(t)	L{F(t)}=f(s)
1	Linear property	$a_1F_1(t)+a_2F_2(t)$	$a_1L\{F_1(t)\}+a_2L\{F_2(t)\}$
2	First shifting property	$e^{at}F(t)$	$f(s-a)$
3	Second shifting property	$G(t)=\begin{cases}F(t-a), t>a\\ 0, t<a\end{cases}$	$e^{-as}f(s)$

The first one is the Linearity property, then we have the First and Second Shifting properties.

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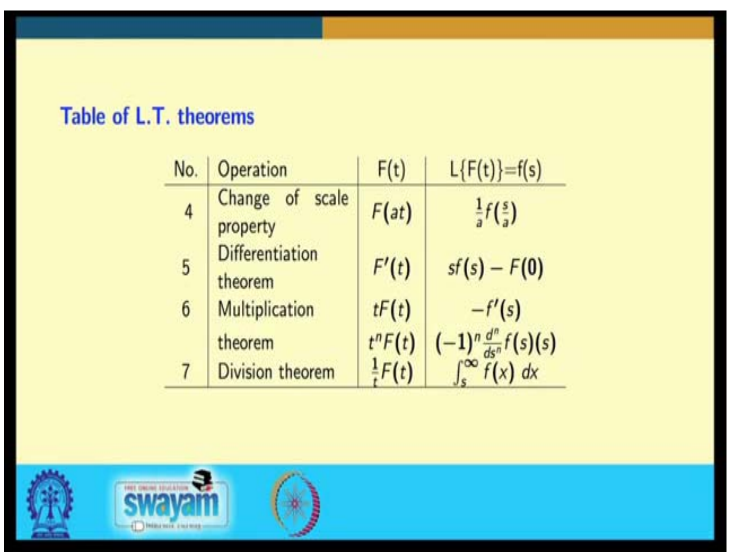

Table of L.T. theorems

No.	Operation	F(t)	L{F(t)}=f(s)
4	Change of scale property	$F(at)$	$\frac{1}{a}f(\frac{s}{a})$
5	Differentiation theorem	$F'(t)$	$sf(s)-F(0)$
6	Multiplication	$tF(t)$	$-f'(s)$
	theorem	$t^nF(t)$	$(-1)^n\frac{d^n}{ds^n}f(s)(s)$
7	Division theorem	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)\,dx$

Next comes the Change of Scale property followed by the Differentiation, Multiplication and Division theorems.

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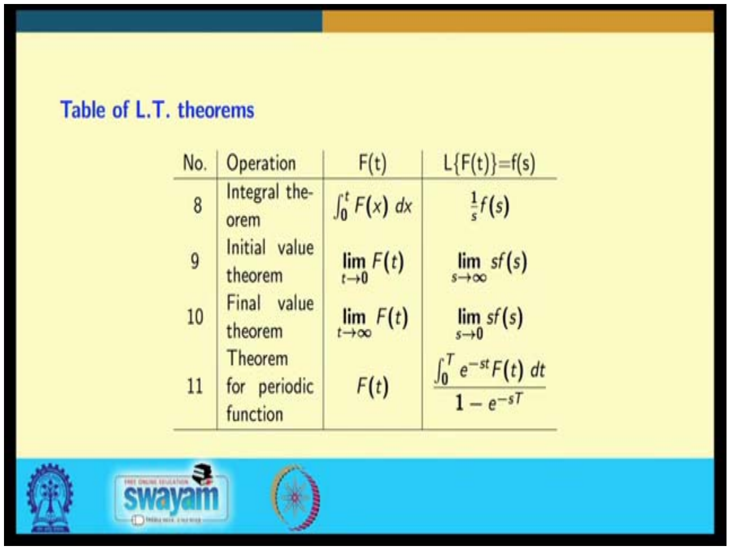

Then we have the Integral Theorem. After this, comes the Initial Value Theorem followed by the Final Value Theorem. Lastly, we discussed the Fundamental Theorem for periodic function.

So, if we just note these two slides, then very easily we can memorize the results of the theorems we have done so far.

Now, let us come to some examples. We want to find the Laplace transform of saw-tooth wave function.

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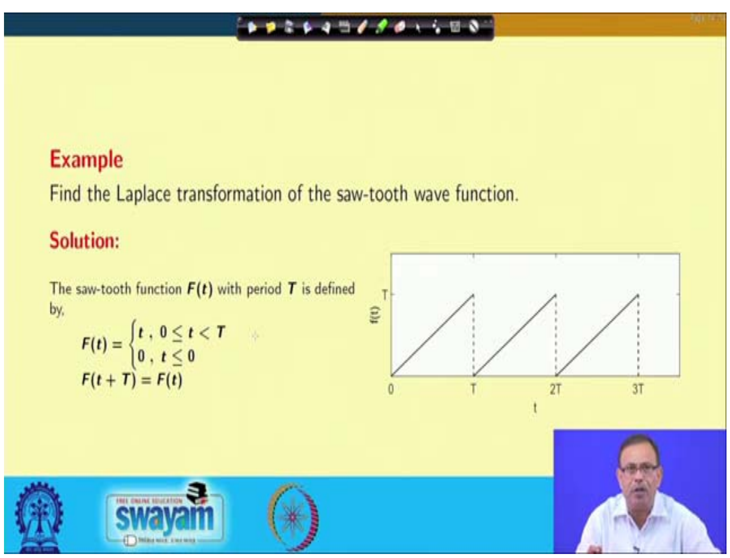

Saw-tooth wave function is a periodic function $F(t)$ with period T defined as

$$F(t) = \begin{cases} t, & 0 \leq t < T \\ 0, & t \leq 0 \end{cases}$$

with $F(t+T) = F(t)$. A graphical illustration of the saw-tooth wave function is represented in the slide. So, next we need to find out the Laplace transform of this function.

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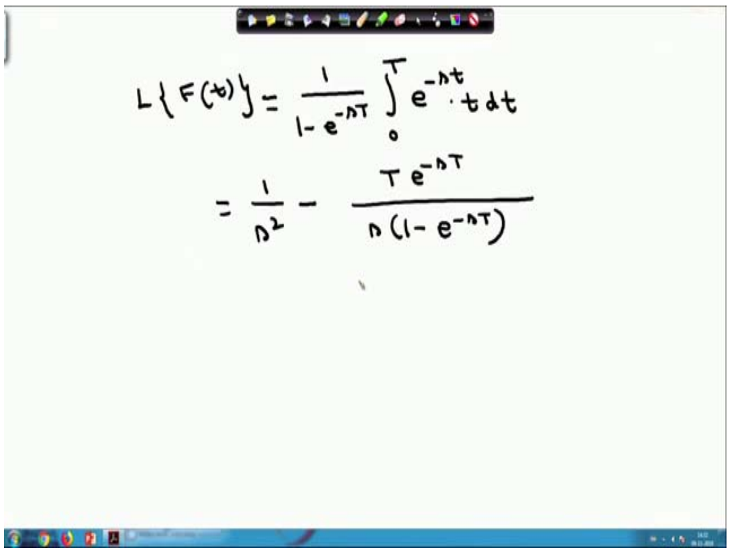

We use the Fundamental Theorem for periodic function as derived earlier:

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}$$
$$= \frac{1}{1 - e^{-sT}} \int_0^T t e^{-st}\, dt.$$

We use integration by parts to solve the above integral. The result in simplified form is presented as:

$$L\{F(t)\} = \frac{1}{s^2} - \frac{Te^{-sT}}{s(1 - e^{-sT})}.$$

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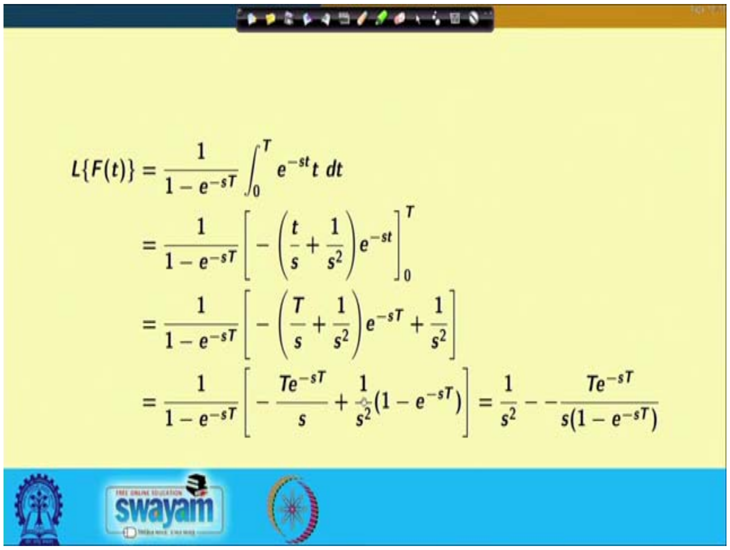

So, effectively we do not have to evaluate the Laplace transform of the periodic function using the normal process. Instead, we are just using the theorem for periodic functions and directly obtaining the result.

Now in the next example, we want to find out the Laplace transform of the function

$$F(t) = \begin{cases} 1, & 0 \leq t < 2 \\ -1, & 2 \leq t \leq 4 \end{cases}$$

where $F(t)$ is a periodic function with period $T = 4$.

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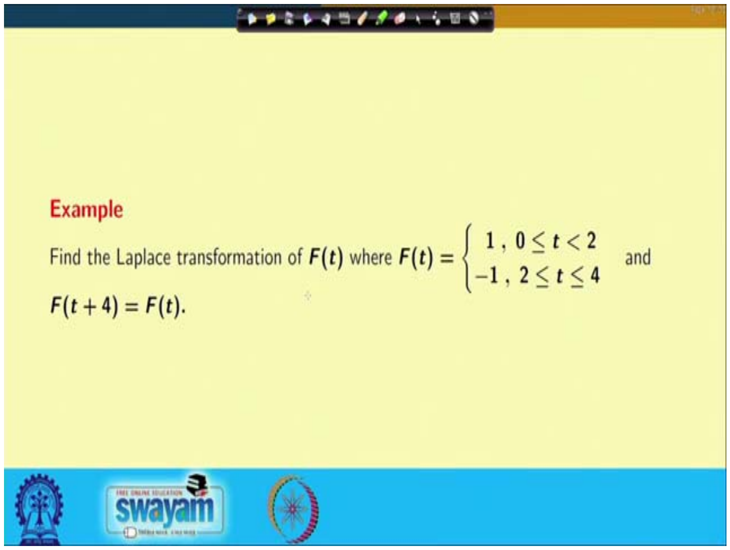

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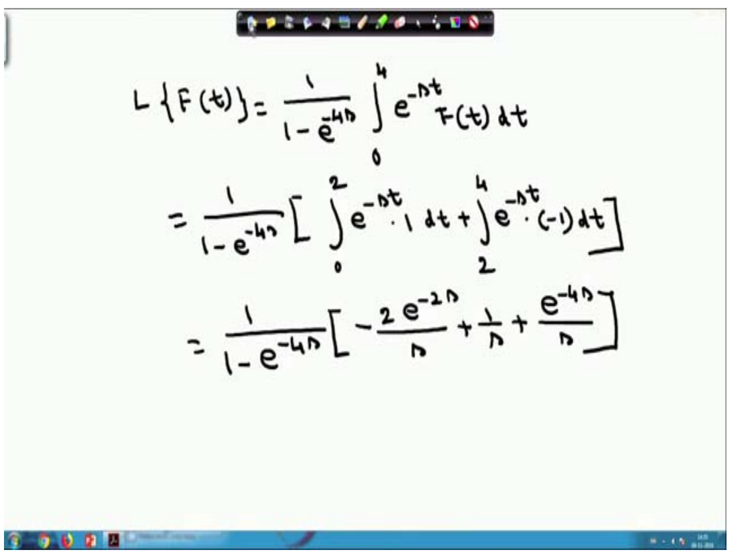

According to the Fundamental Theorem for periodic functions, we have:

$$L\{F(t)\} = \frac{\int_0^4 e^{-st} F(t)\, dt}{1 - e^{-4s}}$$

since $T = 4$ in this case. Therefore, we break the limits of the integration as per the definition of the function as:

$$L\{F(t)\} = \frac{1}{1 - e^{-4s}} \left[\int_0^2 e^{-st}. 1\, dt + \int_2^4 e^{-st}. (-1)\, dt \right]$$

$$\Rightarrow L\{F(t)\} = \frac{1}{1-e^{-4s}}\left[\int_0^2 e^{-st}\,dt - \int_2^4 e^{-st}\,dt\right].$$

This can easily be integrated to obtain the following:

$$L\{F(t)\} = \frac{1}{1-e^{-4s}}\left[-\frac{2e^{-2s}}{s} + \frac{1}{s} + \frac{e^{-4s}}{s}\right].$$

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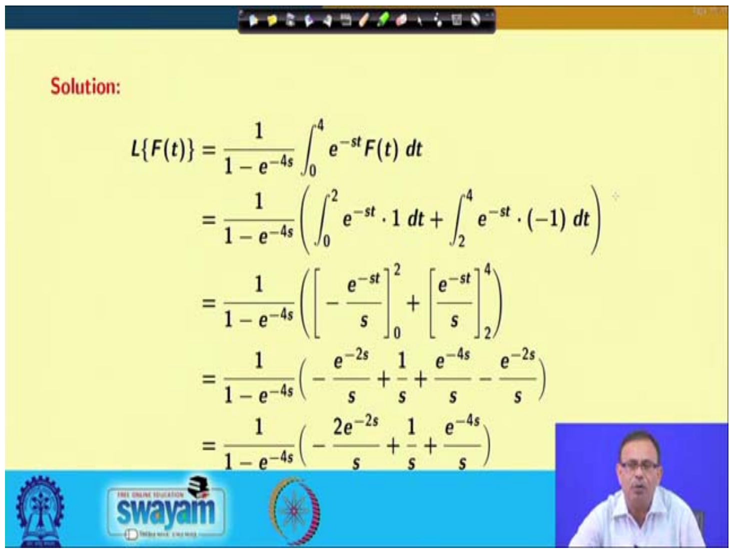

Let us take one more example. We try to find the Laplace transform of periodic square wave function.

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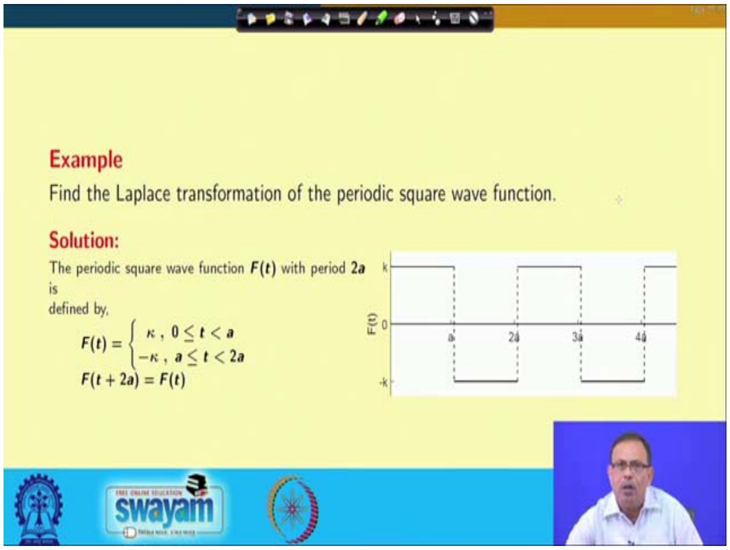

Periodic square wave function is defined as

$$F(t) = \begin{cases} \kappa, & 0 \le t < a \\ -\kappa, & a \le t < 2a \end{cases}$$

where $F(t)$ is a periodic function with period $2a$. A graphical illustration of the Periodic square wave function is represented in the slide.

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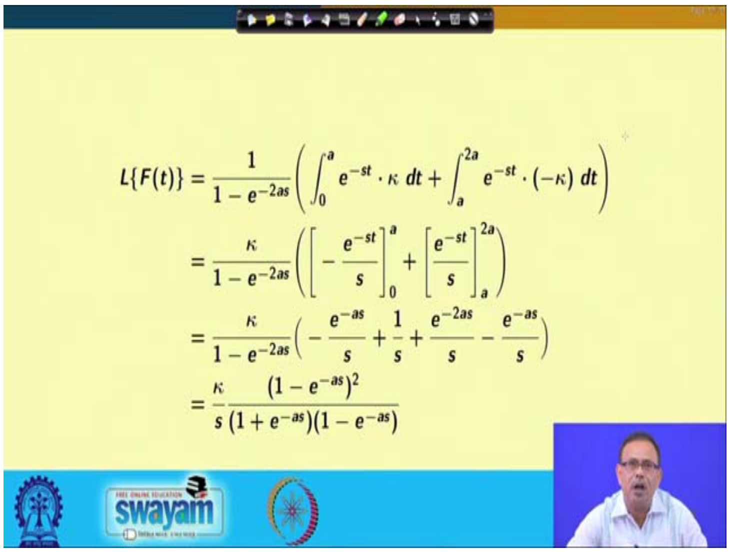

Just as we proceeded in the previous cases for the periodic functions, here also

$$L\{F(t)\} = \frac{\int_0^{2a} e^{-st} F(t)\, dt}{1 - e^{-2as}}$$

since $T = 2a$ in this case. Therefore, we break the limits of the integration as per the definition of the function as:

$$L\{F(t)\} = \frac{1}{1 - e^{-2as}}\left[\int_0^{a} e^{-st}.\kappa\, dt + \int_a^{2a} e^{-st}.(-\kappa)\, dt\right]$$

Now, again the rest of the process is the evaluation of the integrals only and finally, we will obtain the result as follows:

$$L\{F(t)\} = \frac{\kappa}{s}\tanh\left(\frac{as}{2}\right).$$

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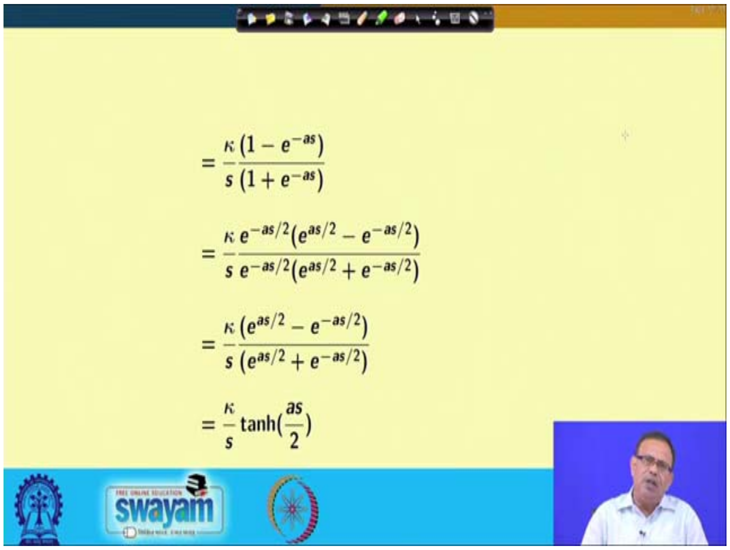

So, we have discussed about the Initial Value Theorem, Final Value Theorem and the Fundamental Theorem for periodic functions. Thank you.