Transform Calculus And Its Applications In Differential Equations
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Lecture - 60
Evaluation of Z - Transform of some functions

So, in the last lecture of this series let us start with the one to more properties to be very precise initial value theorem and final value theorem or Z-transform.

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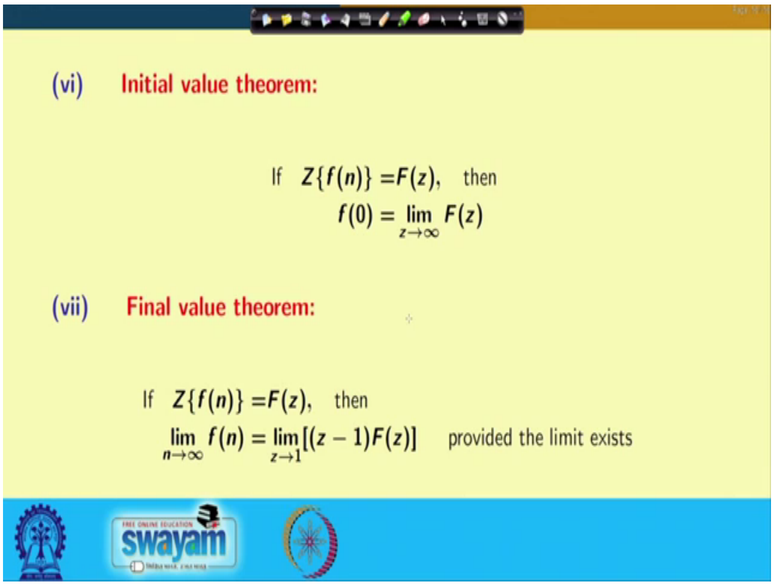

So, initial value theorem says that if Z-transform of f n equals F z, then f 0 equals limit z approaches infinity capital F z.

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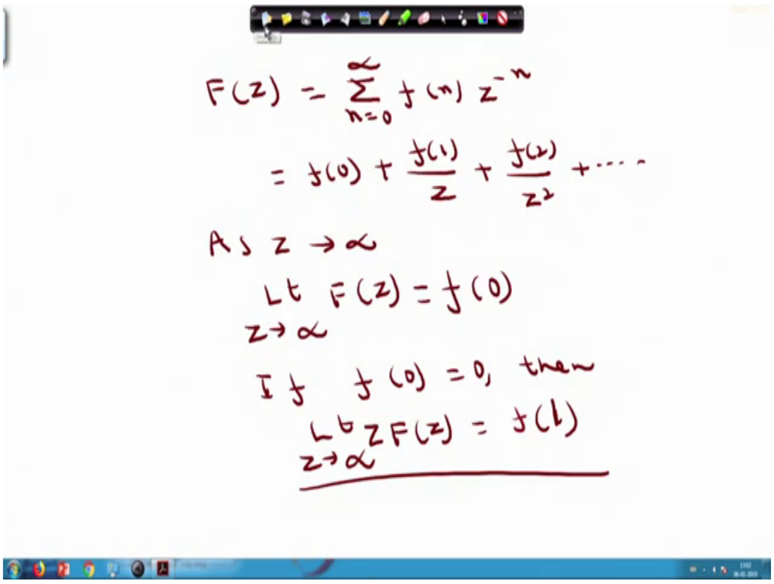

So, to prove this one, we will just do it little quickly, F z equals your summation n equals 0 to infinity f n into z to the power minus n. So, if I simply expand the series, this I can write down f 0 plus f 1 by z plus f 2 by z square like this way. So, as z approaches infinity means limit z approaches infinity F z, all terms will vanish except the first term that is f 0. So, this completes the proof of the initial value theorem that limit z approaches infinity, F z equals f 0.

And please note here from here if your f 0 equals 0, then we can prove this thing, limit z approaches infinity F z this will be equals to f of 1. Please note this one I am not giving the proof of this that if f 0 equals 0, then limit z approaches infinity Z into F z sorry limit z approaches infinity Z F z will be equals to f 1.

Now, let us come to the next theorem that is final value theorem. If Z-transform of f n equals F z, in that case limit n approaches infinity f n equals limit z approaches 1 z minus 1 into F z provided the limits exist.

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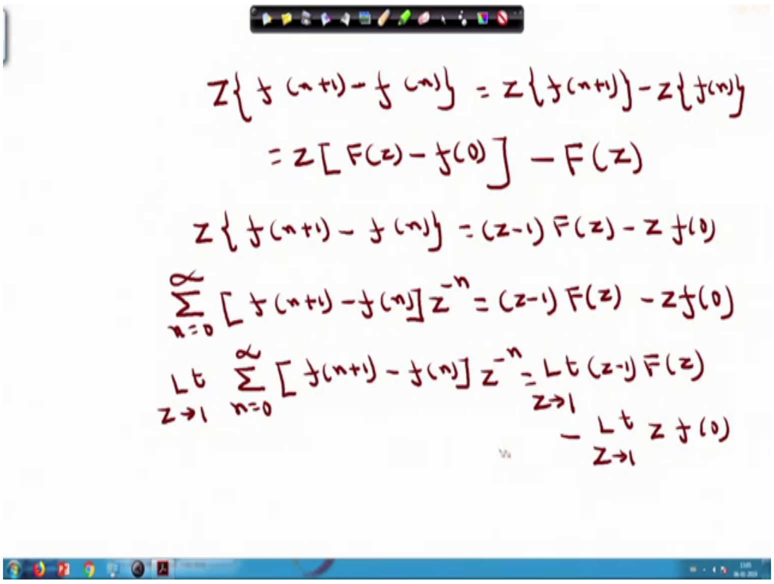

So, let us give the proof of this. We have this thing Z transform of; we have this one Z-transform of f of n plus 1 minus f n f of n plus 1 minus f n, this equals you can write down Z-transform of f n plus 1 minus Z-transform of f n from the linearity property, we can write down this equals z into capital F z minus f of 0.

Please note that Z-transform of f of n plus 1 this equals z, z into capital F z minus f 0 this we have proved earlier minus Z-transform of f n I can write down capital F z. Equivalently if you see this we can write down, Z-transform of f of n plus 1 minus f of n, this is equals to I can write down z minus 1 into F z minus z into f 0. Just simplifying the earlier term this equals you can write down.

Now, from here if I take the summation, summation n equals 0 to infinity f of n plus 1 minus f n into z to the power minus n that is Z-transform of f of n plus 1 minus f n this I am writing using definition as n equals 0 to infinity the function that is f of n plus 1 minus f n into z power minus n. This is equals z minus 1 into F of z minus z into f of 0. So, from here as limit z approaches 1 summation n equals 0 to infinity f of n 1 minus f n into z to the power minus n, this equals limit z approaches 1 z minus 1 into F z minus limit z approaches 1 z into f 0. So, basically we are taking as z approaches 1 on the both side.

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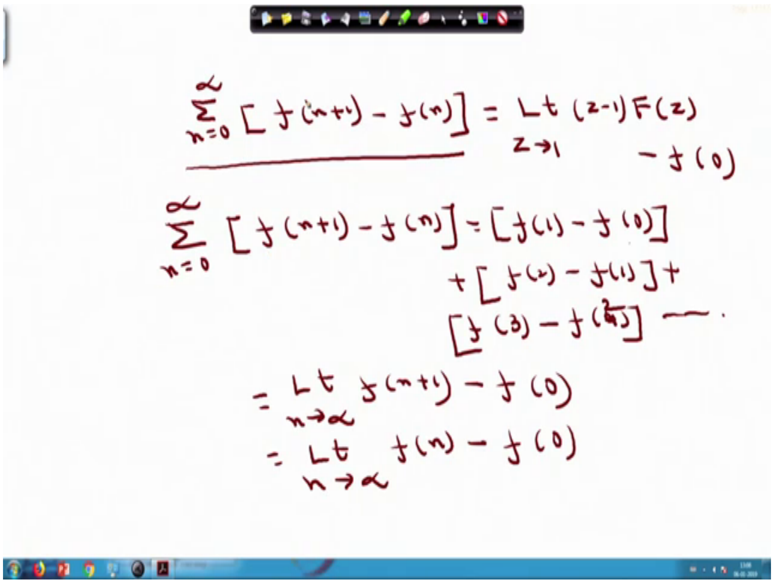

So, this again I can write it as summation n equals 0 to infinity after left hand side taking the limit n equals 0 to infinity f of n plus 1 minus f of n as z approaches 1 z to the power minus n will vanish, and this is equals to limit z approaches 1 z minus 1 into capital F z minus f of 0.

Now, if you see what you are getting summation n equals 0 to infinity f of n plus 1 minus f n, this equals I can write down if I z approaches 1 F z this equals what I can write down this is nothing but f 1 minus f 0 plus if I just expand this summation, f 2 minus f 1 then plus f 3 minus f 4 like this way it will continue sorry this will be f 2, so this will be 3 minus 2 like this way it will continue.

And so that ultimately, what you will get; all terms will be cancelled. There will be only two terms; one is limit n approaches infinity f of n plus 1 and your f 0 would not be canceled. So, all terms will be cancelled. Here f 1 is canceling if 2 is cancelling like this way except this f 0 and the last term that is limit in a is infinity f n plus 1 all other terms will vanish.

So, therefore, the left hand side that is this one summation n equals 0 to infinity f of n plus 1 minus f n this equals you can write down limit n approaches infinity, and f n plus 1 then can be replaced by f n minus f 0. So, once I am getting this, if I substitute this value what I will obtain?

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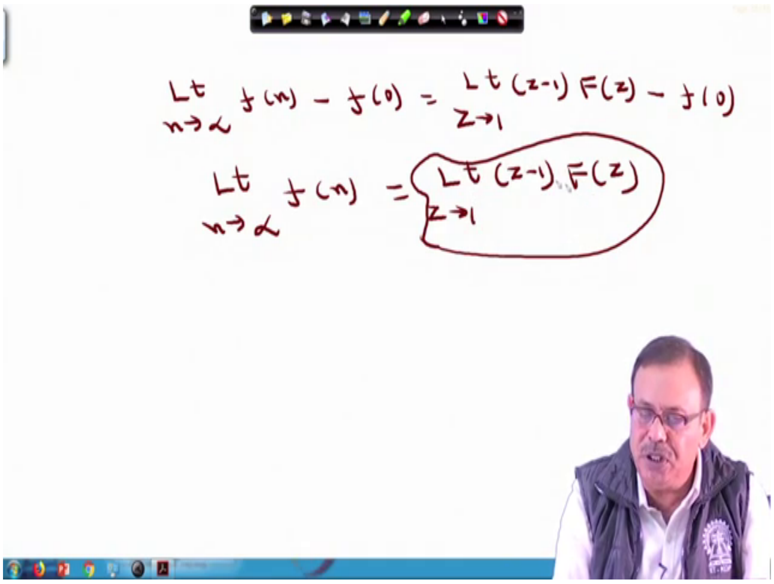

On the left hand side, there will be limit n approaches infinity f n minus f 0, this is equals limit z approaches 1 z minus 1 into F z minus f 0. Therefore, what we can say limit this f 0 will be cancelled and limit n approaches infinity f n this is equals limit z approaches 1 z minus 1 into F z provided the limits both the limits exist. And this completes the proof.

Now, one can ask one question that why not this value is vanishing limit, why not this value is 0 limit z approaches 0 z minus 1 into F z? Let me give one to compare examples from where it will be clear that depending upon the value of capital F z, this value may not be 0, although you have the term limit z approaches 1 z minus 1 into F z.

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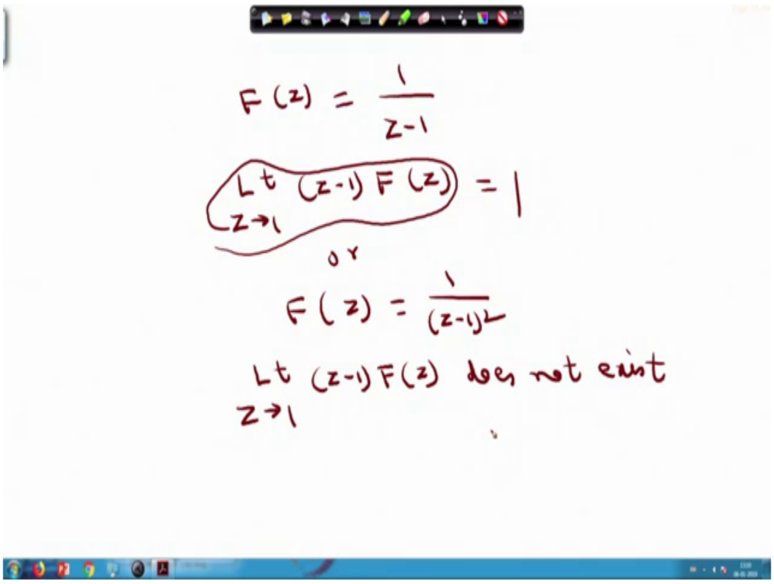

Say your F z is 1 by z minus 1. In that case, what happens limit z approaches 1 z minus 1 into F z this is equals to 1 this is not 0. Or if you consider F z equals 1 by z minus 1 whole square, then limit z approaches 1 z minus 1 into F z that is it will be 1 by z minus 1 that limit does not exist. That is the reason we have told that limited approaches 1 z minus 1 into F z is this value we have to calculate. So, these are the basic properties of the Z-transform.

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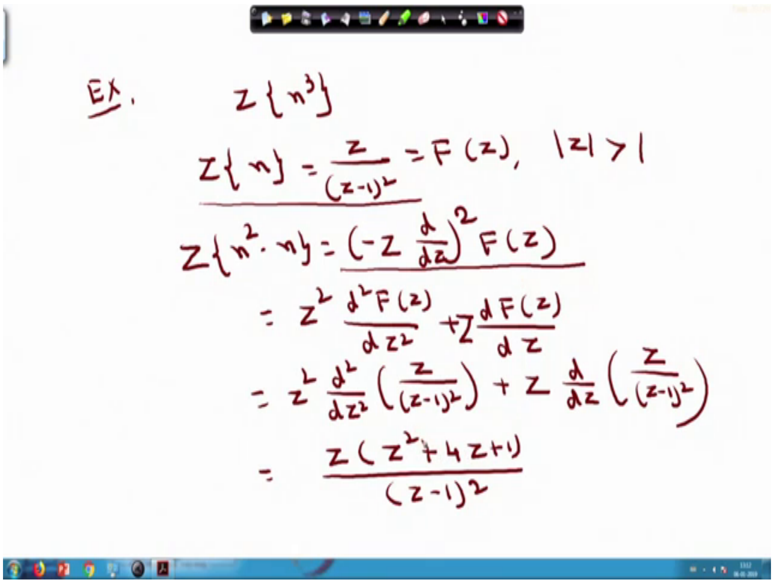

And using these properties, now let us solve some problems quickly so that you can better understand how to find out the Z-transform of various functions using the properties. The first is your Z-transform of n cube we want to find out, we know this thing already we have done it. Z-transform of n is z by z minus 1 whole square. So, this is equals capital F z; and modulus of z is greater than 1, this we have done earlier in the last lecture itself, we have done this thing.

So, now, from here using the property Z-transform of n square into n; this equals I can write down minus z d dz whole square of capital F of z. And so minus z d dz whole square into capital F z, this again we have done earlier using the property I can write down. So, this equals z square d square F z by dz square plus it will be d F z by dz. And already you know what is your capital F z capital F z is z by z minus 1 whole square. So, this actually d square dz square of z by z minus 1 whole square this is one term, and then here it will be sorry here one z will come. So, it will be z into d dz of z by z minus 1 whole square.

So, simply you have to differentiate for the first case twice, for the second case once. And if you just differentiate it your ultimate result we will be this. Please check this one of your own whether it is correct or not. So, the result will be Z-transform of n cube is equals to z into z square plus 4 z plus 1 divided by z minus 1 whole square. So, just I am using the property over here. Since I know this Z-transform of n is z by z minus 1 whole square using the property, I am just by differentiation I am getting the Z-transform of n cube.

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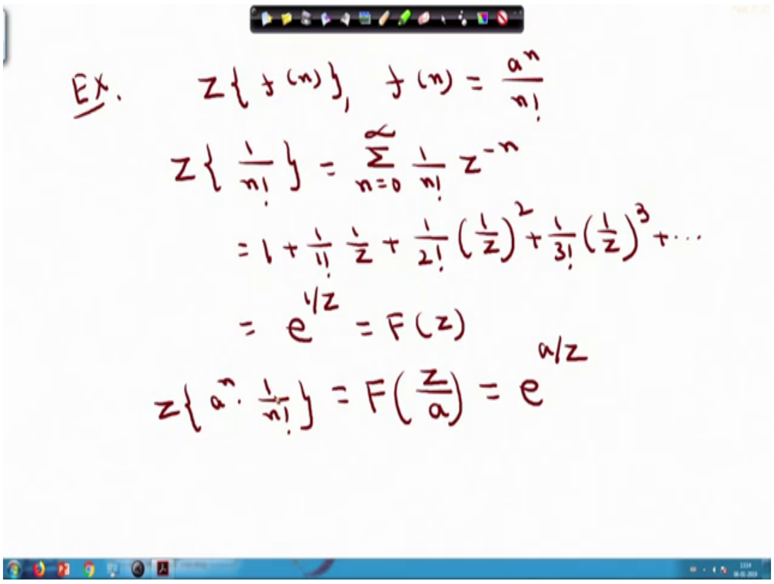

Next is say we want to find out the Z-transform of f n, where your f n is defined as a to the power n divided by factorial n, where your f n is a to the power n by factorial n. Now, again Z-transform of 1 by factorial n, this I can write as from definition summation n equal 0 to infinity 1 by factorial n into z to the power minus n. If I expand the series, I will obtain 1 plus 1 by factorial 1 into 1 by z plus 1 by factorial 2 into 1 by z whole square plus 1 by factorial 3 into 1 by z whole cube, and like this way it will continue.

And if you see closely this series, this series is nothing but e power 1 by z. If I make the expansion of e power 1 by z, I will obtain this series. This equals capital F z say. Therefore, Z-transform of 1 by factorial n is nothing but e power 1 by z. So, Z-transform of a to the power n into 1 by factorial n; using scaling property, I can say this is nothing but capital F of z by a. And capital F of z e power 1 by z, so that capital F of z by a will be equals to e power a by z.

So, you see first in am finding the Z-transform of 1 by factorial n. And using this particular property that Z-transform of 1 by factorial n e is power 1 by z, then using scaling property we are saying what is the Z-transform of a to the power n into 1 by factorial n. So, I hope it is clear.

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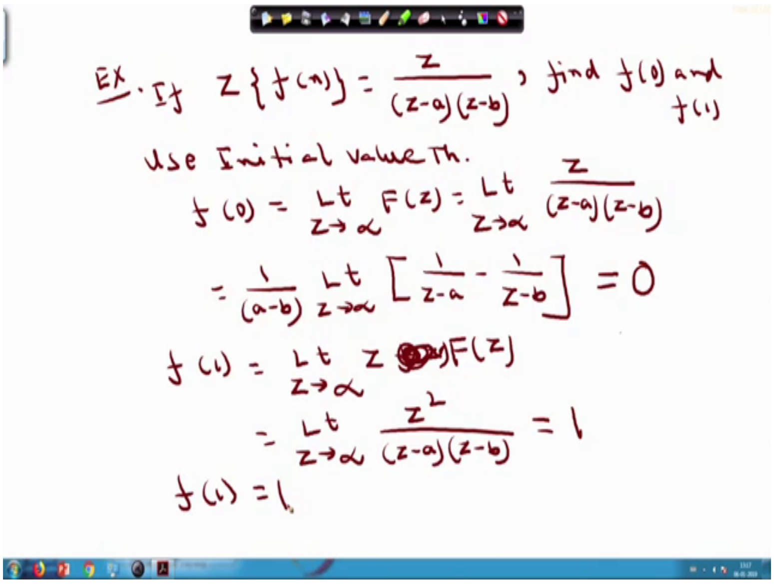

Next let us take another example. It is given that I am just doing the opposite one now, if Z-transform of f n this is equals z by z minus a into z minus b, then you find out f 0 and f 1 that or in other sense what we want to say is that if Z-transform of your function is known to us, how to find out the value of f n, because here we have given the Z-transform of a function and the inverse one we can find out.

Now, you use initial value theorem, using initial value theorem, what you can write down f 0 equals limit z approaches infinity F z, and limit z approaches infinity F z is nothing but z by z minus a into z minus b. This equals if you break it, you will obtain 1 by a minus b into limit z approaches infinity 1 by z minus a minus 1 by z minus b. Therefore, both the limits will be 0 limit z approaches infinity 1 by z minus a will approach 0 also limit z approaches infinity 1 by z minus b this will also put 0. So, the value is 0. Therefore, is simply using initial value theorem the value of f 0 is 0.

Similarly I know that f 1 is limit z approaches infinity z f z limit z approaches infinity sorry this is not small f z, but this is z into capital F z. So, this is equals limit z approaches infinity if I substitute the value z square by z minus a into z minus b. If I evaluate the limit simply z square I can both numerator and denominator I can divide. And in that case, I will obtain the result as this one; that is f of 1 is 0 the sorry f of 1 is 1. Therefore, by initial value theorem, simply I am saying f of 0 is 0. And using this property that f 1 equals limit z approaches infinity F z and by evaluating the limit we are saying that f 1 this is equals to 1.

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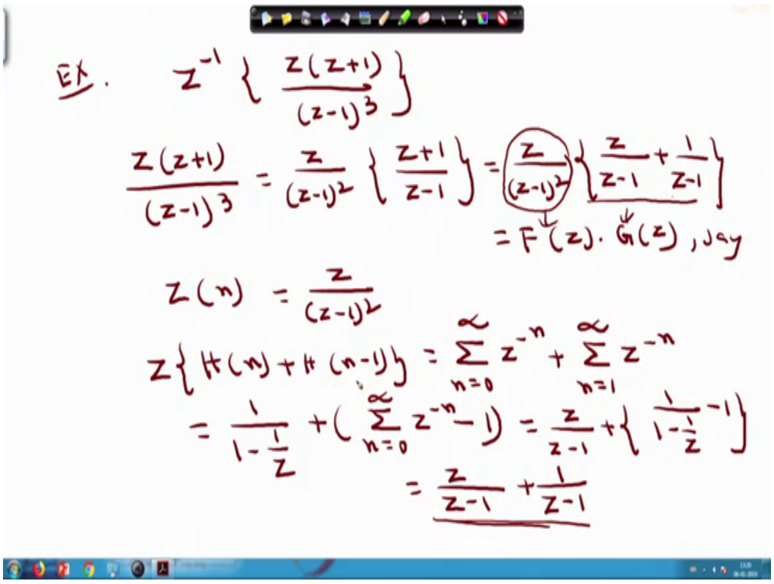

Let us take some other example that is inverse Z-transform of this function z into z plus 1 by z minus 1 whole cube, we have obtained the Z-transform now we are deriving, we want to derive the inverse Z transform also of a function, so that you can do it of your own. You take this function z into z plus 1 divided by z minus 1 whole cube, this equals you can write down z by z minus 1 whole square into z plus 1 by z minus 1, so that this I can break it again z by z minus 1 whole square into z by z minus 1 plus 1 by z minus 1. And this I can write it as F z into G z. So, F z I know, G z I know, these are I am assuming say.

So, from here I want to find out what would be z. Now, you know this thing Z-transform of n is z by z minus 1 whole square. So, basically your F z is this quantity and your G z is basically this quantity. So, I want to find out the function, whose Z-transform is this. Already we know these things we have done earlier that Z-transform of n is z by z minus 1 whole square.

And also Z-transform of H of n plus H of n minus 1, where H is the Heaviside function. So, this equals summation n equals 0 to infinity z to the power minus n plus summation this will be n equals 1 to infinity z to the power minus n, this we have done depending upon the value of n and n minus 1.

So, summation n equals 0 to infinity z to the power minus n, this is nothing but 1 by 1 minus 1 by z. This we have done for the second part. I can write down this thing as summation n equals 0 to infinity z to the power minus n. So, 1 I have added here; so I have to subtract 1 here. So, this equals this I can write down z by z minus 1 plus this part again I can write down 1 by 1 minus z minus 1.

And this if I simplify this will be equals to z by z minus 1 plus this will become 1 by z minus 1. So, this is; this term is nothing but this one. So, therefore, I obtained the function whose Z-transform is this two. So, what I can write down; your if n is here, what will be your f n, f n will be n; and what will be your g n, g n will be H n plus H n minus 1.

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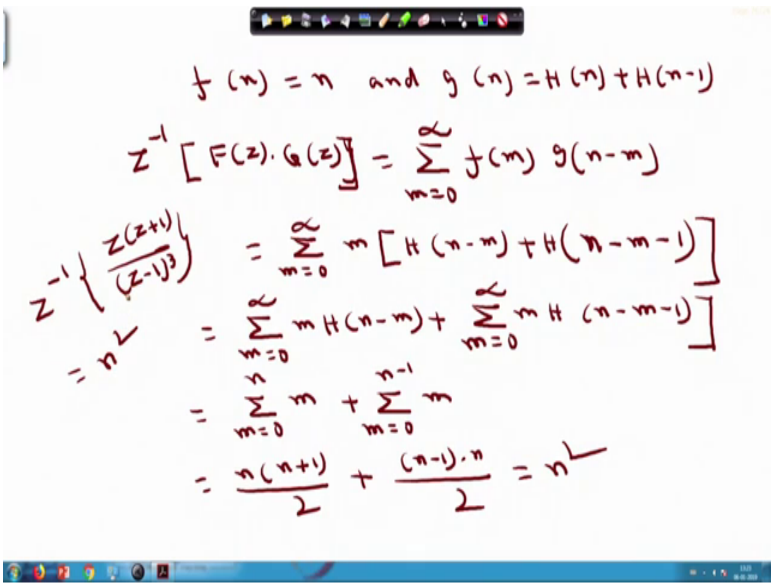

So, let me write down this thing. Your f n is equals to 1 and your g n is equals to H n plus H n minus 1. I hope the method was clear. We are first finding out that function, whose transform; whose Z-transform is that function of z. From there I am writing f n is this g n is this. Now, using convolution theorem, I can write down inverse Z-transform of capital F z into G of z, this is equals to I can write down summation m equals 0 to infinity f m into g of n minus m.

This equal summation m equals 0 to infinity m f m is m and g of n minus m will be H of n minus m plus H of n minus m minus 1 from here. So, I obtained this. So, if I break it, I will obtain two series; one is m equals 0 to infinity m into H of n minus m; other one will be summation m equals 0 to infinity m into H of n minus m minus 1. So, once I am getting it, so this I can write down m equals 0 to n m, because for other cases from in to infinity, it is 0; only 0 to n it is 1.

Similarly, here for m equals 0 to n minus 1, it is 1; for other values, it is 0. So, m equals 0 to n minus 1 into m. If I evaluate, this is n into n plus 1 by 2 plus this will be n minus 1 into n by 2. And if I simplify this, I will obtain the value as n square. Therefore, according to the problem your z inverse this one, your problem was to find out z inverse z into z plus 1 by z minus 1 whole cube. So, this value is n square that is the value of n square if I take Z-transform of n square is this. So, I hope it is clear using inverse z-transform how to find out the function itself that is quite clear from here.

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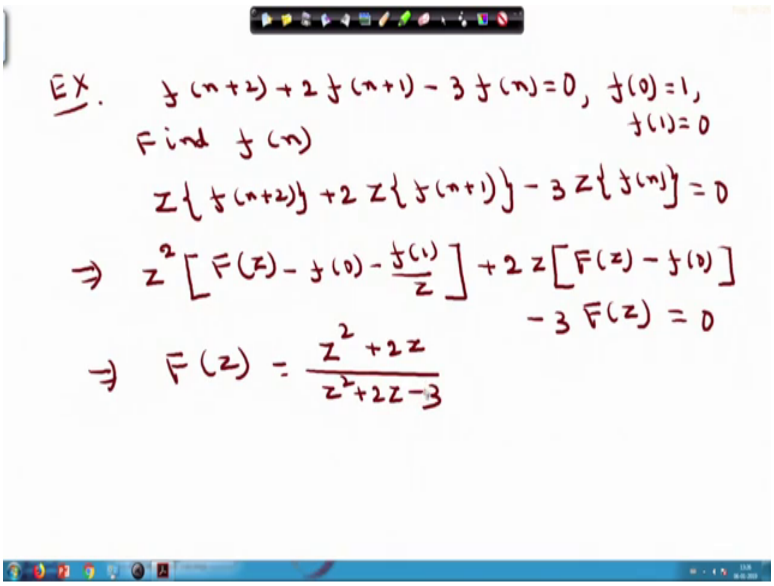

Let us take some other type of example again. It is given that f n plus 2 plus 2 into f of n plus 1 minus 3 f n this is equals 0. This equation is given. Also it is given f 0 equals 1, f 1 equals 0. So, one equation in terms of function of n is given, f of n plus 2 plus twice f of n plus 1 minus 3 f n equals 0, where f 0 equals 1, f 1 equals 0. And we have to find out what is the value of f n.

So just like solving the differential equation what we have done using Laplace or Fourier transform, here also if one discrete functions equation is given from there how to find out the value of f n. Now, you take the Z-transform on both side of the given equation, so that you can write down Z-transform of f of n plus 2 plus twice Z-transform of f of n plus

1 minus 3 into Z-transform of f n this is equals to 0. Now, from the definitions, already we know what is the Z-transform of f of n plus 2, Z-transform of f of n plus 1, Z-transform of f n, these things are known to us we have done it earlier.

So, if I simply follow those formulas, I can write down z square into Z-transform of f n plus 2 is equals to z square into capital F z minus f 0 minus f 1 by z plus 2 into z into Z-transform of f of n plus 1 is capital F z minus f 0. And this is 3 into transform of f n the transform of f n is capital F z this is equal to 0.

So, now, you know the values of f 0 and f 1; f 0 and f 1 if I know the values of these two, then I can substitute it and ultimately I can find out what is my F z. My F z will be then z square plus twice z by z square plus twice z minus 3, you please check it up whether it is proper or not, substitute f 0 equals 1 and f 1 equals 0. And then by simplifying I will obtain F z equals z square plus twice z by z square plus twice z minus 3.

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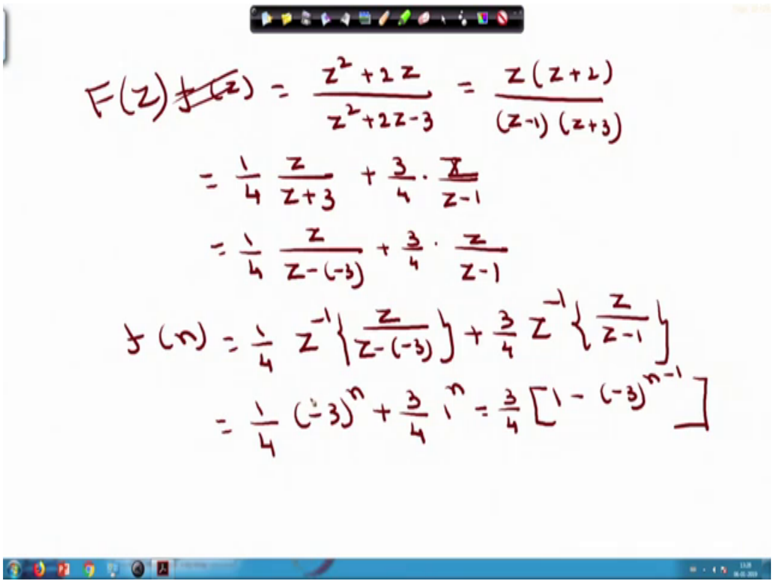

So, I will just; let me rewrite that is F z equals z square plus twice z divided by z square plus twice z minus 3. This I can write it as z into z plus 2 divided by z minus 1 into z plus 3. And using componendo dividendo, if you break it up you will obtain 1 by 4 into z by z plus 3 plus 3 by 4 into 1 by z minus 1. And this on the simplification term 1 by 4 into z minus 3; I can write down plus 3 by 4 into z by z minus 1 3 by 4 into z by z minus 1, this will be actually z will come here; this will be z by z minus 1.

So, F z I am getting this so sorry this is I have written wrongly; this will be capital F z. Please note this one not small f z capital F z that is Z-transform of f n, so that now from here f n, I can write down simply 1 by 4 z inverse of z minus, minus 3 plus 3 by 4 z inverse of z by z minus 1. And we know the values of these things 1 by 4 into this will be minus 3 whole to the power n plus 3 by 4 into 1 to the power n, and this if you wish, you can write down 3 by 4 into 1 minus, minus 3 to the power n minus 1.

So, like this way whenever I am finding if I have been given a equation in terms of f n and some conditions are provided. In this particular format by substituting the values of f n f n plus 1, f n plus 2, I can ultimately write down capital F z equals a function z, then you substitute it you take the inverse Z-transform and find out the value of f n.

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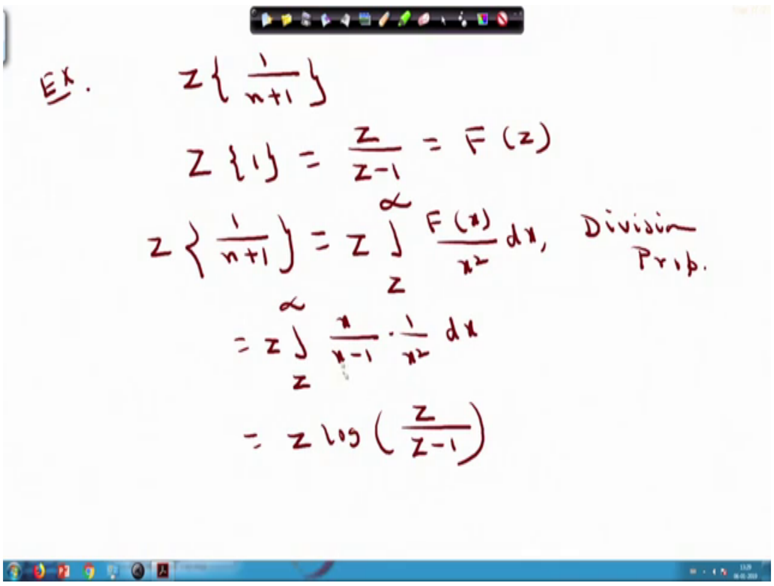

Let us solve one more problem. We want to find out the Z-transform of 1 by n plus 1. We know that the Z-transform of 1 is equals to z by z minus 1. So, this equals say F z. So, therefore, Z-transform of 1 by n plus 1 that is the division property now. Using the division property, this equals write down z to infinity F x by x square dx. So, please note that here we have used the division property. So, simply what I have to do now, z into z to infinity you know what is capital F z that is z by z minus 1, so x by x minus 1 into 1 by x square into dx.

If you evaluate the integral, I am just writing directly z into log of z by z minus 1, because this I can evaluate very easily. So, like this way whenever as and when required if you see, I have used here division property. So, please note one thing that whenever I have the problem, I have to find out the Z-transform of a function or the inverse Z-transform whatever properties we have done using those properties very easily we can find out the Z-transform, also we will provide more assignments on each of them and I hope that the course will be really useful for you all.

Thank you.