Transform Calculus and its Applications in Differential Equations
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Lecture – 06
Explanation of properties of Laplace Transform using Examples

In the last lecture, we have covered certain properties of Laplace transform, such as: if we know the Laplace transform of a function, then how to find out the Laplace transform of the n^{th} derivative of that function that is Laplace transform of $F^n(t)$ or the Laplace transform of the integration of that function, that is Laplace transform of $\int_0^t F(x)dx$.

Or, if we multiply a function by t i.e., if new function becomes $tF(t)$, then also we can find out the Laplace transform of $tF(t)$ knowing Laplace transform of $F(t)$. Or, if we divide a function by t i.e., new function becomes $\frac{F(t)}{t}$, then also we can evaluate the Laplace transform of $\frac{F(t)}{t}$; these things we have covered in the last lecture. Let us go through certain examples as how to use these properties to find out the Laplace transform of various complicated functions.

So, the first one is to prove that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$ using Laplace Transform.

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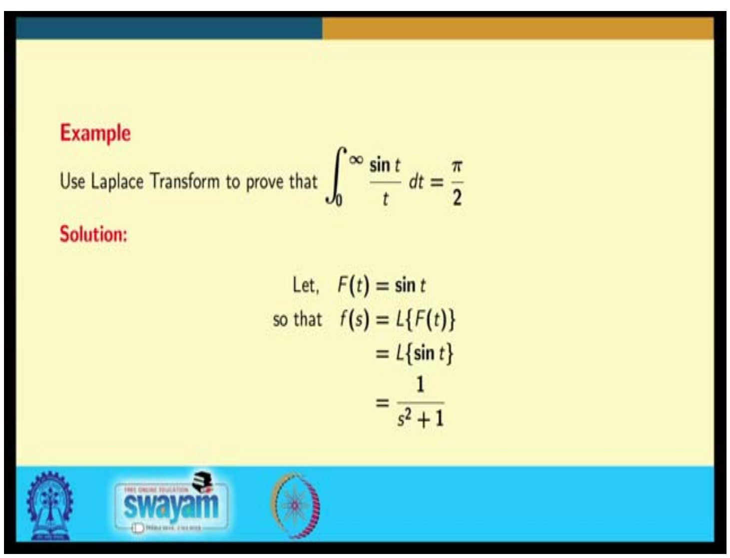

Those who are familiar, this particular integral $\int_0^{\infty} \frac{\sin t}{t} dt$ can be solved using integral method but the solution is tedious, of course. However, we can find out the solution using Laplace transform very easily.

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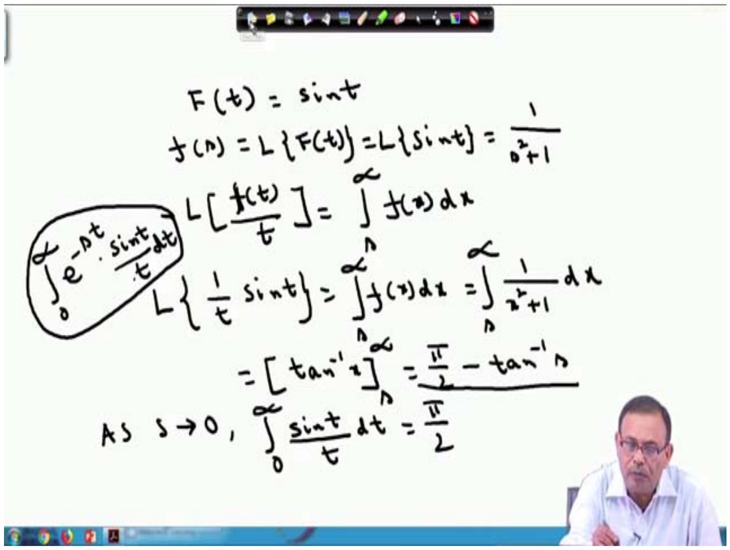

So, we assume at first, $F(t) = \sin t$. Therefore,

$$f(s) = L\{F(t)\} = L\{\sin t\} = \frac{1}{s^2 + 1}$$

Now, by the theorem for division by t, we have $L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x)\, dx$. Applying the same, we can write,

$$\begin{aligned} L\left\{\frac{\sin t}{t}\right\} &= \int_s^{\infty} \frac{1}{x^2 + 1} dx \\ &= [\tan^{-1} x]_{x=s}^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1} s. \end{aligned} \quad (1)$$

Now, we use the definition of Laplace Transform on the LHS so that

$$L\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1} s$$

$$\Rightarrow \int_0^{\infty} e^{-st} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} s.$$

We now substitute $s = 0$ in the equation to obtain the following result:

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

which proves the desired result.

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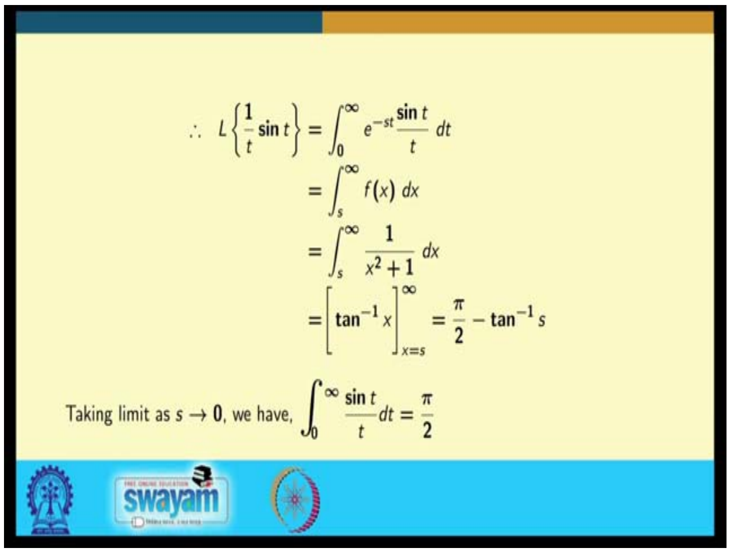

Let us take the next example. We want to evaluate the integral $\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t} dt$.

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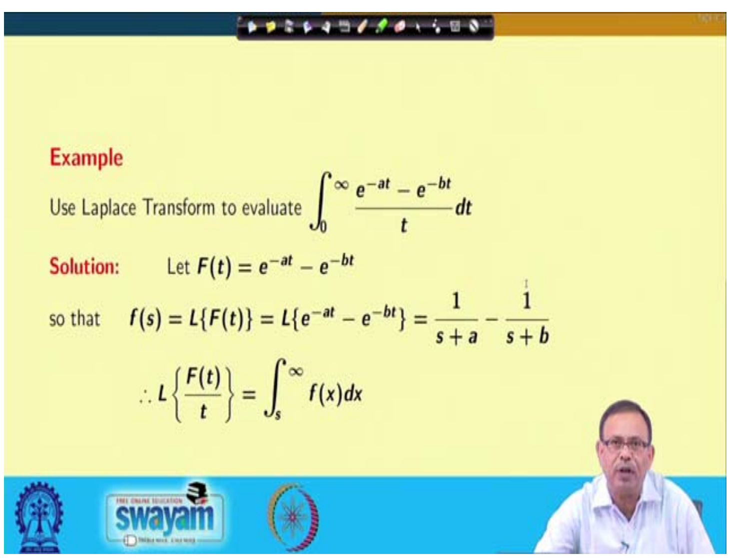

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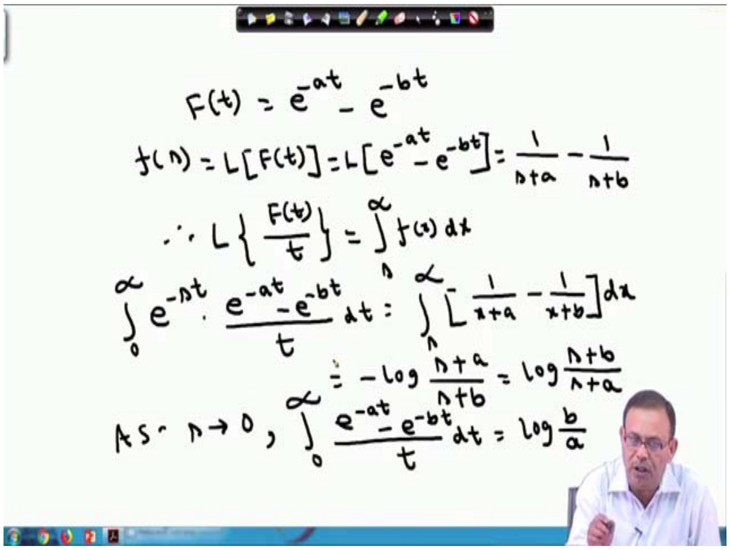

So, initially we assume that $F(t) = e^{-at} - e^{-bt}$. Therefore,

$$\begin{aligned} f(s) \; &= L\{F(t)\} \\ &= \; L\{e^{-at} - e^{-bt}\} \\ &= \frac{1}{s+a} - \frac{1}{s+b}. \end{aligned}$$

Therefore, using the division property,

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x)\, dx. \qquad (2)$$

Or in other sense, we can write (2) as

$$\int_0^{\infty} e^{-st}\, \frac{F(t)}{t} dt = \int_s^{\infty} \left[\frac{1}{x+a} - \frac{1}{x+b}\right]\, dx$$

$$\begin{aligned} \Rightarrow \int_0^{\infty} e^{-st}\, \frac{e^{-at} - e^{-bt}}{t} dt &= \int_s^{\infty} \left[\frac{1}{x+a} - \frac{1}{x+b}\right]\, dx \\ &= \left[\log \frac{x+a}{x+b}\right]_s^{\infty} \\ &= 0 - \log \frac{s+a}{s+b} \\ &= \log \frac{s+b}{s+a}. \end{aligned}$$

But, we have to evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$. Therefore, we can make as $s = 0$ on both sides, then $e^{-st} = 1$, therefore we obtain

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\frac{b}{a}.$$

So, effectively without solving the integral directly, using the properties of Laplace transform, the integrals can be evaluated easily.

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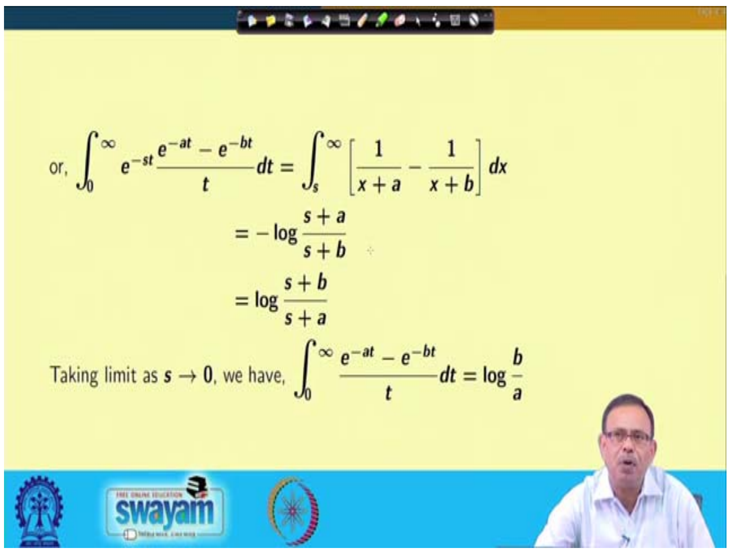

Let us see the next example where we need to find out the value of $\int_0^\infty t\, e^{-3t} \sin t \;\, dt$.

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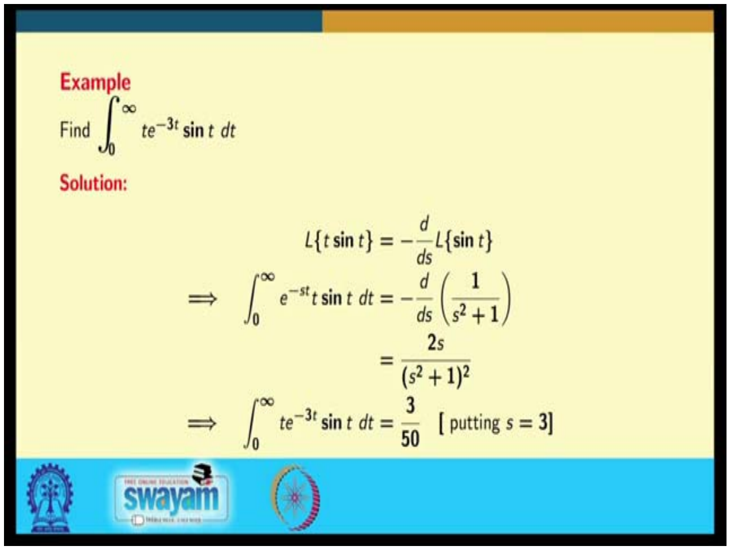

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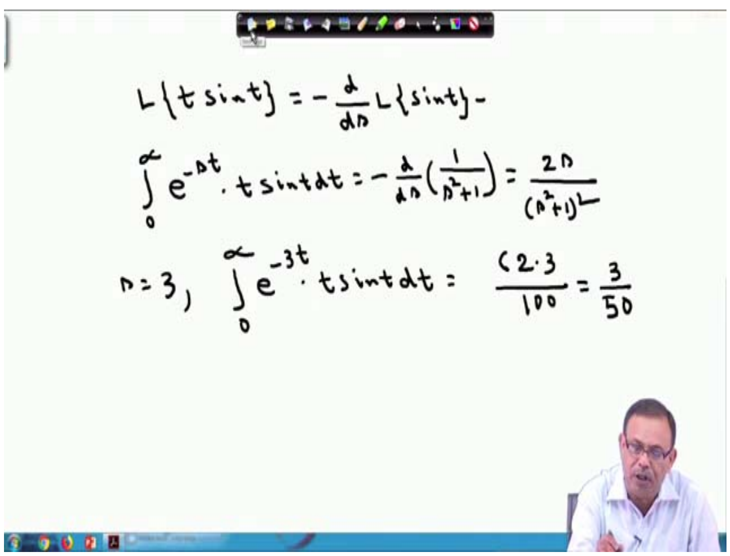

In this case, we know $L\{t \sin t\} = \int_0^\infty e^{-st} t sin\, t\, dt$ (by definition). This is what we need to evaluate but for $s = 3$.

By the property of multiplication by t, we have,

$$L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}. \qquad (3)$$

We know Laplace transform of $\sin t$ is $\frac{1}{s^2+1}$. Therefore,

$$-\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$
$$= \frac{2s}{(s^2+1)^2}.$$

From (3), we have,

$$L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$
$$\Rightarrow \int_0^\infty e^{-st} t\sin t\, dt = \frac{2s}{(s^2+1)^2}.$$

But we have to evaluate the integral $\int_0^\infty t\, e^{-3t}\sin t\ dt$. So, we substitute $s = 3$ in the above equation so that

$$\int_0^\infty t\, e^{-3t}\sin t\ dt = \frac{3}{50}.$$

Let us take the next example that is $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$.

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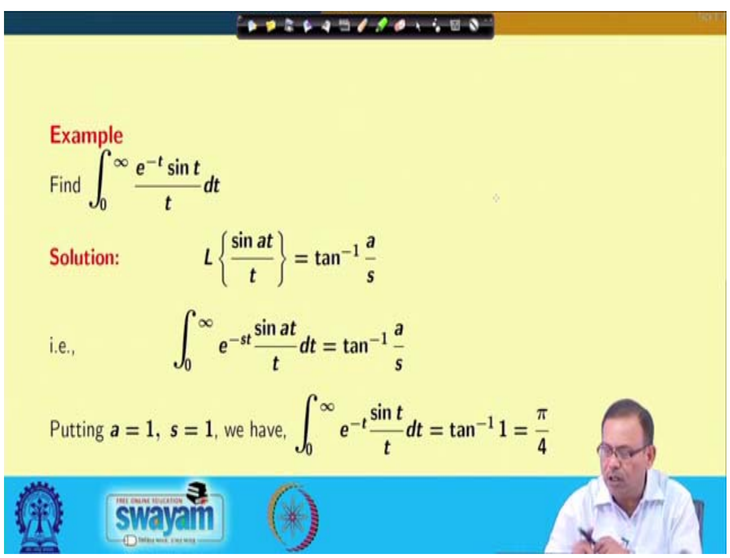

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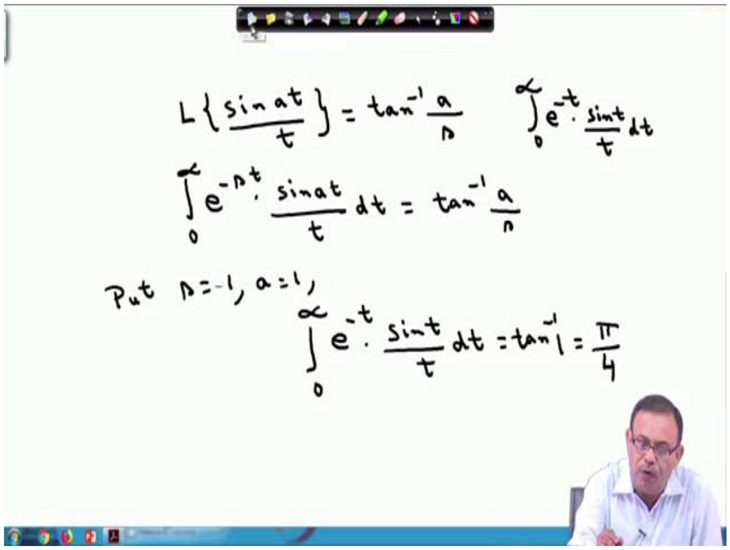

We have already obtained that $L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{s}$. We can write this, using definition as follows:

$$\int_0^\infty e^{-st}\frac{\sin at}{t}dt = \tan^{-1}\frac{a}{s}.$$

Now, we have to find out the value of $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$. So, obviously, we will put $s = 1$, $a = 1$ here to obtain:

$$\int_0^\infty \frac{e^{-t}\sin t}{t}dt = \tan^{-1} 1 = \frac{\pi}{4}.$$

Now, let us take another example where we need to find Laplace transform of $\int_0^t \frac{\sin x}{x} dx$.

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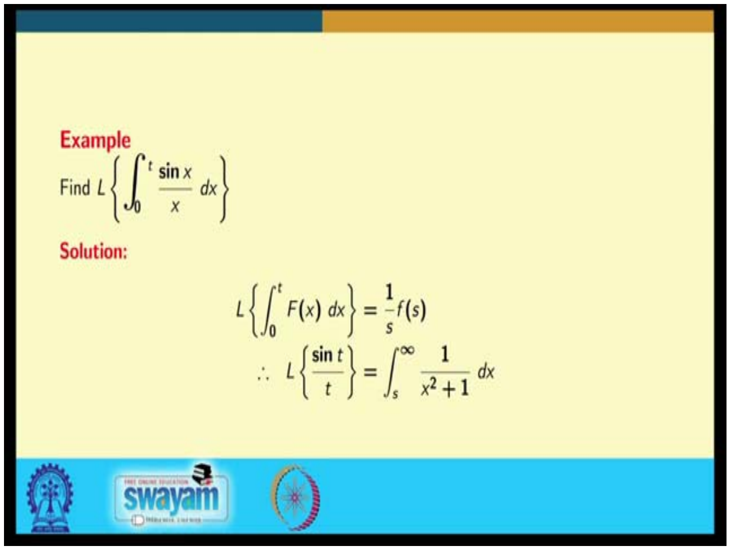

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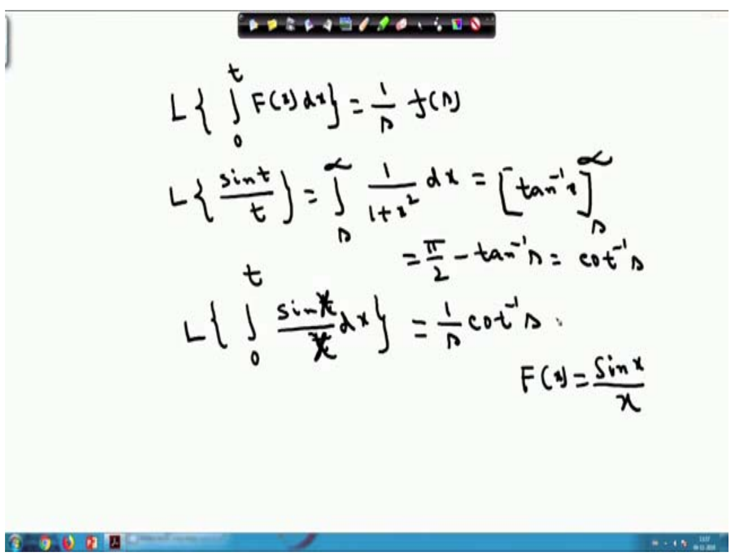

From the Laplace Transform of integral of a function as discussed in previous lectures, we know, $L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s} f(s)$, where $f(s) = L\{F(t)\}$. In this case, our $F(t) = \frac{\sin t}{t}$. As already obtained, from (1), we can write,

$$L\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$$

So, $f(s) = L\{F(t)\} = \cot^{-1} s$. Therefore,

$$L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s} f(s)$$

$$\Rightarrow L\left\{\int_0^t \frac{\sin x}{x} dx\right\} = \frac{1}{s} \cot^{-1} s.$$

Again we see how easily we are finding out the solution using the properties of Laplace transform.

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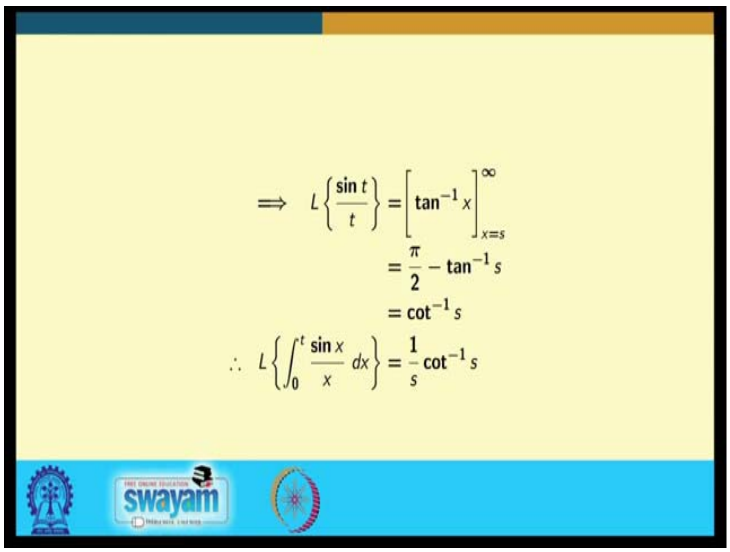

Now, we move to the next problem. To solve $\int_0^\infty t^3 \, e^{-t} \sin t \, dt$.

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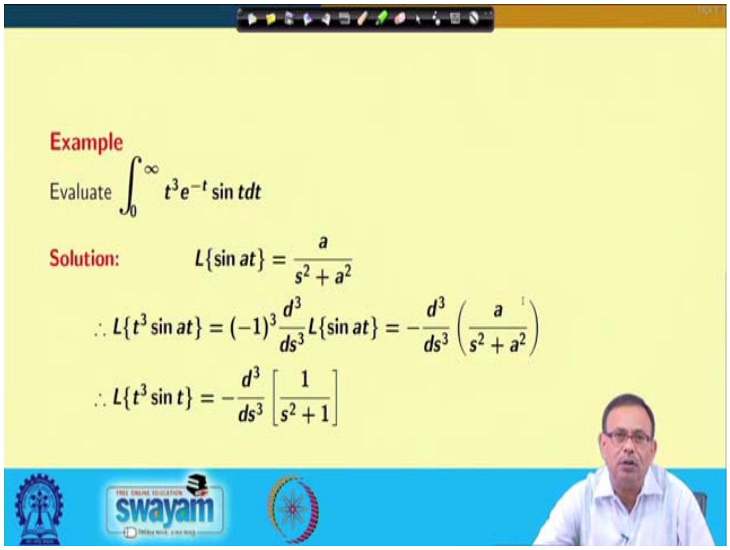

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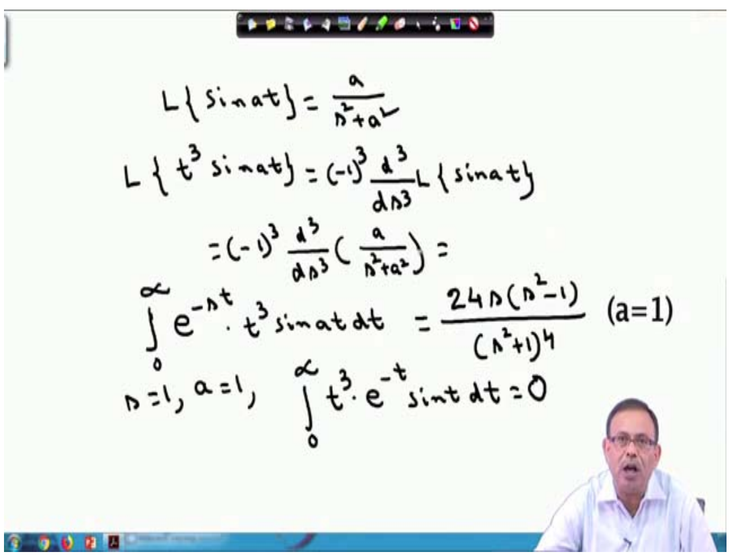

In this case, we start with Laplace transform of $\sin t$ as we know already

$$L\{\sin t\} = \frac{1}{s^2+1}.$$

Using the formula for multiplication by powers of t, we can write,

$$L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} L\{\sin t\}$$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right).$$

So, we have to differentiate it thrice to obtain the result as follows:

$$L\{t^3 \sin t\} = \frac{24s(s^2-1)}{(s^2+1)^4}.$$

Using definition of Laplace transform,

$$\int_0^\infty e^{-st} t^3 \sin t \ dt = \frac{24s(s^2-1)}{(s^2+1)^4}.$$

We have to find out the value of $\int_0^\infty t^3 \ e^{-t} \sin t \ dt$. So, we put $s = 1$ on both sides of the above equation to obtain

$$\int_0^\infty e^{-t} t^3 \sin t \ dt = 0.$$

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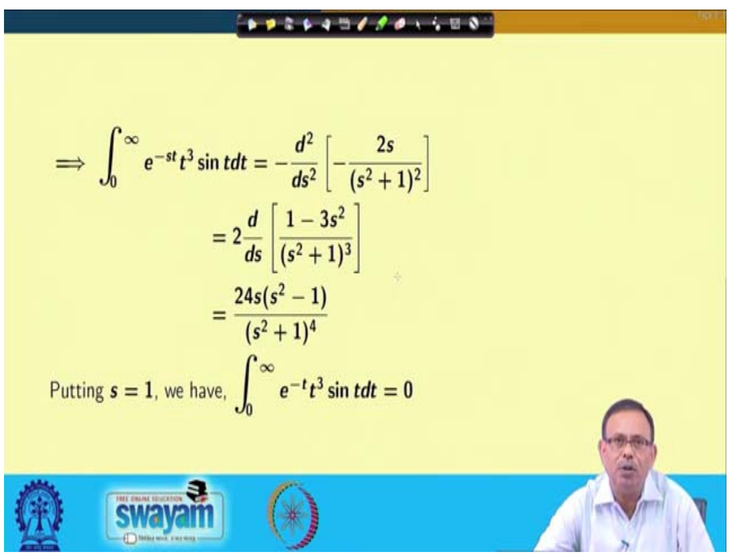

Let us take the next example.

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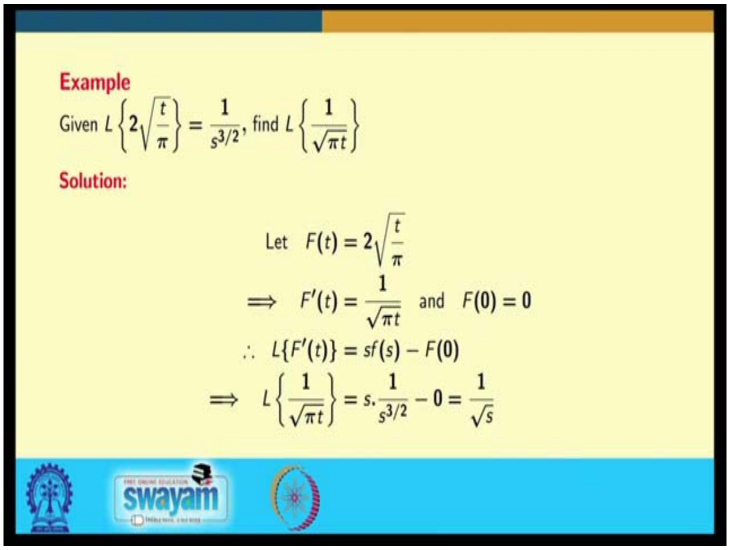

It is given that $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$. We have to find out $L\left\{\frac{1}{\sqrt{\pi t}}\right\}$.

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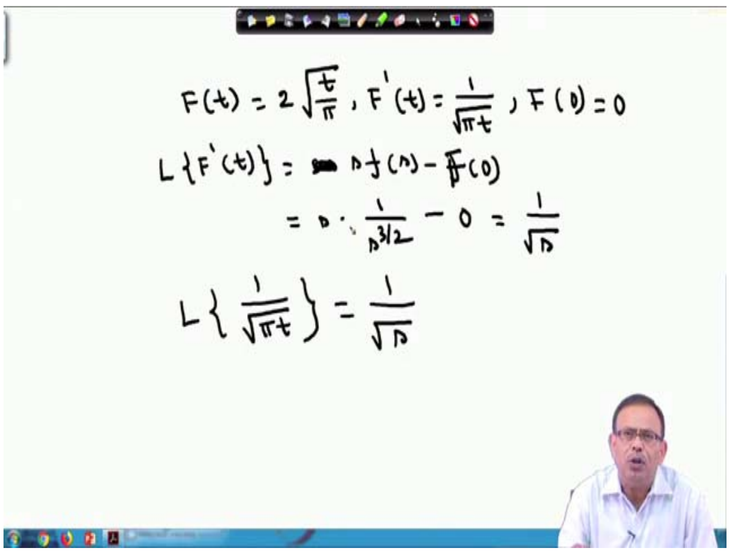

We assume $F(t) = 2\sqrt{\frac{t}{\pi}}$ which on differentiation gives $F'(t) = \frac{1}{\sqrt{\pi t}}$ and also clearly, we have $F(0) = 0$.

Therefore, by the given condition, we see that $L\{F(t)\} = L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ is known to us.

And our aim is to obtain $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = L\{F'(t)\}$. Therefore, using the Laplace transform of derivative of a function, we have

$$L\{F'(t)\} = s\,f(s) - F(0), \quad \text{where}, f(s) = L\{F(t)\}$$

$$\Rightarrow L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s \cdot \frac{1}{s^{3/2}} - 0$$

$$= \frac{1}{\sqrt{s}}.$$

We now move to the next example where we need to find $L\{H(t)\}$ and $L\{H'(t)\}$ where $H(t)$ is given by,

$$H\{t\} = \begin{cases} t+1\ , & 0 \le t \le 2 \\ 3\ , & t > 2 \end{cases}$$

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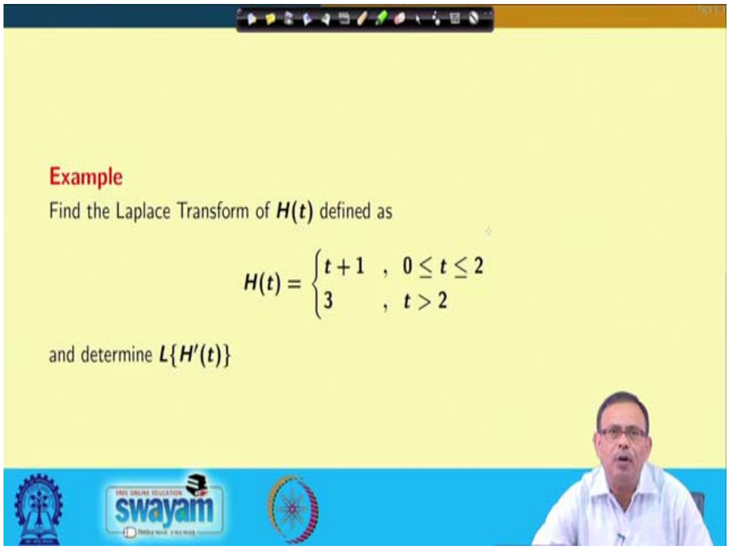

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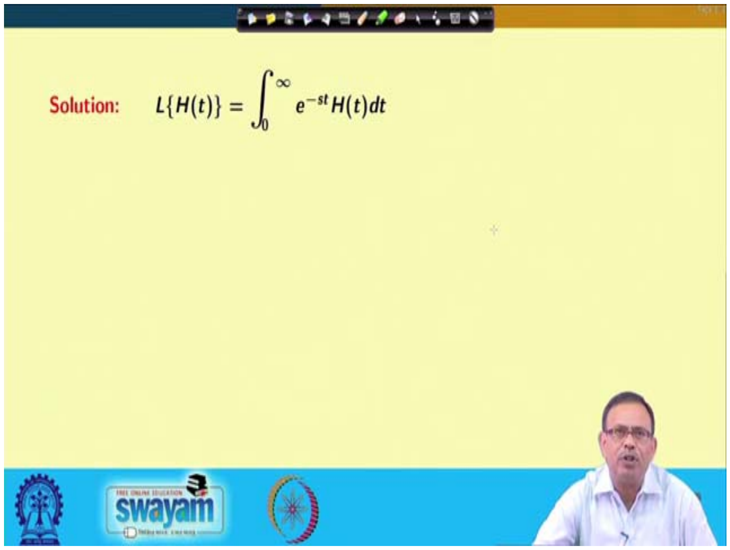

So, we start with Laplace transform of $H(t)$, then only we can go for the Laplace transform of $H'(t)$.

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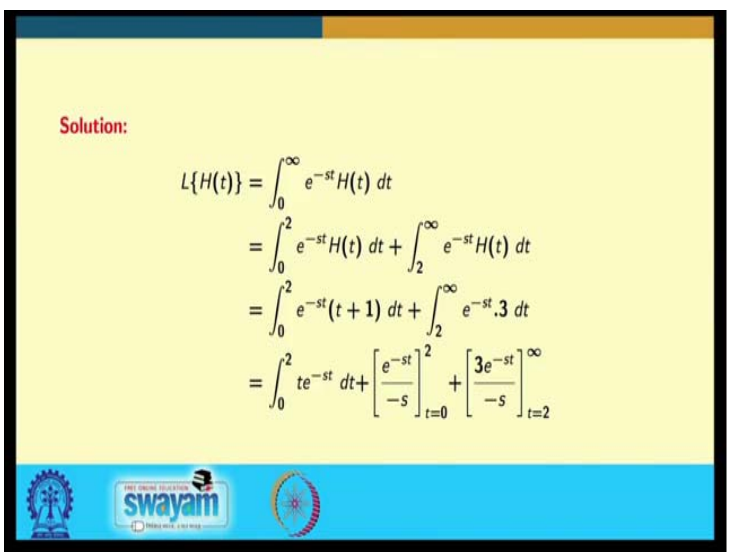

We have from definition of Laplace Transform,

$$L\{H(t)\} = \int_0^\infty e^{-st} H(t) dt$$

$$= \int_0^2 e^{-st} H(t) dt + \int_2^\infty e^{-st} H(t) dt.$$

In the first part, value of the function $H(t)$ is $(t+1)$ whereas, in the second part, the value of the function is equal to 3. So, if we evaluate the integral, we will obtain

$$L\{H(t)\} = \int_0^2 e^{-st}(t+1)dt + \int_2^\infty 3e^{-st}dt.$$

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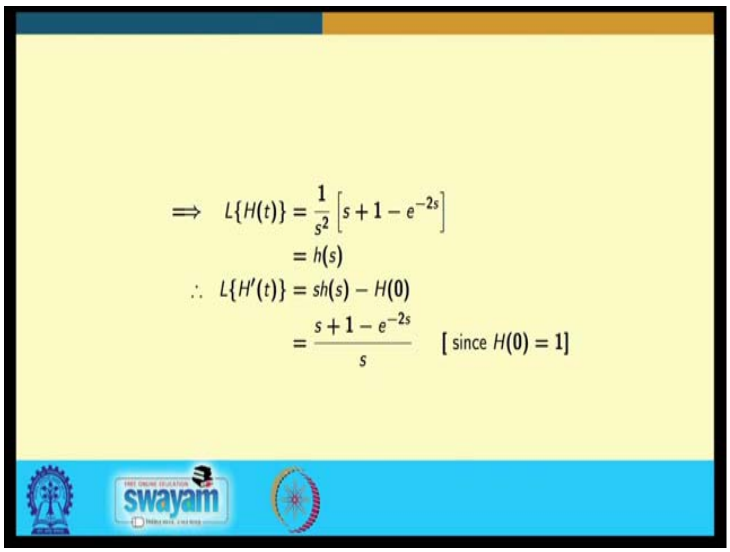

The above integrals can be easily evaluated to obtain the following result:

$$L\{H(t)\} = \frac{1}{s^2}(s+1-e^{-2s}).$$

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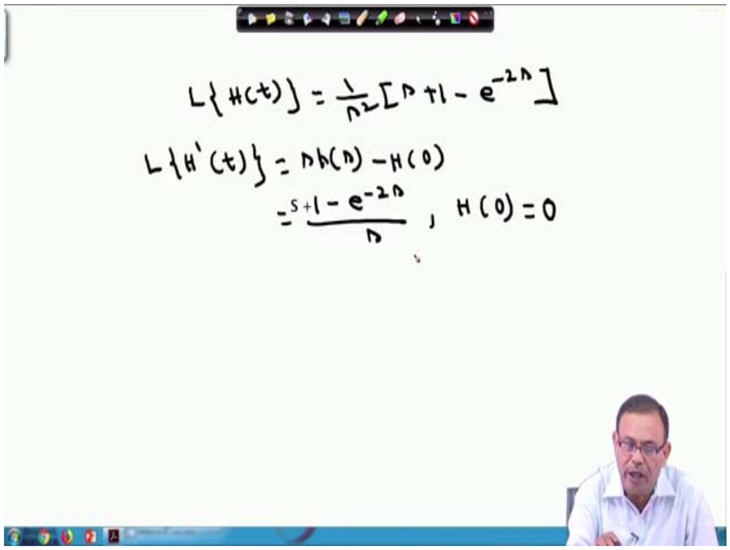

Now we need to evaluate Laplace transform of $H'(t)$. By the property of Laplace transform of derivative of a function,

$$L\{H'(t)\} = sL\{H(t)\} - H(0)$$
$$= \frac{1 - e^{-2s}}{s}, \quad [\because H(0) = 1].$$

In the next example, we have to evaluate $L\left\{\int_0^t \frac{1-e^{-2x}}{x}\, dx\right\}$.

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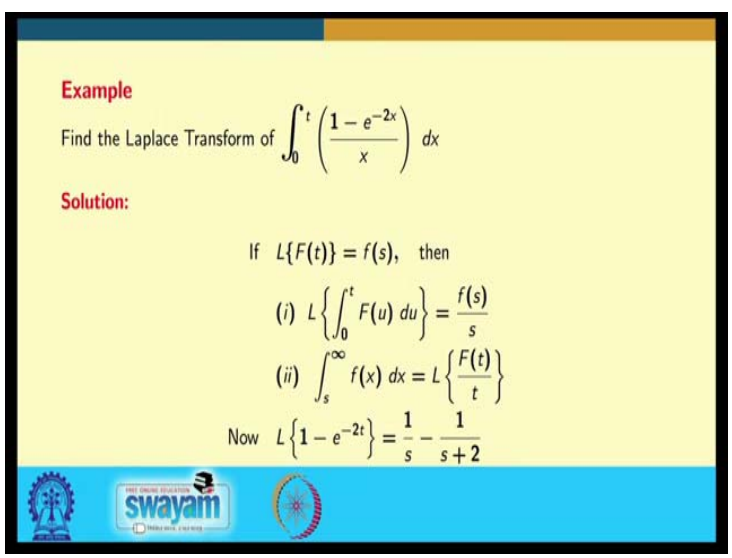

If Laplace transform of $F(t)$ is $f(s)$, then we know the following two formulas which we are going to use:

$$L\left\{\int_0^t F(u)\ du\right\} = \frac{f(s)}{s} \quad \text{and}$$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\ dx.$$

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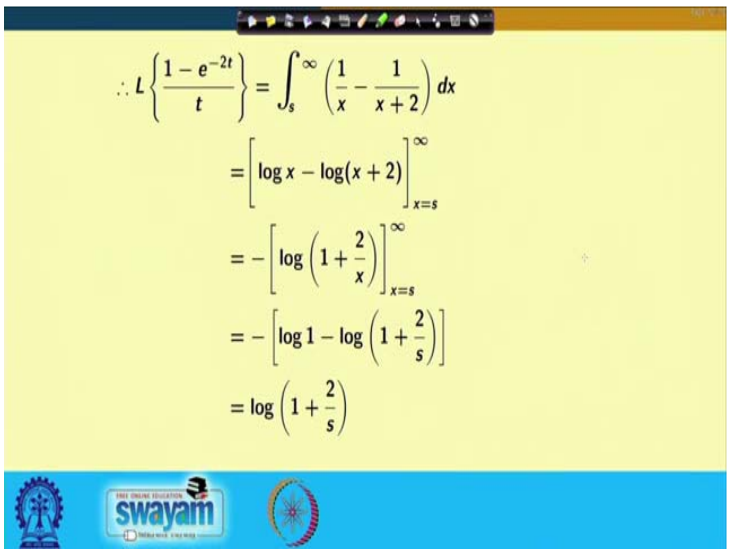

We assume $F(t) = 1 - e^{-2t}$. Therefore, $L\{F(t)\} = L\{1 - e^{-2t}\} = \frac{1}{s} - \frac{1}{s+2} = f(s)$.

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\ dx$$

$$\Rightarrow L\left\{\frac{1-e^{-2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{x} - \frac{1}{x+2}\right) dx$$

$$= \log\left(1 + \frac{2}{s}\right) \qquad \text{(after simplification)}$$

$$\therefore L\left\{\int_0^t \frac{1-e^{-2x}}{x}\ dx\right\} = \frac{1}{s} L\left\{\frac{1-e^{-2t}}{t}\right\} \qquad \text{(using formula for LT of integral)}$$

$$= \frac{1}{s}\log\left(1 + \frac{2}{s}\right)$$

So, using the properties, we can find out the Laplace transform of a function or evaluate an integral very easily. In the next lectures, we will go through some more properties and their applications. Thank you.