Transform Calculus and its Applications in Differential Equations
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Lecture - 59
Properties of Z - Transform

So, if you remember in the last lecture we started the Z-transform, we give the definition of Z-transform and inverse Z-transform, and some simple application, some simple examples we have done. In this lecture, we will start with the properties of Z-transform various properties and after that we will see how to find out the Z-transform of various functions using some examples.

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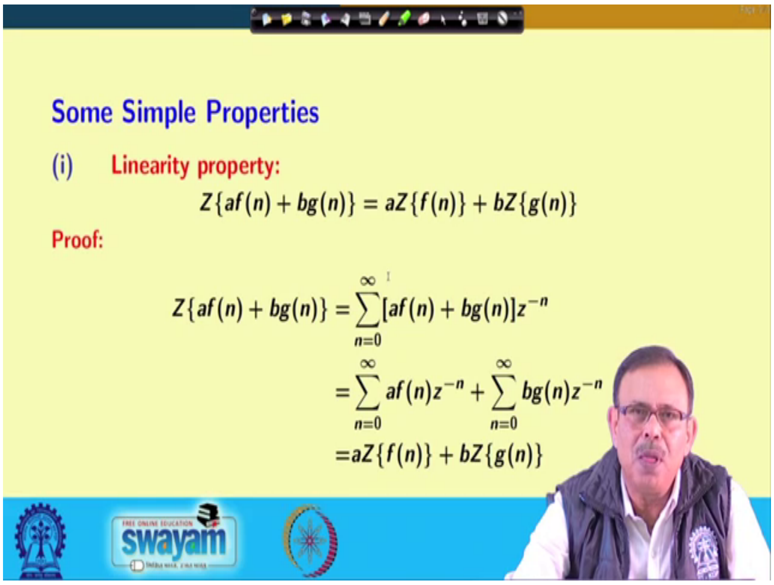

So, let us see the first property that is linearity property. Z-transform of a f n plus b g n equals a into Z-transform of f n plus b into Z-transform of g n.

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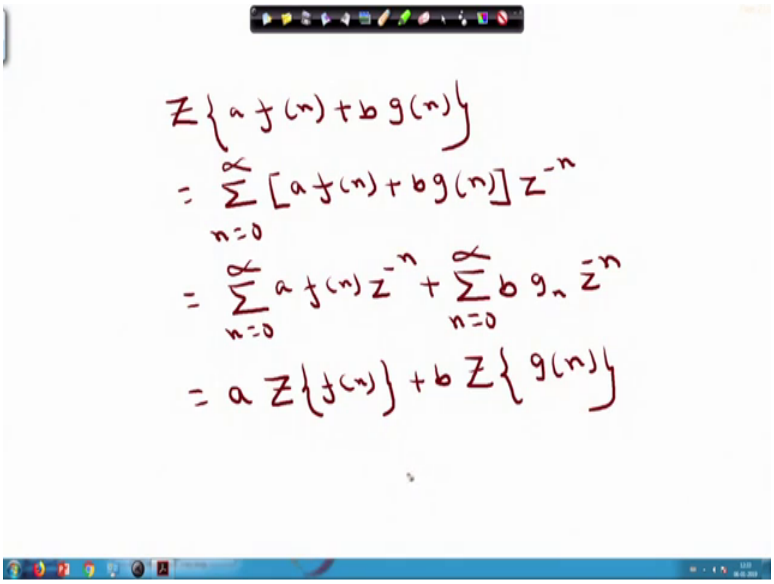

The proof is similar as we have done earlier. Z-transform of a f n plus b g n, this is equals, this is capital Z, this equals you can write down from the definition summation n equals 0 to infinity the function itself, the function here is a f n plus b g n into this z to the power minus n.

So, simply I can break it as into two parts summation n equals 0 to infinity a f n into z to the power minus n plus summation n equals 0 to infinity summation n equals 0 to infinity b g n into z to the power minus of n. And this is nothing but a into summation n equals 0 to infinity f n z to the power minus n, it is the Z-transform of f n plus b can come outside, therefore this will be equals to b into Z-transform of g of n And this completes the proof that the linearity property Z-transform of a f n plus b g n is equals to a into Z-transform of f n plus b into Z-transform of g n.

Let us see it quickly here. So, Z-transform of a f n plus b g n using the definition of Z-transform summation n equals 0 to infinity the function that is a f n plus b g n in to z to the power minus n. So, I can break it into two parts. And the first part is nothing but a into Z-transform of f n plus b into Z-transform of g of n.

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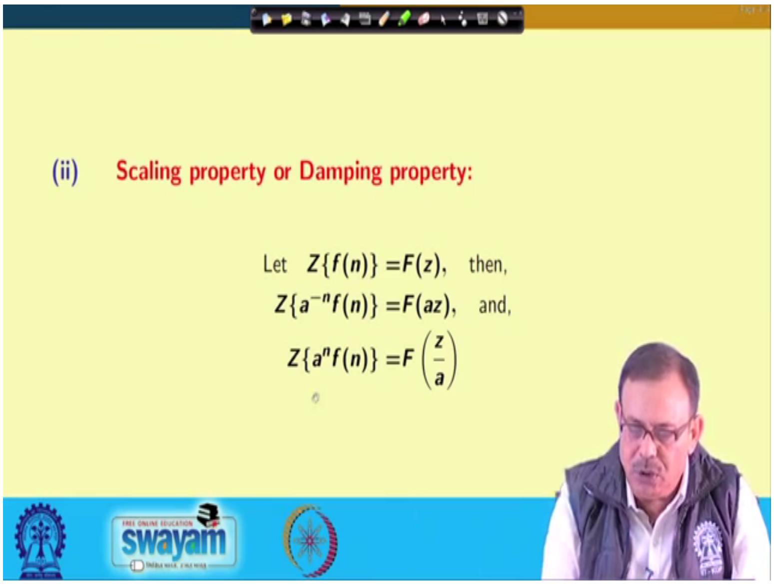

Next property is scaling property or damping property. Let f of z of f n that is Z-transform of f n if it is equals to F z say, in that case Z-transform of a to the power minus a n f n this is equals to F of a z.

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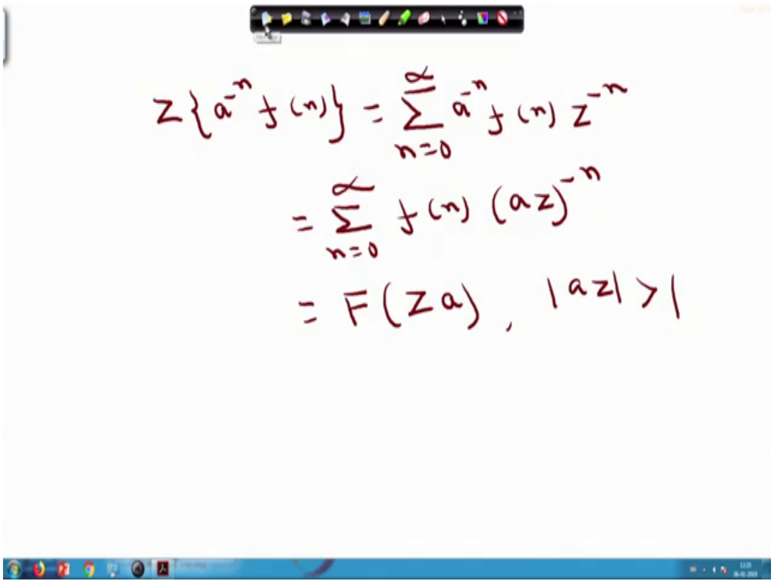

So, to prove this one, we are starting from the left hand side that is Z-transform of a to the power minus n into f n, this is equals to you can write down from the definition summation n equals 0 to infinity a to the power minus n f n into z to the power minus n. The function is a to the power minus n into f n into z to the power minus n. So, this is equals n equals 0 to infinity, I can write down f n into a z to the power minus n this a and z, I can club it together and this will be equals to a z to the power minus n. And therefore, your function is z is replaced by a z, so that I can write down that the F of z a obviously, for convergence modulus of a z should be greater than 1.

Therefore, we are saying that the a capital F of z a where capital F is the Z-transform of the function. Therefore, the proof is complete that is Z-transform of a to the power minus n f n equals the F of z a, where F of z is the Z-transform of the function f n.

The next one is Z-transform of a to the power n f n equals F of capital F of z by a, where Z-transform of f n is F of z.

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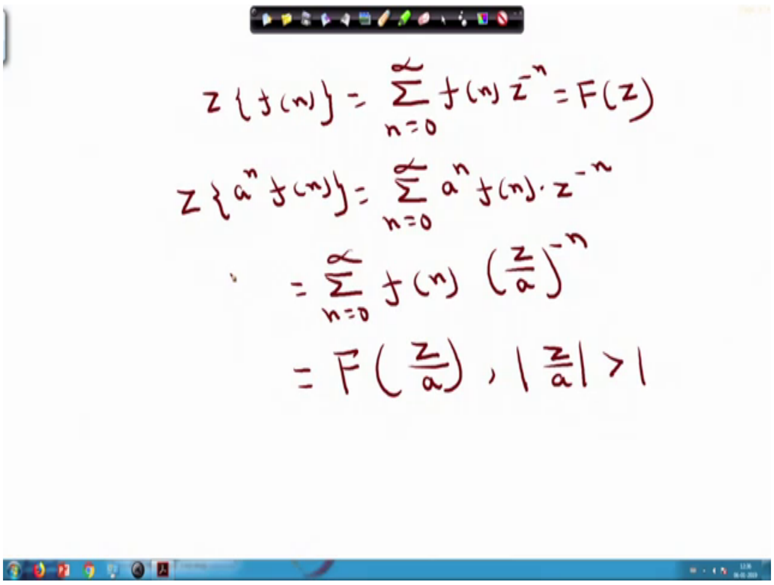

So, now let us prove the next part that is this one. So, it is given that Z-transform of f n, we know it this is equals nothing but n equals summation n equals 0 to infinity f n into z to the power minus n, and this we have assumed as capital F of Z. Now, Z-transform of a to the power n f n this is equals to summation n equals 0 to infinity a n f n function is a n f n in to the power minus n. And this equals you can write down summation n equals 0 to infinity f n into z by a to the power minus n z by a to the power minus n a to the power minus n, so that it will become a to the power n.

And this is again from the definition of the Z-transform, this I can write down capital F of z by a, so that where your where this modulus of z by a should be greater than 1 for convergence purpose. So, this completes our proof that Z-transform of a to the power n f n this is equals capital F of z by a, so this completes the proof. And this is the scaling property or the damping property.

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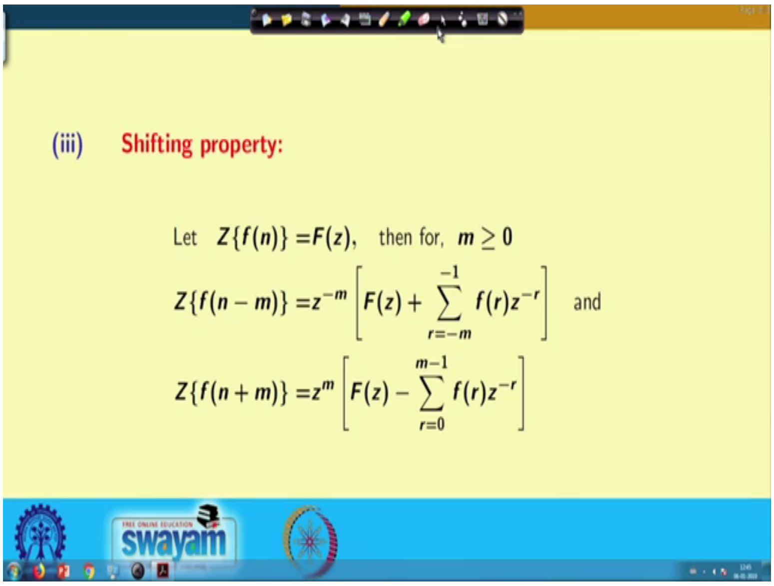

The next property is the shifting property that is if Z n Z-transform of f n equals F z, then we can solve these things. Z-transform of f of n minus m will be equals to z to the power minus m into capital F z plus summation r equals minus 1 minus m to minus 1 f r z to the power minus r. And capital Z of Z-transform of f of n plus m the earlier one was n minus m, and Z-transform of f of n plus m will be equal to z to the power m into F z minus summation r equals 0 to m minus 1 f r in to z to the power minus r.

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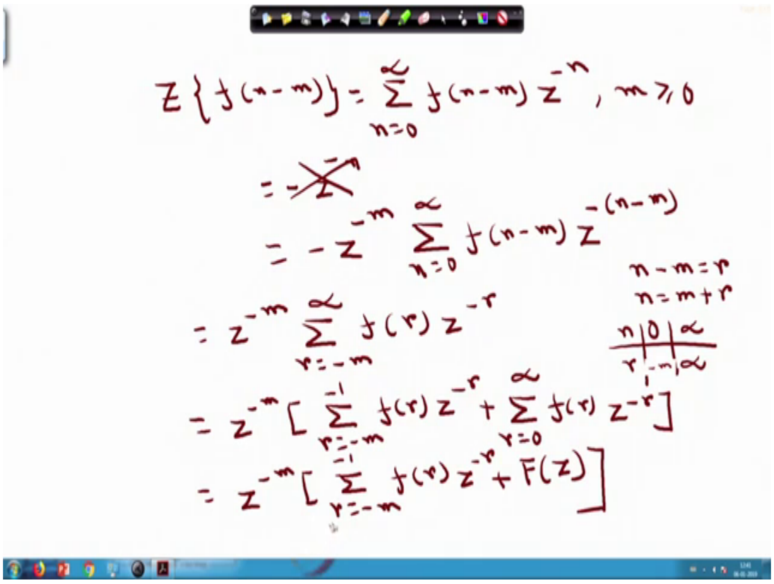

Let us see the proof of this thing. For the first part Z-transform of f of n minus m, this is equals to summation n equals 0 to infinity f of n minus m into z to the power minus n, z is greater than sorry m is greater than equals 0, this we are assuming, m is greater than equals 0. So, this equals you can write down minus z to the power minus n sorry instead of this minus z to the power minus m I can make it here. So, that summation n equals 0 to infinity function of n minus m into z to the power minus n is there since I have made it here. So, n minus m so that this z to the power minus m will be cancelled.

Now, once I am writing this, let us assume that n minus m this is equals to say r, so that your n is m plus r. So, you are n and the r if you take whenever n is 0; yourr will be 0; when n is infinity from here your r is infinity, so that the summation over r will be from 0 to infinity only. So, this equals I can write down z to the power minus m into summation r equals minus m to infinity f of r n minus m is r into z to the power minus r.

And this equals I can write down this equals z to the power minus m into summation r equals, I can break it into two parts minus m to minus 1 minus m to minus 1 f of r z to the power minus r plus r equals 0 to infinity summation r equals 0 to infinity f of r into z to the power minus r. So, this equals I can write down z to the power minus m into summation r equals minus m to minus 1 this will remain as it is that is f of r z to the power minus r plus this part r equals 0 to infinity f of r z to the power minus r this is actually the Z-transform of r.

So, this I can write down capital F of z right. Z-transform of the function f n is capital F Z. So, this part is capital F Z. And this part is r equals minus m to minus 1 f of r z to the power minus r. So, this completes the proof of the first part that is Z-transform of f of n minus m, this is equals to the z to the power minus m into summation r equals minus m to minus 1 f of r z to the power minus r plus capital F of Z.

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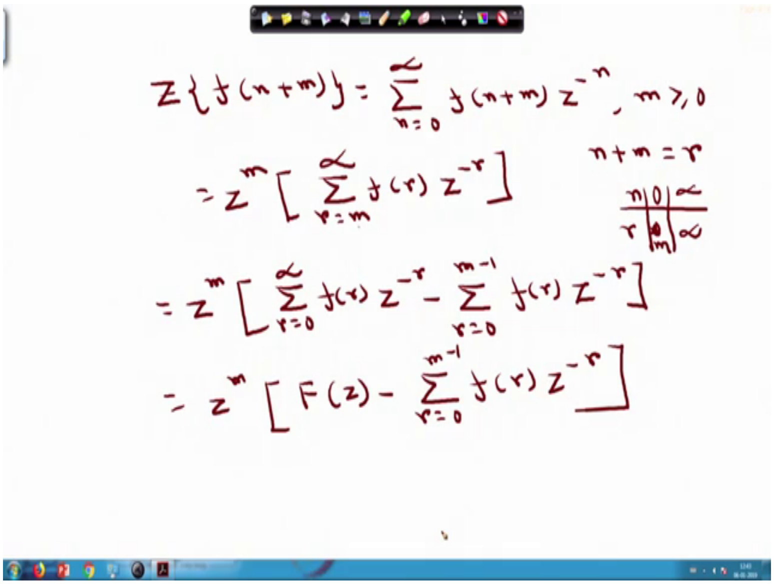

Now, let us see the proof of the second part that is Z-transform of f of n plus m. We will follow the similar steps whatever we have done earlier. So, summation over n equals 0 to infinity; f of n plus m z to the power minus n, m is greater than equals 0 we can write down. Again following the same process, I am not repeating. We can assume that n plus m equals to r. And this instead of z to the power minus m, here it will be z to the power m into summation over r equals this will become m to infinity f of r z to the power minus r.

Now, here since I have taken n plus m equals r. So, whenever n is 0 your r is 0 and whenever you are n sorry whenever n is 0, your value of r is not 0, but it is m; and whenever n is infinity your r is infinity, so that summation over r is from here from m to infinity. And this I can write it as z to the power m into summation r equals 0 to infinity f of r z to the power minus r minus summation r equals 0 to m minus 1 f of r z to the power minus r.

The first part is nothing but the capital F of z this r equals 0 to infinity f r in to z to the power minus r, this is F of z minus r equals 0 to m minus 1 F z plus f of r in to the power minus r. So, this completes the proof that Z-transform of f of n plus m equals z to the power m into capital F of z minus summation r equals 0 to m minus 1 f of r z to the power minus r.

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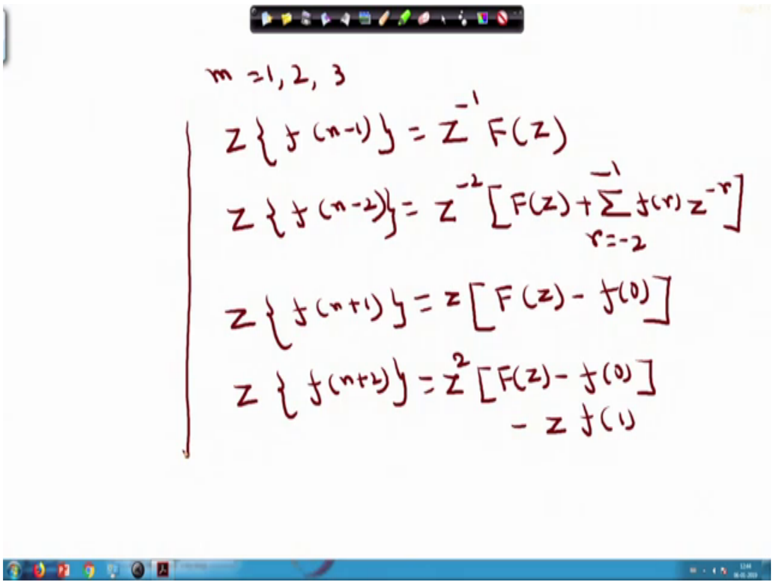

For a particular cases, if on these values, if I take m equals 1, 2 and 3, then from here I can write down Z-transform of f of n minus 1, this is equals z to the power minus 1 capital F of z. Similarly, Z-transform of f of n minus 2, this will be equals to z to the power minus 2 into F of z plus summation r equals minus 2 to minus 1 f of r into z to the power minus r. And similarly z of f of n plus 1 this will be equals to z into capital F z capital F z minus f 0.

And z of f of n plus 2 simply I am putting substituting on the given earlier results z squared into capital F z minus f 0 minus z of z into f of 1. Please note that this particular formulas, these are very useful whenever we will try to solve the problems. So, please note this particular formulas where Z-transform of f of n minus 1 z to the power minus 1 capital F of z like this way, we can write down Z-transform of f of n minus 2, Z-transform of f of n plus 1, and Z-transform of f of n plus 2, this will be useful whenever we will try to solve the problems.

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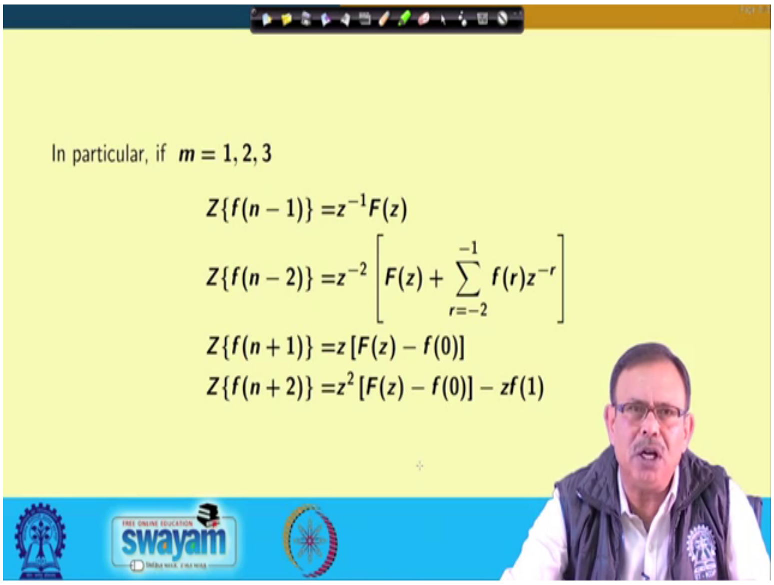

Now, so this we have proved. And from here we have given. In particular, these values are true what I told you just now these values I can simply. If I substitute m equals 1, 2, 3, I will obtain Z-transform of f of n minus 1, Z-transform of f of n minus 2, Z-transform of f of n plus 1 and transform of f of n plus 2.

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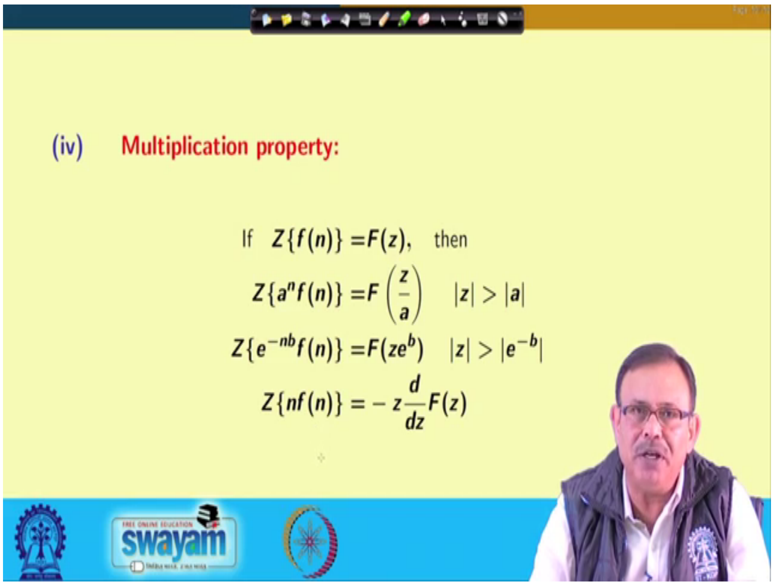

Now, multiplication property, if the Z-transform of f n equals F z, then Z-transform of a to the power n f n, this is equals F of z minus a. Let us see the proof of this one.

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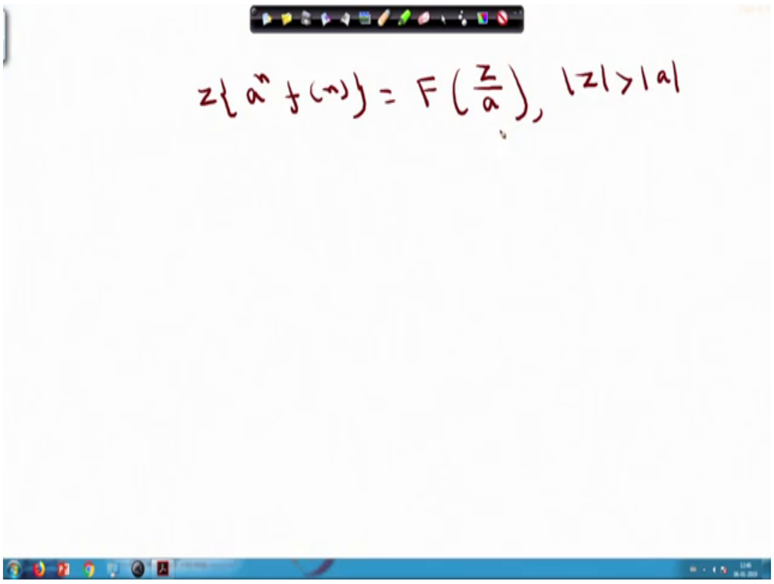

Your Z-transform of this thing already if you remember z to the power Z-transform of a to the power n f n this is equals capital F of z by a, where modulus of z greater than modulus of a. This already we have proved in the scaling property. So, Z-transform of a to the power n f n equals capital F of z by a. So, the first part is over. For the second part is Z-transform of e to the power minus n b f of n equals capital F of z of e power b, where modulus z is greater than e power minus b.

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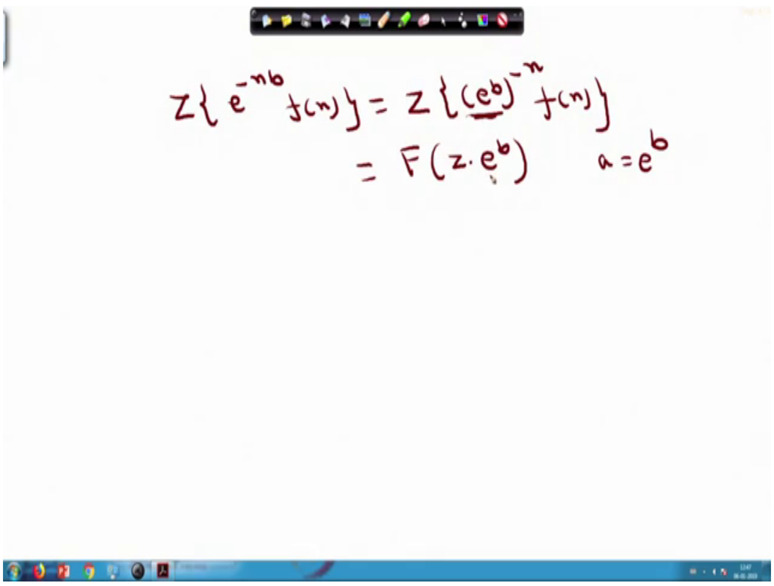

To prove this thing; we have the Z-transform left hand side we are considering Z-transform of e power minus n b into f of n, this is equal Z-transform of this I can write down e power b to the power minus n into f of n. And this is equals nothing but capital F of z into e power b. Basically by scaling property, we can write down this thing, here we are assuming a equals e power b, this is your a. So, a power minus n, therefore, using scaling property I can show that Z-transform of e power minus n b into f n this equals capital F of z into e of; z into e power b.

The next one is the third one on this is Z-transform of n into f n if you multiply by n here. In the first case you multiplied the f n by a to the power n; in the second case, you multiplied f n by e power minus n b. And in the third case we are multiplying n by f n, so that Z-transform of n f n equals minus z d dz of capital F z.

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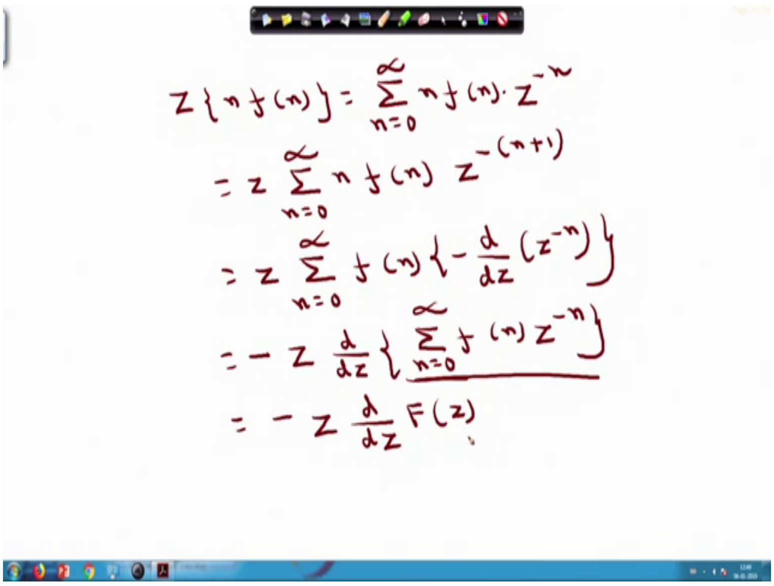

Let us see the proof of this one. Z-transform of n into f n, we are starting from here from the definition object transform, I can write down summation n equals 0 to infinity n f n into z to the power minus n. This equals I can write down I can multiply z here and summation over n equals 0 to infinity n f n into z to the power minus n plus 1, so that z to the power minus 1 and I multiply it by z. So, this equals I can write down z equals summation n equals 0 to infinity f n into minus d dz of z to the power minus n minus d dz of z to the power minus n if I make it then this term will come back to us.

So, this equals I can simply this equals I can write down minus z can come out d dz of summation n equals 0 to infinity f n z to the power minus n. Here please note that f n is independent of z, so that I can bring this d dz outside. So, this equals I can write down minus z into d dz of summation n equals 0 to infinity f n z to the power minus n. And this equals minus z d dz and this term this is nothing but capital F z this already we have done it. So, therefore from here, we can write down that Z-transform of n f n this is equals minus z into d z of F z which completes the proof.

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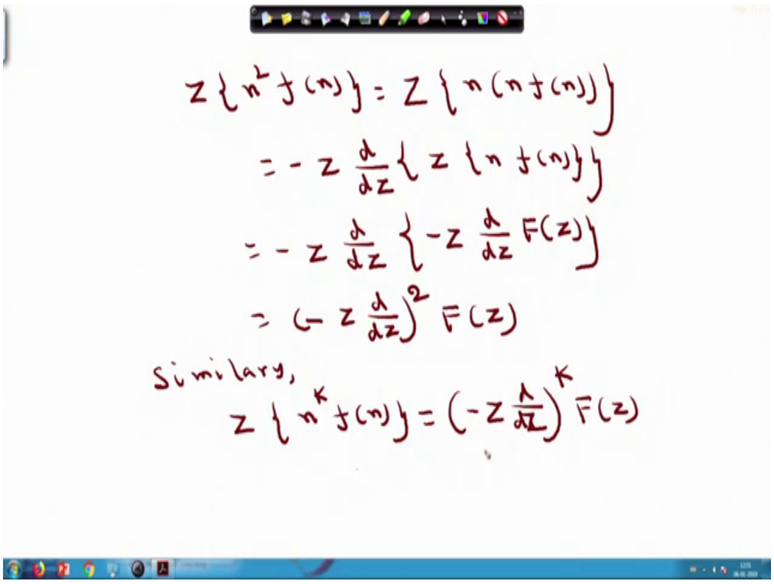

Similarly, on the same way if I take Z-transform of n square into f n Z-transform of n, so this equals I can write down n into n f n, so that now your f n is n f n. So, if I use the earlier result, then this equals I can write down minus z into the d d z of z into this z into n f n this equals minus z d dz. This I can write down as minus z d dz of F z again, again I am using the earlier property just whatever we have proved.

So, this equals minus z into d d z whole square of F z. And in the same fashion if I continue, similarly we can say that Z-transform of n to the power k into f n this will be equals to minus z dz to the power k into capital of F z. So, therefore similarly we are writing Z-transform of n to the power k into f n equals minus z d dz to the power k into F z.

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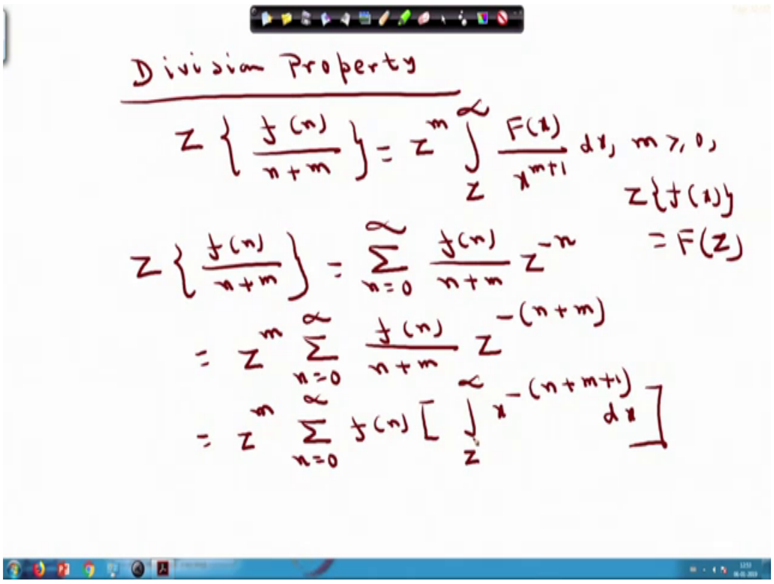

Now, there is another property, which I call as division property, we call it as division property. Division property says us Z-transform of f n by n plus m, this is equals to z to the power m into z to infinity F x capital F already we have discussed by x to the power m plus 1 into dx, where m is greater than equals 0 and Z-transform of f x equals or f n equals F of z. So, you have to prove that Z-transform of f n by n plus m that is you are dividing now the function in the earlier cases we are multiplying the with the function, here we are dividing the function by n plus m. And this will be equals to z to the power m integral over z to infinity F x by x to the power m plus 1 into dx.

To prove this one, let us start with the left hand side Z-transform of f n by n plus m, this is equals from definition summation n equals 0 to infinity f n by n plus m into z to the power minus n. So, here again I can write it z to the power m I can multiply here n equals 0 to infinity f n by n plus m, so that it becomes z to the power minus of n plus m.

So, this I can write down again z to the power m summation n equals 0 to infinity f n into this in terms of integral, I can write it z to infinity x to the power minus n plus m plus 1 into dx. So, please note this one if I evaluate the integral I will simply obtain the z to the power minus of n plus m divided by n plus m, so that I am writing this integral it I am replacing this z to the power minus n plus m by n plus m by this given integral. And this again in the let see in the next slide.

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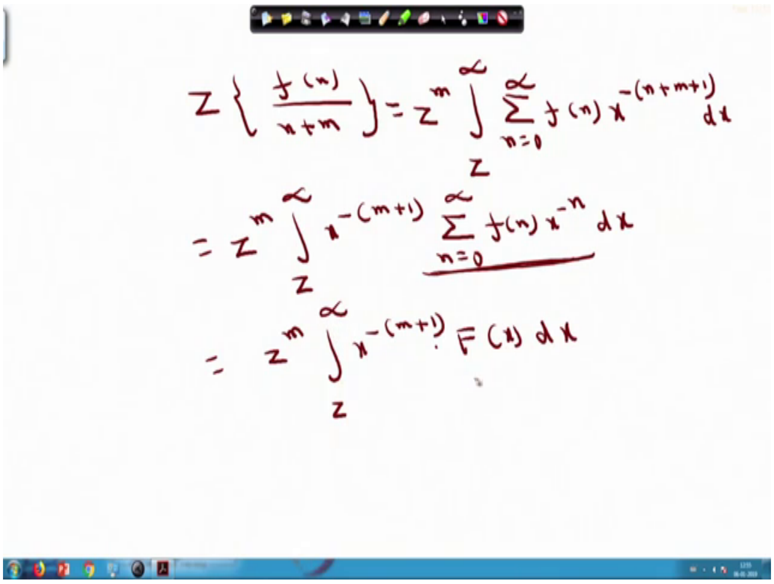

So, Z-transform of f n by n plus m, this is equals to z to the power m into z to infinity the earlier integral I can write down just I am interchanging the summation and integration n equals 0 to infinity f n into x to the power minus n plus m plus 1 into dx. And this equals again you can write down z to the power m into z to infinity x to the power minus m plus 1, I can bring outside here. And this I can write down summation n equals 0 to infinity f n into x to the power minus n into dx.

This equals z to the power m z to infinity x to the power minus m plus 1 that is x to the power minus m plus 1 into this is nothing but your capital F x from the definition, so this is equals this. Therefore, Z-transform of f n by n plus m this equals to the z to the power m into summation z to infinity F x by x to the power m plus 1 into dx. So, this completes the proof of the division property.

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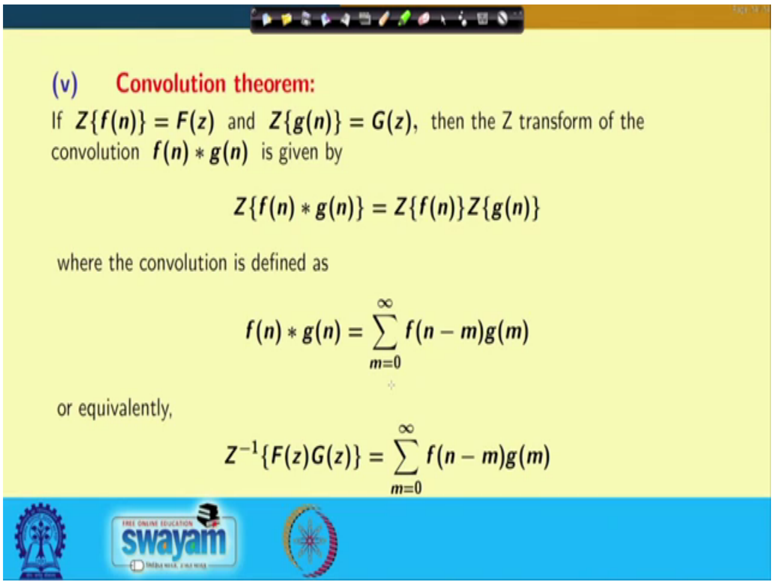

Next is the convolution theorem. If Z-transform of f n equals F z and Z-transform of g n equals capital G z, then g transform of convolution of two functions f n and g n is given by Z-transform of f n convolution g n equals Z-transform of f n into Z-transform of g n; where the convolution is defined like this of two functions f n convolution g n equals summation m equals 0 to infinity f of n minus m into g m.

And so this is basically keeping in parity with the Laplace transform, Fourier transform and the Mellin transform for the convolution also it is following the same thing. And equivalently we can say that Z inverse F z capital F z into capital G z this is equals summation m equals 0 to infinity f of n minus m into g of m.

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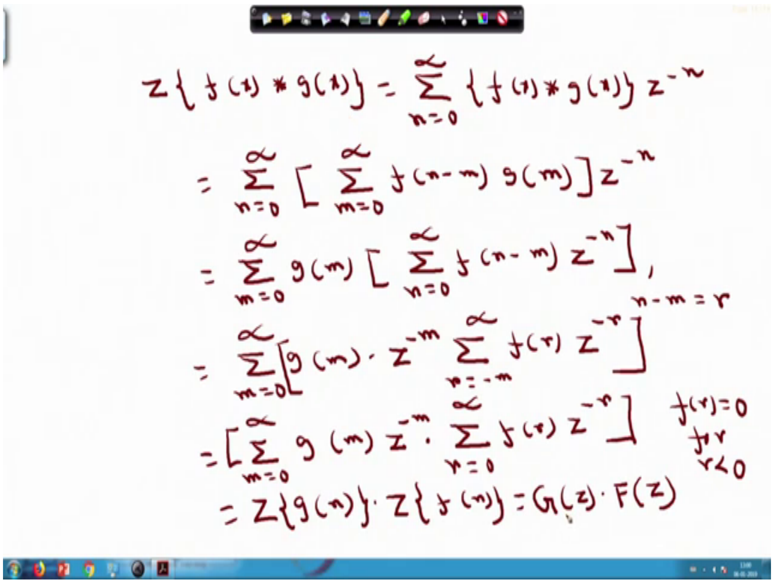

Quickly let us see the proof of this one convolution. Z-transform of f x star g x this is equals to from the definition; summation n equals 0 to infinity f x star g x into z to the power minus n. This equals summation n equals 0 to infinity. I am substituting the convolution definition from the convolution definitions m equals 0 to infinity f of n minus m into g of m into g to the power minus n.

This equals now interchanging the order of the summation, this equals I can write down summation m equals 0 to infinity g m into summation n equals 0 to infinity f of n minus m into z to the power minus n. Now here you substitute n minus m equals r say. If I substitute, this n minus m equals r in that case this will become m equals 0 to infinity g m into this I can put it entire thing within third bracket z to the power minus m will come here, n is replaced by r plus m. So, z to the power minus m will come, then summation r equals minus m to infinity f of r into z to the power minus r.

So, once I am getting this, this n has been replaced by r plus m. So, one I am getting z to the power minus m; another one I am getting z to the power minus r. So, this equals I can write down summation m equals 0 to infinity g m z to the power minus m will be as it is z to the power minus m will be as it is. This I can write down as r equals 0 to infinity f r into z to the power minus r.

Please note that this is; we have this f r is 0 for r less than 0. Since f r is 0 for r less than 0, therefore, r equals minus m to infinity can be replaced by r equals 0 to infinity. So, I can break it into two parts very simply, one is summation m equals 0 to infinity g m in to z to the power minus m, which is nothing but your Z-transform of g of n directly we can write down from the definition.

And similarly for the second part summation r equals 0 to infinity f r into z minus r; this I can write down Z-transform of f n. So, therefore, I am getting Z-transform of g n is capital G z in capital G of z into Z-transform of f n that is equals to capital F of z. And this completes the proof that Z-transform of f x star g x equals g z into capital F z. So, in the next class, we will just do one or two more properties and some examples on Z-transform.