Transform Calculus and its Applications in Differential Equations
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Lecture – 58
Introduction to Z – Transform

So, let us start with one-two more examples on Mellin transform, how to find out the Mellin transform of some functions. And then we will shift to the next topics the last topics of this lecture series that is on Z-transform.

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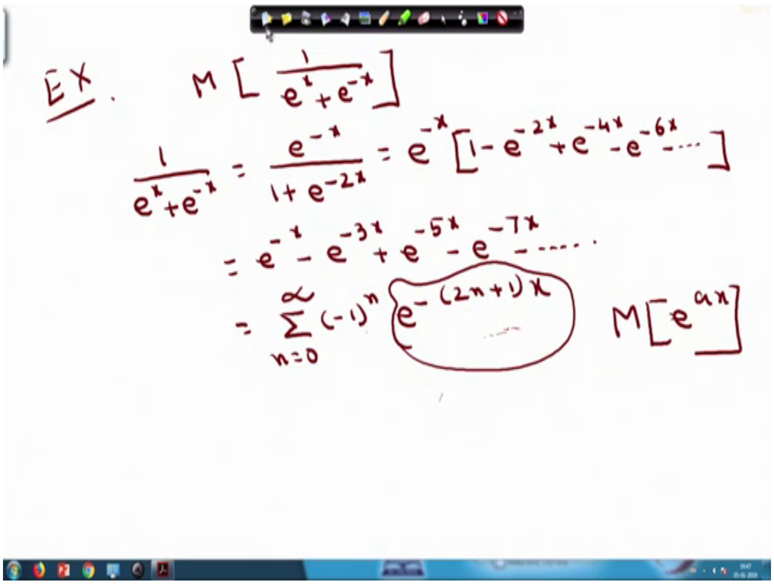

So, let us find out the Mellin transform of this function, 1 by e power x, just I am trying to give exposure of different kind of functions; Mellin transform of 1 by e to the power x plus e to the power minus x. Now, if you take this function 1 by e to the power x plus e to the power minus x, this equals you can write down e power minus x by 1 plus e to the power minus 2 x, so that if you 1 minus x into if you expand this function 1 plus e to the power minus 2 x minus 1, you can write down 1 minus e to the power minus 2 x plus e to the power minus 4 x minus e to the power minus 6 x like this way, it will go on.

And this equals, if I multiply e to the power minus x with this series, e power minus x minus e power minus 3 x plus e power minus 5 x minus e power minus 7 x minus like this way, I will continue. And this is equals, I can write down in terms of series summation n equals 0 to infinity minus 1 to the power n into e power minus twice n plus

1 into x, so that if you see whenever n is equals to 0, you are getting the first term minus 1 to the power 0 that is 1 e power minus x. Whenever n is 1, then in the summation this value will be minus, and this will be e power 2 plus 1 3 x.

Next one whenever n is 2, this your minus 1 to the power 2 will be plus, and this will be e to the power minus 2 into 2 plus 1, so that 5 x. So, this 1 plus 1 by e power plus x into e power minus x, this I am representing in terms of a series summation n equals 0 to infinity minus 1 to the power n into e power minus twice n plus 1 into x, so that the finding the Mellin transform of 1 by 1 plus 1 by e to the power x plus e power minus x, which is equivalent to finding the Mellin transform of summation n equals 0 to infinity minus 1 to the power n e power minus 2 into n plus 1.

And I hope you can understand why we are transforming these in terms of e power, because already you know this thing Mellin transform of e power ax or e power minus ax is known to us. Since, Mellin transform of e power ax is known to us. So, Mellin transform of this part, we can find out very easily.

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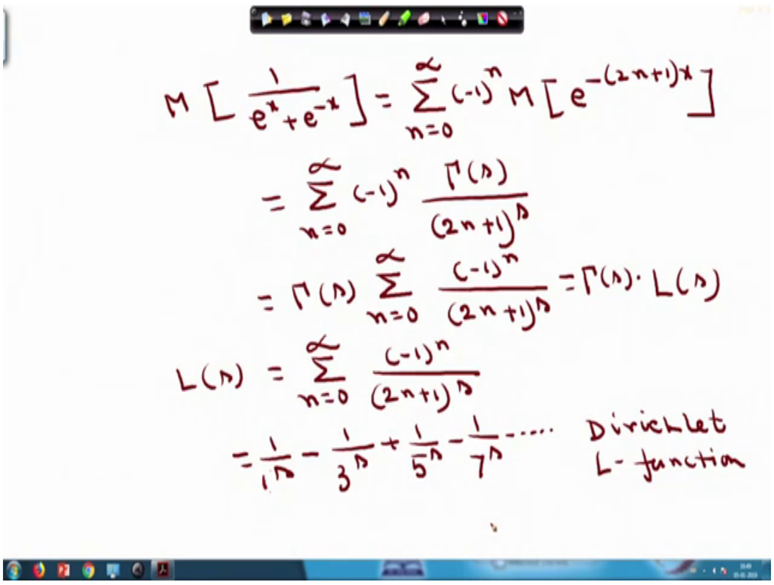

So, from here we can write down Mellin transform of 1 by e power x plus e power minus x, this equal summation n equals 0 to infinity minus 1 to the power n Mellin transform of e power minus twice n plus 1 into x. And from Mellin transform of e power x, this equals you can write down summation n equals 0 to infinity minus 1 to the power n gamma s a is to per 2 n plus 1. So, therefore it will be 2 n plus 1 whole to the power s.

So, once I am writing this, this equals gamma s I can always bring outside of the this one, so n equals 0 to infinity minus 1 whole to the power n by twice n plus 1 whole to the power s. This equals you can write down gamma s into L s, where your L s I can write down this series that is summation n equals 0 to infinity minus 1 to the power n by twice n plus 1 whole to the power s.

So, Mellin transform of 1 by e power x plus 1 by e power minus x is equals to of the form gamma s into L s, where L s equals n equals 0 to infinity minus 1 to the power n divided by twice n plus 1 whole to the power x. And this series actually this is nothing but if you evaluate it 1 by s minus 1 by 3 to the power s plus 1 by 5 to the power s minus 1 by 7 to the power s like this way, it is continuing is known as Dirichlet L function.

This series is known as Dirichlet in function, so that in other words we can say that the Mellin transform of 1 by e to the power x plus e to the power minus x can be represented in terms of Dirichlet L function, where Dirichlet L function is 1 by s minus 1 by 3 to the power s plus 1 by 5 to the power s minus 1 by 7 to the power s like this.

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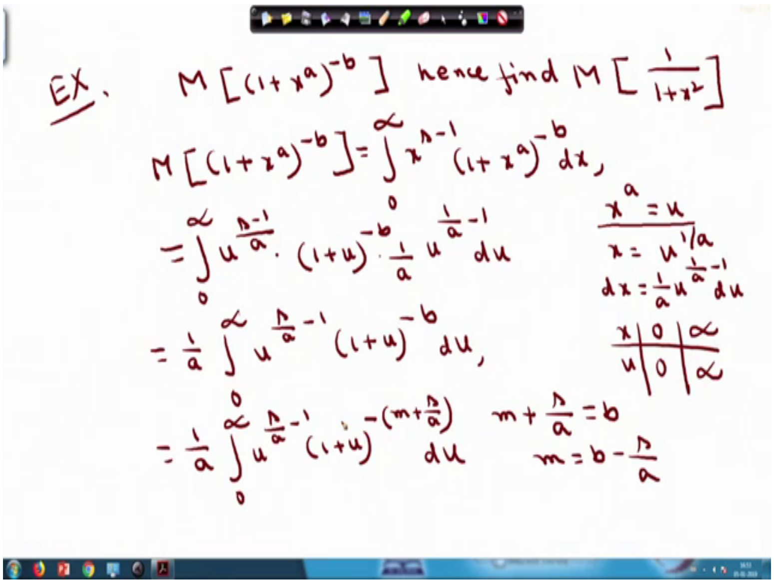

Let us do the next example on Mellin transform. We want to find out the Mellin transform of 1 plus x to the power a to the power minus b. And hence we want to find out the value of this one Mellin transform of 1 by 1 plus x square or in other sense, I am finding a generalized Mellin transform of a function. And then by substituting different values of a and b, I can find out the Mellin transform of some other related functions of also.

So, Mellin transform of this one that is 1 plus x to the power a to the power minus b, from definition I can write down this is equals to 0 to infinity x to the power s minus 1 into 1 plus x to the power a to the power minus b into d x. Let us make a substitution say x to the power a equals u, so that your x is u to the power 1 by a, d x is equals to 1 by a u to the power 1 by a minus 1 into d u. For x 0, your u will be 0, for x infinity, your u also will become infinity, so that the limit of the integration will remain same 0 to infinity.

So, if I substitute this one x to the power a equals u, in that case I will obtain 0 to infinity x to the power s minus 1, so that u to the power it will be s minus 1 by a into 1 plus u to the power minus b, x to the power a is u into d x I have to replace that is 1 by a u to the power 1 by a minus 1 into d u.

If I simplify it this thing, then in that case this will be equals to 1 by a will come outside 0 to infinity u to the power s by a minus 1 into 1 plus u to the power minus b into d u. Now, if you put m plus s by a, this is equals to b that is your m is equals to b minus a say. I am assuming m plus s by a this is equals to b, so that your m will be equals to this thing. So, this integral you can write down as this one 1 by a 0 to infinity u to the power s by a minus 1 into 1 plus u to the power b kind be written as m plus s by a into d u.

So, effectively now your Mellin transform of 1 plus x to the power a to the power minus b, I am writing as this integral, 1 by a 0 to infinity u to the power s by a minus 1 1 plus into 1 plus u to the power minus m into s plus a m plus s by a into d u.

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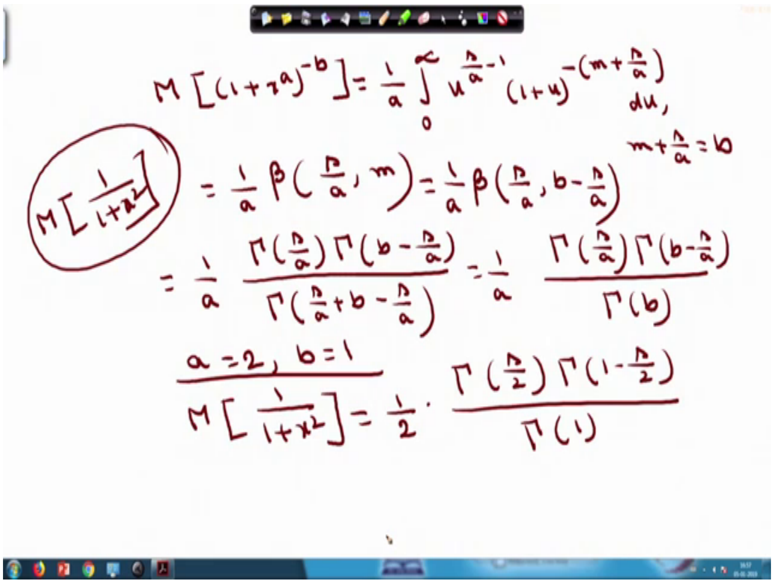

I am rewriting this in the next one, so that you can understand it in the better way. So, Mellin transform of 1 plus x to the power a to the power minus b, this equals we are writing it as 1 by a 0 to infinity u to the power s by a minus 1 into 1 plus u to the power minus m plus s by a into d u, where if you recall, we have assumed m plus s by a this is equals to b.

And if you see this function 0 to infinity x to the power m minus 1 1 plus m to the power in format, this is nothing but I can simply represented in terms of beta function like this that is beta function of s by a comma m. This s by a is coming here, and this is m, so this I can write down beta s by a comma m. This I am not showing how it is coming, because directly from integral, I can obtain this particular value. This integral value if I evaluate in terms of beta function, I can represent it. So, this m if I replace, so this will be 1 by a beta s by a comma b minus s by a, a I am replacing the original value. So, it is in the form of beta m n.

So, if I use the formula in terms of gamma function, I can write down this is equals to gamma s by a gamma b minus s by a divided by gamma m plus n, so that gamma s plus a plus b minus s by a, so that after simplification this will be gamma s by a gamma b minus s by a divided by gamma b. So, the Mellin transform of 1 plus x to the power a to the power minus b equals 1 by a gamma s by a into gamma b minus s by a divided by gamma b.

Now, we want to find out the Mellin transform of this thing, Mellin transform of 1 by 1 plus x square, we want to find out Mellin transform of this. To find out the Mellin transform of this simple you put a equals 2, and b equals 1 in the earlier result. If I put a equals 2, and b equals 1, in that case it will the function will become Mellin transform of 1 by 1 plus x square. And this is equals a is 2, so half into gamma s divided by 2 into gamma b is 1, so 1 minus s by 2 by gamma b that is your gamma 1.

So, therefore Mellin transform of 1 by x square 1 divided by 1 plus x square equals half into gamma s by 2 gamma 1 minus s by 2 into gamma 1. So, by finding the Mellin transform of a general function from there by changing the values of the parameters different parameters of a and v, I can form different functions. And directly I can write down, what is the Mellin transform of that function. So, I hope it is clear that the properties are Mellin transform, and how to find out the Mellin transform of different functions. Also we have done the Mellin transform of some well-known functions, which are very frequently used.

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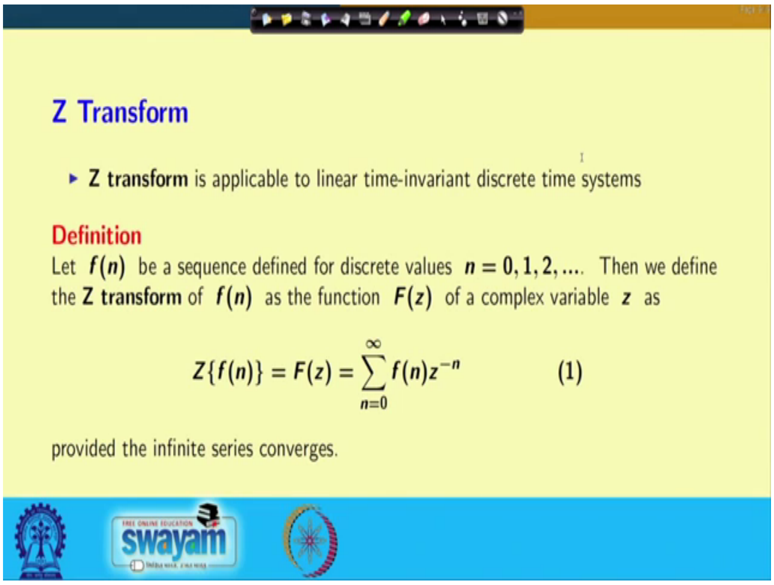

Now, let us come to the last topics that is the Z transform. This Z transform has unique identification, I am just coming. Please see this one. Z transform is applicable to linear time-invariant discrete time systems that is in the time series, whenever you are talking about the time system, and which take the discrete values not continuous values. In earlier all cases, whatever case you have considered, we have taken only the continuous values, but a parameter or a variable can take only discrete values. Therefore, this distance form can be used, whenever it can parameter value can take only the discrete value, then we use the Z transform. And this is linear time-invariant time system in statistics mostly this Z transform is used.

So, first we will give the definition, and the properties after that we will see how to find out the Z transform of some functions. Suppose, let f n be a sequence defined for discrete variables n equal 0, 1, 2 that is they have a sequence for each value of n f n as a particular value. Then we define the Z transform of f n as the function of F z that is where z is a complex variable, please note this one.

The function f n is a sequence, and Z transform of f n is a function of z, where z is a complex variable, and which is defined as Z transform of f n, which we are writing as capital F z equals summation n equals 0 to infinity f n z to the power minus n provided the infinite series one converges, obviously the series has to converse.

So, therefore the Z transform of a function f n is defined as summation n equals 0 to infinity f n z to the power minus n, where the infinite series will must converge. Please note one thing here that we have not given how this particular Z transform is coming.

Just like for the case of Laplace transform for the case of Fourier transform, we explicitly told how to derive the transform of a particular function due to scarcity of time, directly we are giving the definitions, and we will prove certain properties and we will see some examples. So, please note this thing that Z transform of a function f n, where the function f n is nothing but a f n is a sequence, which is defined for discrete variables discrete values of n equals 0, 1, 2 like this. Z transform of f n equals summation n equals 0 to infinity f n z to the power minus n.

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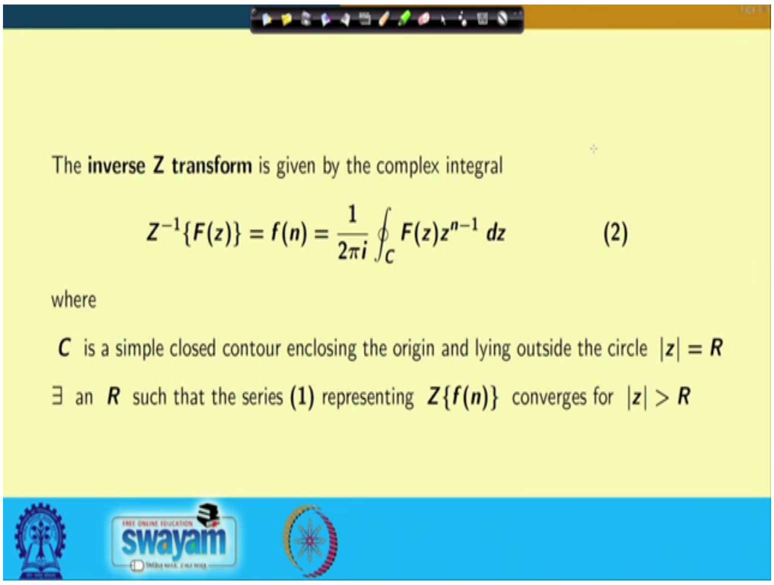

Now, the inverse Z transform, of course it will come on the complex integral because, if you see here, we have told F z, where z is a complex variable. So, therefore inverse Z transform is given by the complex integral. Z inverse F z, which is equals to f n, this equals 1 by twice pi i integral over C F z to the power n minus 1 into F z. Of course, here your value is Z to the power n minus 1 and C is the simple closed contour enclosing the origin and lying outside the circle modulus of z equals R. And please note that there exists some R such that the series 1 representing the Z transform of f n converges for modulus of z greater than R.

So, please note this thing that the inverse Z transform will also can find out by evaluating the contour integral in the form of 1 by twice pi i contour integral over C F z into z to the power n minus 1 into dz, where C is the simple closed contour enclosing the region, enclosing the region and the origin and lying outside the circle modulus of z equals R. And we are choosing R in such a fashion that, the series 1 represented by Z transform of f n converges for modulus of z greater than R. So, please remember this function that the Z transform of f n equals summation n equals 0 to infinity f n into z to the power minus n. So, z to the power n is being here the particular point.

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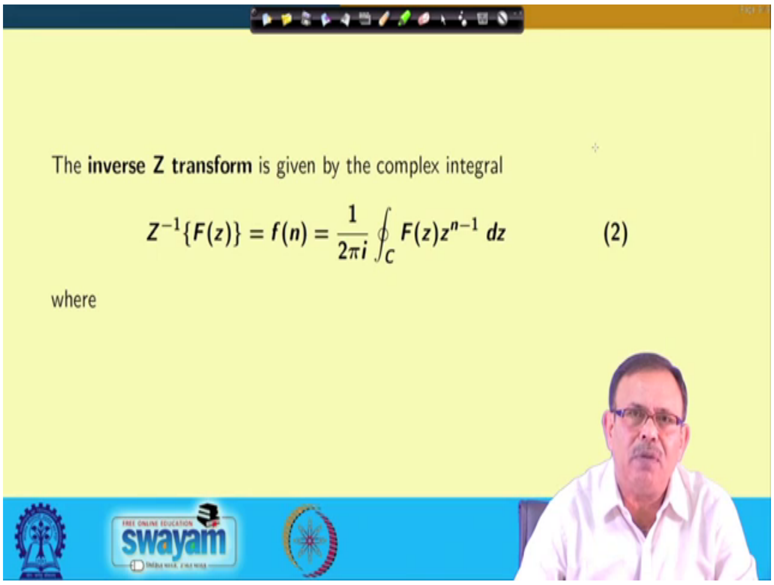

And similarly, the inverse we can find out by this way Z inverse F z equals 1 by twice pi i control integral over C F z into z to the power n minus 1 d z.

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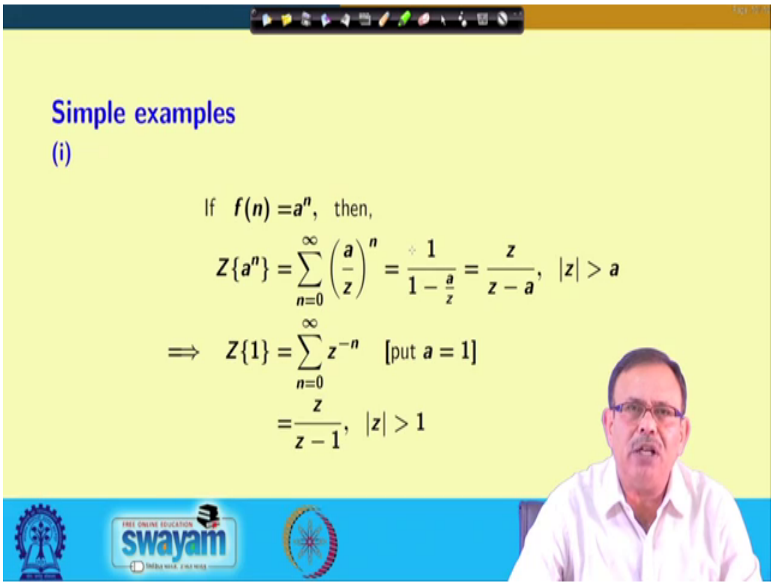

Let us take some a simple examples, where we will try to find out the value of some Z transform of some functions.

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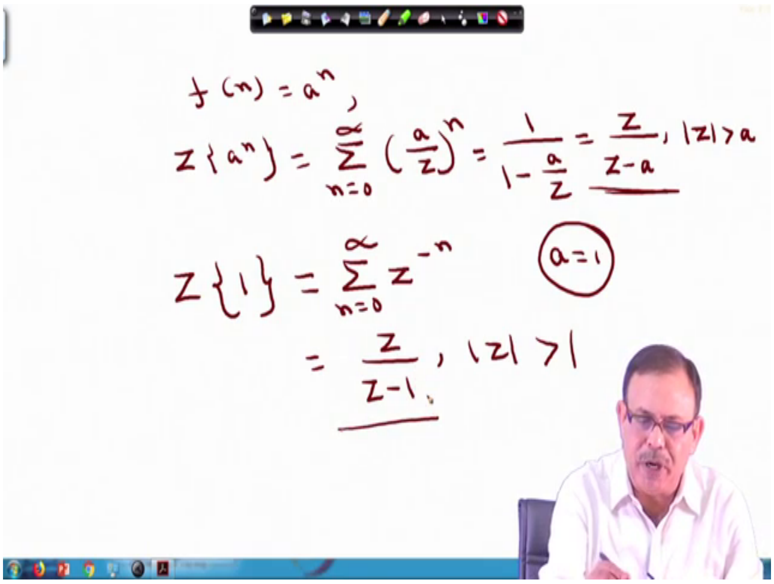

Let us take your f n equals a to the power n say. In that case, Z transform of a to the power n, this equals from the definition summation n equals 0 to infinity f n into z to the power minus n, so that this I can write down summation n equals 0 to infinity a by z to the power n, and if I evaluated this is nothing but 1 by 1 minus a by z, and this equals I can write down z by z minus a, of course modulus of z should be greater than a.

So, therefore the Z transform of a to the power n is this series, this series I can write it in the form of 1 by 1 minus a by z, and which is equals z by z minus a, so that whenever if I have to find out Z transform of 1 that means, I have to simply take a 1. If I take a equals 1, this f n will become 1, so that this is nothing but n equals 0 to infinity z to the power minus n, so it will be from here it value this value will become z by z minus 1, so that you will obtain this, obviously again margit should be greater than 1.

So, therefore please note this one. Z transform of a to the power n is equals to z by z minus a, the definition is quite different, because as you can understand here f n is a sequence, and it has discrete values for n equals 0, 1, 2 like this. So, z transform of a to the power n is z by z minus a, whereas z to the power z transform of 1 simply by substituting a equals 1, I can obtain as z by z minus 1. So, just let us see it quickly z to the power n Z transform of a n from the definition, I can write down summation n equals 0 to infinity a by z out to the power n, and which is equals z by z minus a modulus of z is greater than a. So, Z transform of 1, I can find out by putting a equals 1, so this integral and this is nothing but z by z minus 1.

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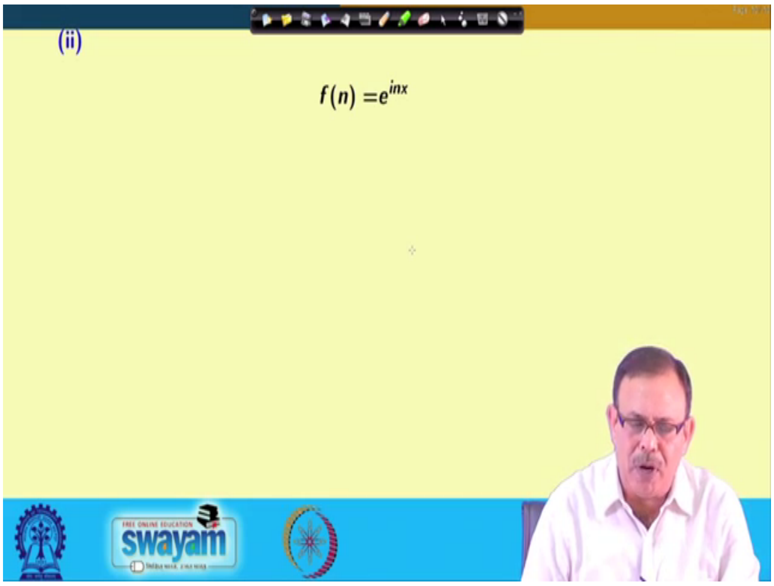

Now, let us see this one. We want to find out the Z transform of f n equals e power i n x.

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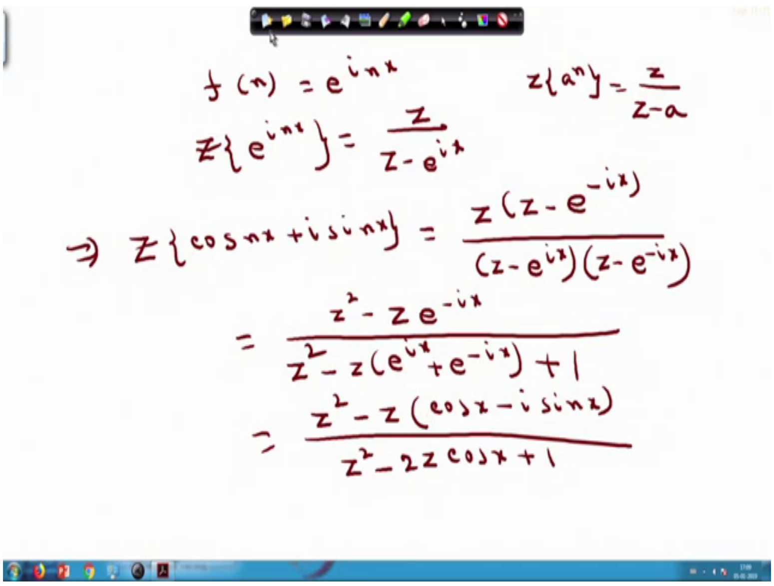

So, whenever we want to find out the Z transform of this f n equals e power i n x. So, Z transform of e power i n x from the earlier example itself, I can tell that z by z minus e power i x, because your Z transform of a to the power n, this is equals to z 2 by z minus a, already we have shown this thing in the last example. So, here a I can substitute as e power i x.

So, if I take a as e power i x, therefore Z transform of your Z transform of this one is equals to z by z minus e power i x, please note the a Z transform. I will use other notation this z, and this z is the parameter value with variable which is a complex variable. Now, from here once I obtain the Z transform of e power i n x, as z by z minus e power i x. So, this from here I can write down, I am writing z like this, so that there is no confusion Z transform of cos n x plus i sin x e power i n x, I can break it into this. This equals I can write down z into z minus e power minus i x I am multiplying on both side, so that z minus e power i x into z minus e power minus i x.

So, on the numerator, you will have z square minus z into e power minus i x. In the denominator if you do it, simply it will be z square minus z into e power i x plus e power minus i x plus 1, if I make it, if I multiply. And you simplification, I will obtain z square minus z into this quantity, so that this equals again z square minus z into e power minus i x, I can represent it in the form of cos n sin, so that this I can write down cos x minus i sin x divided by z square minus z into z square z into e power i x plus e power minus i x, this is nothing but twice cos x, so this I can write down twice z cos x plus 1.

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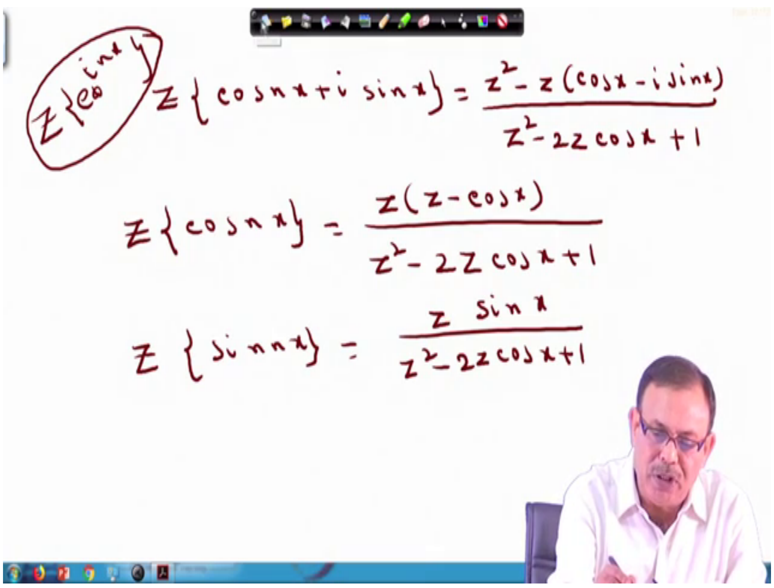

So, now you see z transform of cos x cos n x plus i sin x, so i sin n x, this is equals to z square minus z into cos x minus i sin x divided by z square minus twice z cos x plus 1, so that now I can simply a separate the real part and the imaginary part, so that I can write down your function I got it something like this, Z transform of cos n x plus i sin n x. This is equals to I got it as z square minus z x z into cos x minus i sin n x divided by z square minus twice z cos x plus 1.

So, if I equate the real and imaginary part, I can write down z transform of cos n x will be z into z minus cos x, I will get here divided by the denominator will remain same z square minus twice z cos x plus 1. And z transform of sin n x, this will be equals to your z into sin x divided by z square minus twice z cos x plus 1. So, you see by using this one Z transform of e power i n x e power i n x using this one, I am just telling what is the z transform of cos n x. And what is the z transform of sin n x by simply equating the real part, and the imaginary part.

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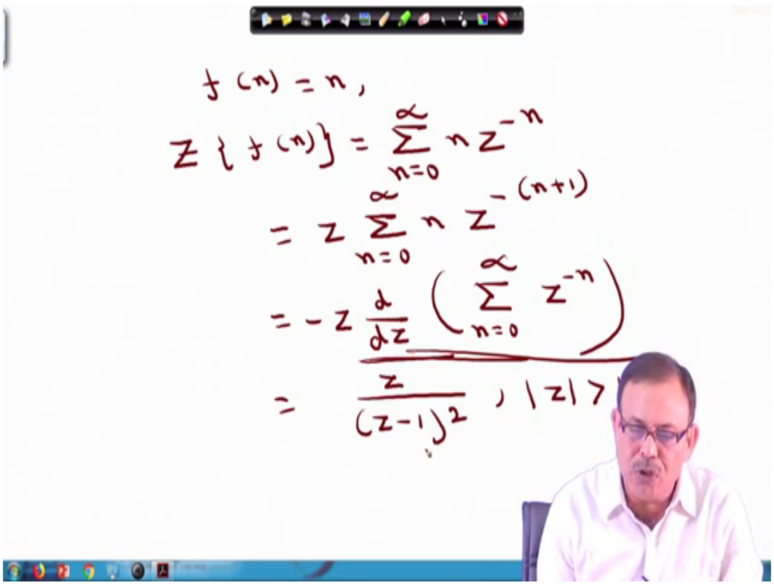

Let us see another thing, let f n equals n, then Z transform of f n, this equals you can write down summation n equals 0 to infinity n z to the power minus 1. This I can write down z into summation n equals 0 to infinity in z to the power minus of n plus 1, so that the this z will be cancelled.

And once I am writing this form, this is nothing but minus d d z of summation n equals 0 to infinity z to the power minus n. If I evaluate this differentiate this d d z, I will get back this thing. And this integral this particular portion, this is nothing but minus of 1 by z minus 1 whole square, so that I can write down that this is equals to z by what z by z minus 1 whole square, where modulus of z is greater than 1. So, whenever f n is equals to n, then I can write down z transform of f n, this is equals z by z minus 1 whole square.

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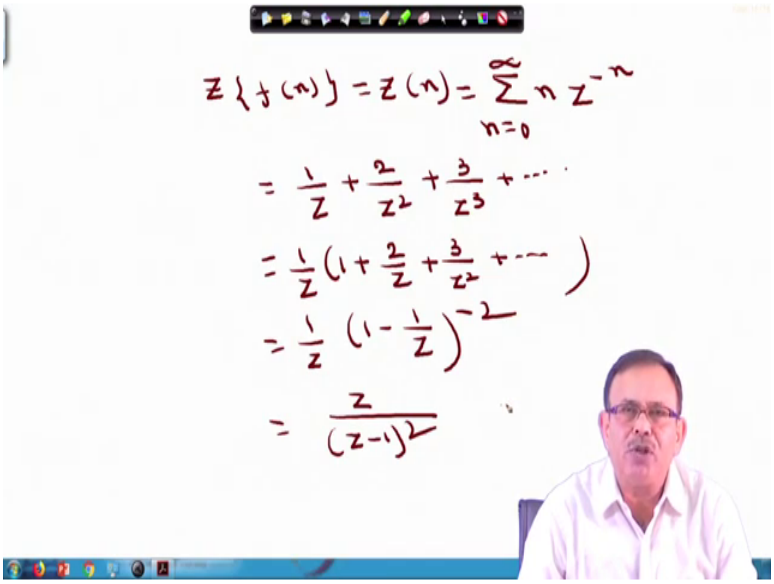

Alternatively, if you do not want to do it like this, if you try to find out Z transform of f n that is f n is nothing but n, this is equals again from definition, we can write down n equals 0 to infinity n z to the power minus n. If I simply evaluate it, I will obtain 1 by z plus 2 by z square plus 3 by z cube like that way I will obtain.

If I simply evaluate this series, so 1 by z if I take common, this will be 1 plus 2 by z plus 3 by z square like this. This equals 1 by z into this integral is nothing but 1 minus 1 by z to the power minus 2, so that I can write down by simplification z by z minus 1 whole square. So, either by that way or by this particular simply by evaluating the series, and then simple manipulations, I can tell that Z transform of n, this is equals to z by z minus 1 whole square. So, in the next class, we will do some properties object transform, and we will see more how to find out the Z transform of some important functions.