Transform Calculus and its Applications in Differential Equations

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Lecture – 57
Examples of Mellin Transform – II

So, in continuation with the last lecture, let us start with some more examples on Mellin transform so that you can understand it in much better way. Last example what we did was Mellin transform of x to the power b into h of x minus a, where h of x minus a is nothing but the Heaviside function, you need to Heaviside function. Now, let us see some well known functions for which we need the Mellin transform and which is very useful in physics and other engineering branches whenever they try to solve some real life practical problems.

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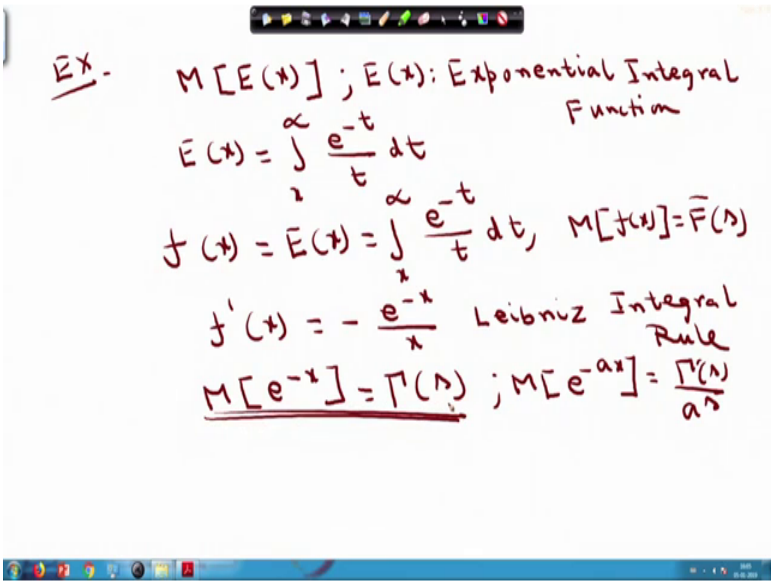

So, the next example what we want to see is, we want to find out the Mellin transform of E of x, where your E x basically is nothing but exponential integral function. This kind of functions appears very frequently and for that reason we are trying to find out the Mellin transform of this one, so exponential integral function. And this function is given by E of x this is equals to x to infinity e power minus t by t into dt. So, we want to find out the Mellin transform of E of x, where E x is nothing but the exponential integral function and is given as E of x equals x to infinity e power minus t by dt.

So, the function is little bit peculiar, but this type of functions appears in real life in various problems. So, you have taken this type of examples, which are very frequently used so that directly you can use the Mellin transform of these particular functions. To find out the Mellin transform of this function, I am starting with say f x, this is equals the exponential integral function, and this is nothing but x to infinity e power minus t by t into dt. And of course, if I take the Mellin transform of f x, this will be equals to F bar s say.

So, now if I take f x equals this integral x to infinity e power minus t by t dt f dash x if I differentiate with respect to x on the both side on the given this thing, then this I can directly write down f dash x equals minus e power minus x by x. And this basically I am using here, I am just writing Leibniz integral rule. So, using Leibnitz integral rule directly I am not specifying or in differentiation under the sin of integration f dash x equals you can write down minus e to the power minus x by x using the Leibniz integral rule.

And once I am writing this, therefore, so Mellin transform of e to the power minus x is equals to gamma s. If you recall in the last lecture itself, we have done Mellin transform of e power minus x equals gamma s by e to the power s. So, from here this we have done in the last lecture itself Mellin transform of e power minus a x is gamma s by a to the power x. From here you can write down Mellin transform of e power minus x, this is equals to gamma x.

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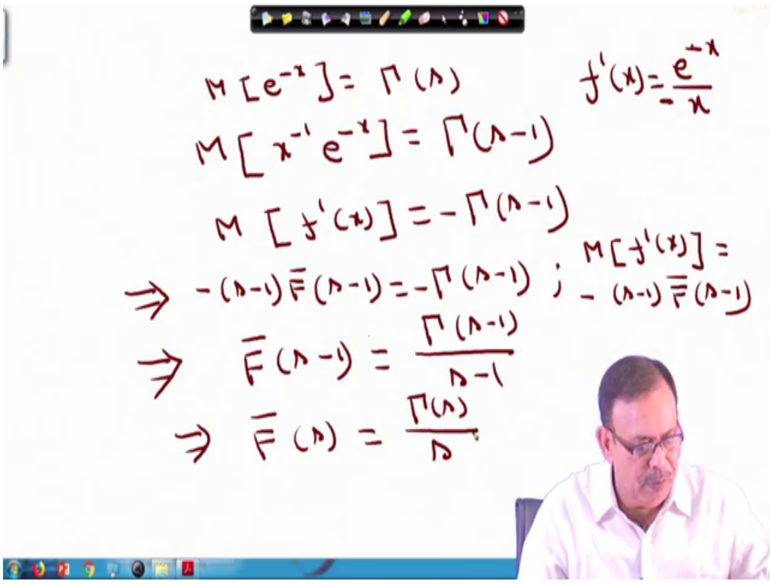

So, once I am getting this from here, you can write down, you have this one Mellin transform of e power minus s is gamma s so that using shifting property now you can write down Mellin transform of x to the power minus 1 e power minus x equals x to the power minus 1. So, it will be the gamma s minus 1.

So, since I know Mellin transform of e power minus x is gamma s using shifting property, we can tell that Mellin transform of x to the power minus 1 e power minus x is nothing but gamma s minus 1, so that this x power minus 1 e power minus x that is e to the power minus x by x this is if you recall in the last slide itself, we have written f dash x is nothing but minus sorry 1 minus sign will come f dash x equals minus e power minus x by x. So, that you can write down Mellin transform of f dash x, this is equals to minus gamma s minus 1.

And since Mellin transform of f dash x is minus s minus 1 gamma s minus 1, so this equals you can write down minus s minus 1 into F bar s minus 1, this is equals to minus gamma s minus 1. If you recall from the Mellin transform of derivatives, we have proved this that Mellin transform of f dash x, this is equals minus s minus 1 into F bar s minus 1. Just I am writing this, if you have forgotten Mellin transform of f dash x equals minus s minus 1 into F bar s minus 1. So, from here I am writing Mellin transform of f dash x equals minus s minus 1 into s by s bar s minus 1 into s bar F bar s minus 1 equals of minus gamma s minus 1.

So, once I am getting this, so therefore you can write down from here F bar s minus 1 this is equals to gamma s minus 1 by s minus 1, the negative sign will be cancelled. So, that if I replace s minus 1 by s, you will obtain F bar s, this calls to gamma s by s, so therefore, you are F bar s that is the required result Mellin transform of e x that is F bar x this is equals to gamma s by s. So, please note this thing that we started from something other place that the Mellin transform of f x equals e x we are considered from there we started. And after that we are finding that Mellin transform of the function e x, this is equals to gamma s by s.

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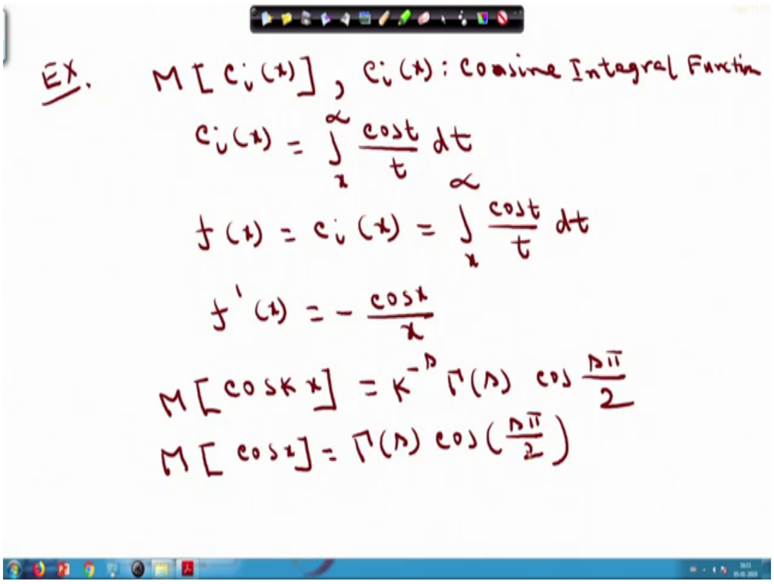

Now, let us take another useful function that is we want to find out the Mellin transform of say we are denoting as C i x, where C i x is nothing but cosine integral function, cosine integral function. So, we want to find out the Mellin transform of C i x, where C i x is the cosine integral function and C i x is defined as x to infinity cos t by t dt. So, cosine integral function is defined as x to infinity cos t by t dt. And we have to find out the Mellin transform of this function. Again just like earlier example, I am starting with f x equals C i x that is this integral x to infinity cos t by t dt.

If I differentiate with respect to x on the both side, then I will obtain f dash x equals minus cos x by x. Again I am doing it using the Leibniz integral formula. So, your f dash x is equals to minus cos x by x. And your Mellin transform of this thing, we have done earlier if you recall Mellin transform of cos k x equals k to the power minus s gamma s into cos s pi by 2. This we have done into the last to last lecture that Mellin transform of cos k x k to the equals k to the power minus s gamma s into cos s pi by 2, so that from here you can write down Mellin transform of cos x this is nothing but your k will not be there so gamma s into cos off s pi by 2. So, please note that am I f dash x is equals to minus cos x by x. Now, first I am finding the Mellin transform of cos x from Mellin transform of cos k x, which we have done earlier, this is equals gamma s into cos of s pi by 2.

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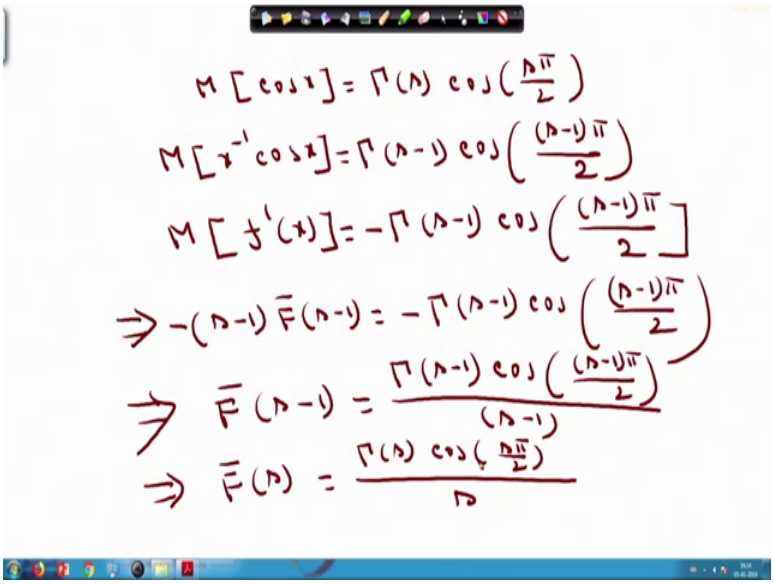

So, that again in the same way, so I am just writing this Mellin transform here, Mellin transform of cos x, we got it as gamma s into cos of s pi by 2. But I am interested on cos x by x so that Mellin transform of x to the power minus 1 into cos x. This equals I can write down gamma s minus 1 into cos of s minus 1 into pi by 2. Again using the shifting property as I described in the last example, in the same way if I know the Mellin transform of cos x, the Mellin transform of x to the power minus 1 cos x, where s will be replaced by s minus 1. So, that Mellin transform of x to the power minus 1 cos x equals to gamma s minus 1 into cos of s minus 1 into pi by 2 so that you can write down as you know Mellin transform of x dash cos x or x x to the power minus 1 cos x that is cos x by x, and this is nothing but f dash x.

So, I am replacing this by this. And this is equals to your minus gamma s minus 1 into cos of s minus 1 into pi divided by 2. So, that from here again this equals Mellin transform of f dash x, this I can write down minus s minus 1 into F bar s minus 1, this is equals minus gamma s minus 1 into cos of s minus 1 into pi by 2. I am not explaining the left hand side, because in the last example itself I have explained that the using the derivative property Mellin transform of f dash x equals minus s minus 1 into F bar s minus 1.

So, that from here you can write down F bar s minus 1 this is equals to gamma s minus 1 into cos of s minus 1 into pi divided by 2 by s minus 1. So, once I am getting this again from here F bar is I can write down as pi s cos of s pi by 2 divided by s. So, therefore, you see the Mellin transform of cosine integral function, which is defined by the integral x to infinity cos t by t dt is equals to gamma s cos of s pi by 2 divided by s. So, Mellin transform of C i t that this cosine integral function is equals to gamma s cos of s pi by 2 divided by s.

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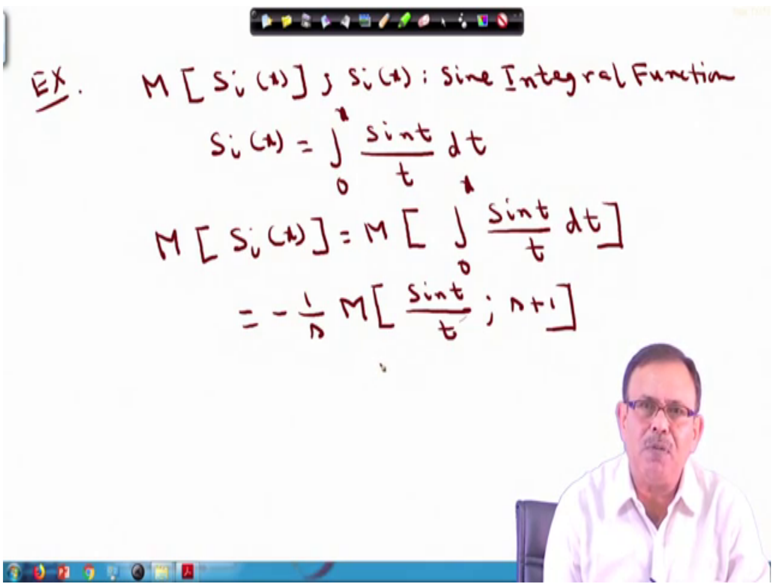

In the next example, let us see for the sin that is Mellin integral of S i x, where your S i x is nothing but again sine integral function; S i x is sine integral function. And this S i x is defined as S i x equals in the same way 0 to x sin t by t dt. So, we want to find out the Mellin transform of sine integral function, where the sine integral function S i x is defined as 0 to x sin t by t dt.

So, once we are doing it Mellin transform of S i x, this is equals to you can write down Mellin transform of 0 to x sin t by t dt, I am just putting the value of S i x. And this equals using Mellin transform on integrals, this equals I can write down minus 1 by s Mellin transform of sin t by t with the parameter s plus 1. So, please note that this Mellin transform of 0 to x sin t by t dt I am writing it as minus 1 by s into Mellin transform us of sorry this M has not come, Mellin transform of sin t by t, where the parameter is s plus 1.

And here we have used the property of the Mellin transform on integrals or in other sense basically at first I have to find out what is the Mellin transform of sin t. Just like in the earlier cases already, you have done Mellin transform of sin k t using that one, I can find out the Mellin transform of sin t. And once I know the Mellin transform of sin t, from there I can find out the Mellin transform of sin t by t. So, Mellin transform of S i x, this is equals I am writing minus 1 by s Mellin transform of sin t by t with the parameter as s plus 1.

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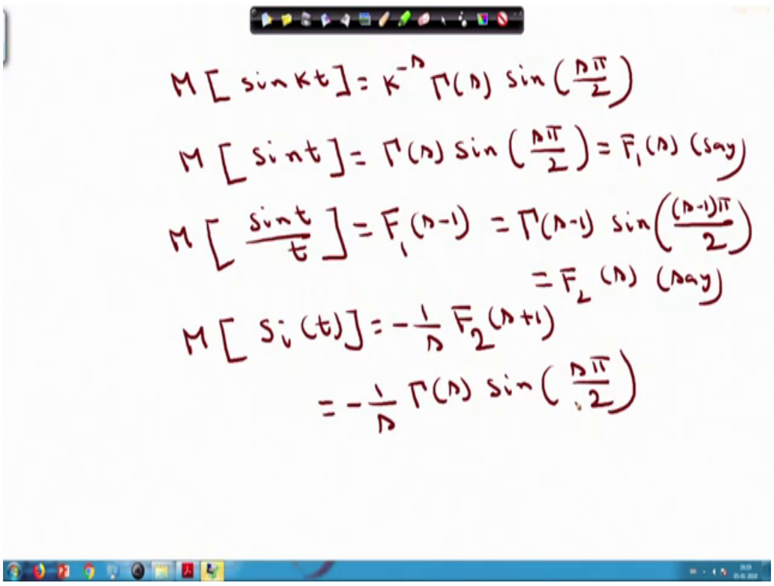

So, you know these things already again we have done it Mellin transform of sin k t. This is equals k to the power minus s gamma s into sin of s pi by 2 we have proved this thing earlier. So that Mellin transform of sin t that is k equals 1, you are putting, so that this will be equals to gamma s into sin of s pi by 2, and this is equal say I am assuming F 1 s, I am assuming F 1 s say. So, from Mellin transform of sin k t I am finding Mellin transform of sin t, which is equals to gamma s sin s pi by 2 and which I am assuming as F 1 s.

Therefore, Mellin transform of sin t by t again from the properties simply this will be equals to F 1 s minus 1. Mellin transform of sin t, if Mellin transform of sin t F 1 s, Mellin transform of sin t will be F 1 s minus 1. And already I know what is F 1 s so by substituting s by s minus 1, I can write down the value of F 1 s minus 1 also and which will be equals to gamma s minus 1 into sin of s minus 1 into pi divided by 2. And say this is equals to again F 2 s another function of s.

So, therefore, Mellin transform of sin t by t is gamma s minus 1 sin s minus 1 into pi by 2, which I am assuming as F 2 x. From here, now, I can write down Mellin transform of S i t, this is equals to nothing but minus 1 by s F 2 of s plus 1, because from here sin t by t is there from the properties we can always write down Mellin transform of S i t this is equals to minus 1 by s F 2 s plus 1, so that F 2 s is known to us. And I can substitute this, so minus 1 by s gamma s sin of s pi by 2. Therefore, the Mellin transform of the sin integral function that is S i t is equals to minus 1 by s gamma s into sin s pi by 2.

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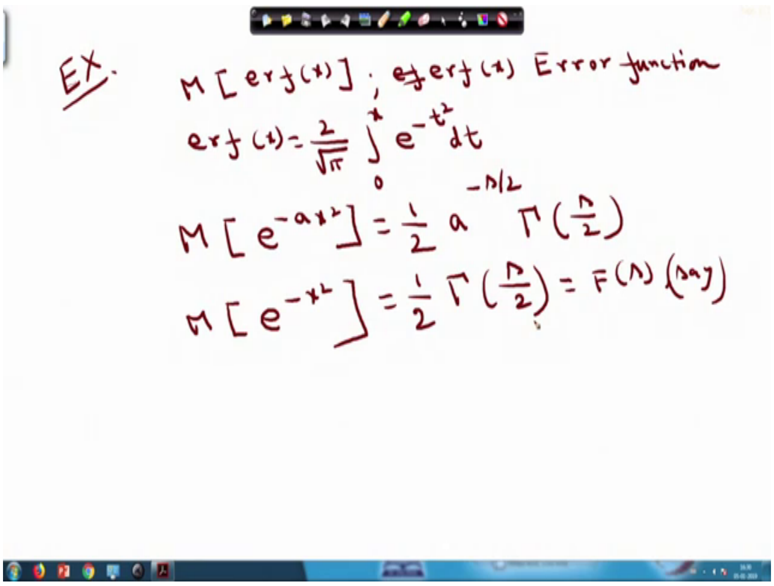

So, now, let us take the next example, the earlier example was on sin integral function. Now, let us take the error function that is we want to find out the Mellin transform of e r f of x again, your e r f of x, e r f of x is error function. And if you recall already, we discussed error function when we started with the Laplace transform, this e r f of x is defined as 2 by root over pi 0 to x e power minus t square into d t. The error function complementary error function, we have defined earlier whenever we did the Laplace transform and the Fourier transform. So, now we want to find out the Mellin transform of error function of x that is e r f of x. So, we start from this thing. If you remember we know the Mellin transform of e power minus x square. This we have done it earlier that is half into a to the power minus a to the power minus s by 2 into gamma s by 2.

So, since e power Mellin transform of e power minus a x square equals half into e power minus s by 2 gamma s by 2. Basically we are interested on e power minus t square is here. Since e power minus t square is here, therefore, we can just find out, we can just find out these from here e power minus a x square is equals to half into a to the power minus s by 2 gamma s by 2.

So, from here you can write down Mellin transform of e power minus x square as I was telling you in e r f function it is given 0 to x e power minus t square. So, we want to you are interested to find out the Mellin transform of e power minus x square. So, since we know Mellin transform of e power minus x square, from here you can write down Mellin transform of e power minus x square, this equals to half into gamma s by 2, here your a is equals to 1, and this equals F s say. So, therefore, Mellin transform of e power minus x square equals to half into gamma s by 2, which is equals F s say.

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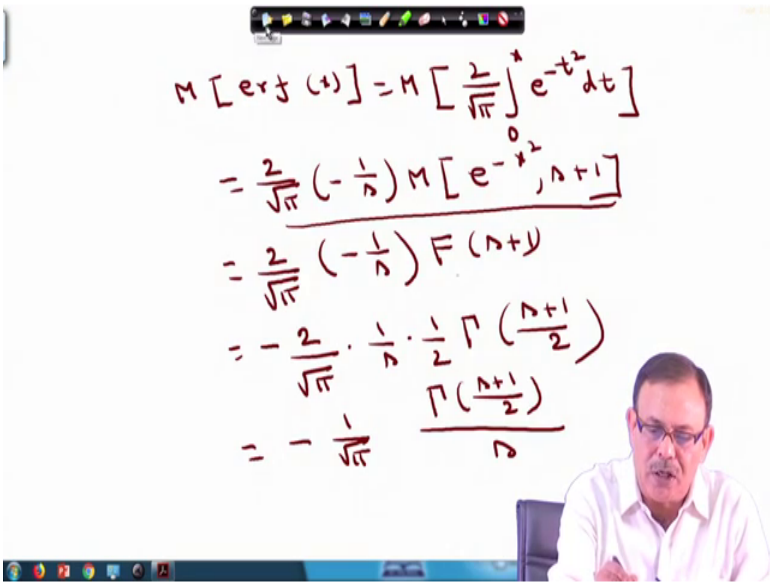

So, from here we are starting that Mellin transform of e r f of x this equals Mellin transform of we are substituting the value of the e r f of x 2 by root over pi 0 to x e power minus t square into dt. And this equals from property we can write down 2 by root over pi minus 1 by s into Mellin transform of e power minus x square plus 1 e power minus x square, where the parameter is s plus 1. This equals 2 by root pi into minus 1 by s into F of s plus 1. So, as you see Mellin transform of e power minus x square, where parameter is s plus 1. And already in the last slide we have shown that Mellin transform of e power minus x square is equals to capital F of s.

Therefore, Mellin transform of e power minus x square with parameter s plus 1 will be equals to F of s plus 1. And this we are writing directly from the property, which we have done earlier so that if I substitute the value, this will be equals to 2 by root pi into 1 by s into half into gamma s plus 1 by 2. And if I simplify it, this will be equals to minus 1 by root pi, 2 will be cancelled, this equals to gamma s plus 1 by 2 by s, this s will come. Therefore, the Mellin transform of error function equals to minus 1 by root pi gamma s plus 1 by 2 divided by s.

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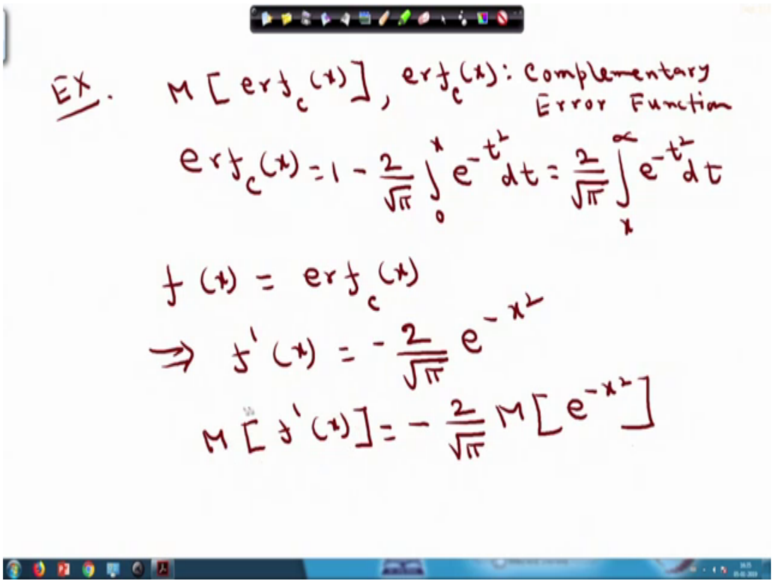

Now, let us see the complimentary error function, Mellin transform of complementary error function that is Mellin transform of e r f of c of x, which is write s again complementary error function also we have done earlier when we were studying the Laplace transform. So, e r f c x is complementary error function. So, this complimentary error function is given by e r f c of x, this is equals to I can write down that is 1 minus error function of x 2 by root pi 0 to x e power minus t square dt. This equals directly also you can write down 2 by root over pi x to infinity e power minus t square d t.

So, complementary error function equals 1 minus 2 by root pi 0 to x e power minus t square dt that is 1 minus error function of x or I can write it as 2 by root pi x to infinity e power minus t square dt. So, I have to find out the Mellin function, Mellin transform of the complementary error function of x.

Now, I am starting with this f x equals again I am considering e r f c of x just, we have done it for the earlier example so that from here, you can write down f dash x is nothing but minus 2 by root pi minus 2 by root pi into e power minus x square. I am using this so minus 2 by root pi e power minus this. Therefore, Mellin transform of f dash x this is equals to minus 2 by root pi Mellin transform of e power minus x square I can write down. This Mellin transform of f dash x, this is equals minus 2 by root pi Mellin transform of e power minus x square, I know the Mellin transform of e power minus x square, I know that Mellin transform of f dash x this I can write down in other property.

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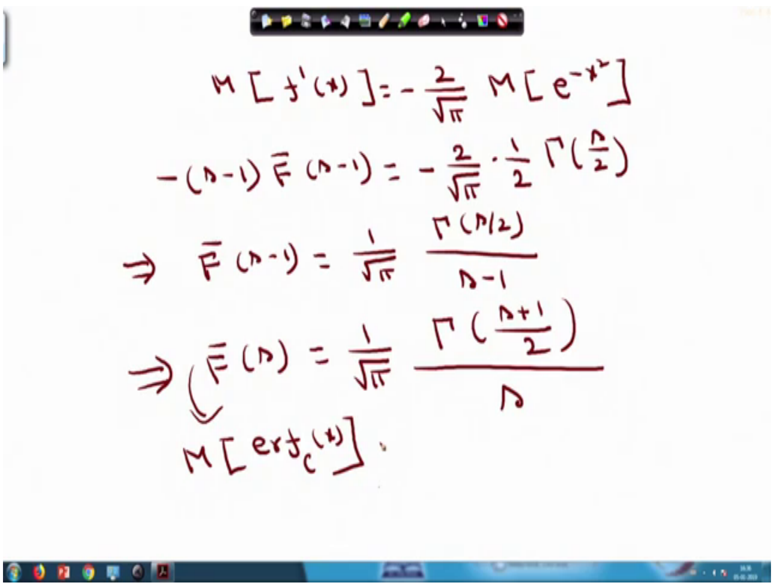

So, I am just using this one, I am writing again Mellin transform of f dash x, this is equals to minus 2 by root pi Mellin transform of e power minus x square. So, once I am writing this from here, Mellin transform of s f dash x, I can write down minus s minus 1 into F bar s minus 1 this we have used earlier also. So, this equals minus 2 by root pi Mellin transform of e power x square will be half into gamma s by 2, because Mellin transform of e power minus x square is half into a to the power minus s by 2 into gamma s by 2, a is 1. So, that Mellin transform of e power minus x square will be equals to half into gamma s by 2.

So, from here I can write down F bar s minus 1, this is equals to 1 by root pi gamma s by 2 will come, this 2 will be cancelled divided by s minus 1. So, that your F bar s, this is equals to I can write down 1 by root pi gamma s plus 1 by 2, s will be replaced by this divided by s, so that the Mellin transform of complimentary error function, because this is nothing but the Mellin transform of e r f c of x complimentary error function of x, this is equals to 1 by root pi gamma s plus 1 by 2 divided by s.

So, in the next class, we will continue with some more example on Mellin transform. And then in the subsequent lectures, we will go through another important transform, which we call as Z-transform which is being used in basic many statistical methods also.