Transform Calculus and its Applications in Differential Equations
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Lecture – 56
Example of Mellin Transform – I

So, in the last lecture, we started the Mellin transform. After doing the Laplace and Fourier transform we give the definition of Mellin transform. We have shown how to find out the Mellin transform of some simple function, and we started the properties, the scaling properties, differential property and all these things we have done. Today also we will start with the some more properties.

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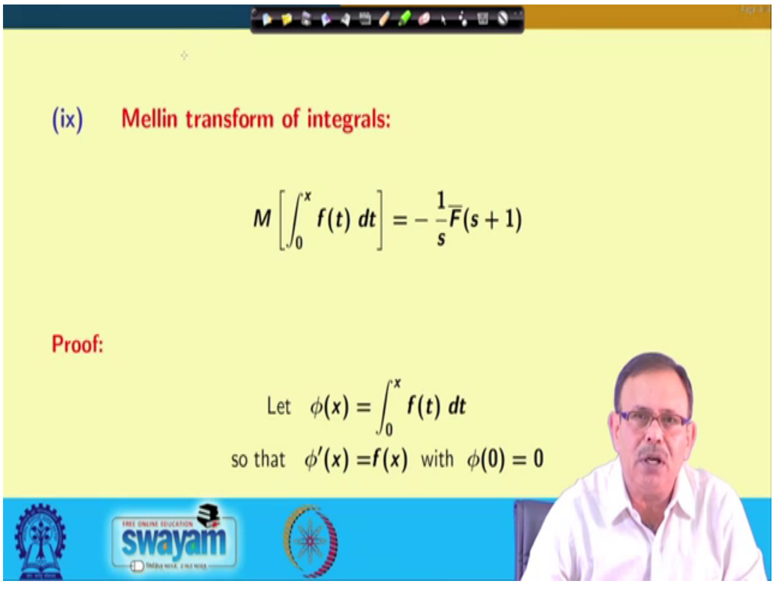

The first property is on Mellin transform of integrals that is Mellin transform of 0 to x f t dt, this is equals to minus 1 by s F bar s plus 1.

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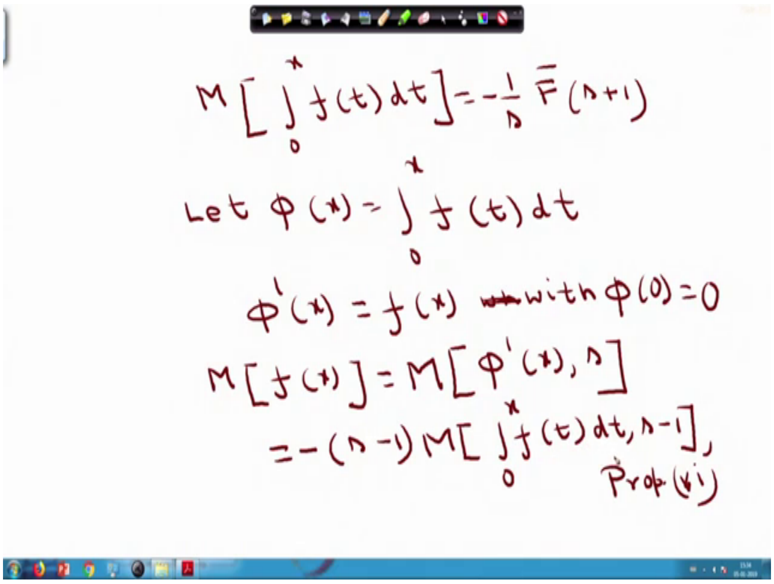

To give the proof of this one we have to start say, so basically you have to proof Mellin transform of 0 to x f t dt, this is equals to minus 1 by s into F bar s plus 1. To prove this one, let us start with say let phi x this is equals 0 to x f t dt. So, once you are writing phi x equals 0 to x f t dt, so that automatically whenever you will find out the phi dash x that is the differentiation with respect to integration, this will be equals to f x with your f 0 equals your phi 0, this is equals to 0.

So, we are assuming phi x equals 0 to x f t dt, where your if I take the differentiation with respect to x, then phi dash x will be equals to f x where your phi 0 is equals to 0. So, that now you can write down Mellin transform of f x you know it, this is equals to Mellin transform of f x is nothing but phi dash x. So, Mellin transform of phi dash x where the variable is s. So, you are writing Mellin transform of f x, this is equals Mellin transform of phi dash x and this is equals to s.

This equals you can write down minus s minus 1 into Mellin transform of 0 to x f t dt, but the variable will be s minus 1, this we are getting from the property 6, please note one; note this one. From property 6, directly we can write down Mellin transform of phi dash x this is equals minus s minus 1 Mellin transform of 0 to x f t dt, where the variable is s minus 1. So, basically the derivative property that is property 6, we can obtain this one. So, once I am writing this Mellin transform of f x equals minus s minus 1 and Mellin transform of 0 to x f t dt, the variable is s minus 1.

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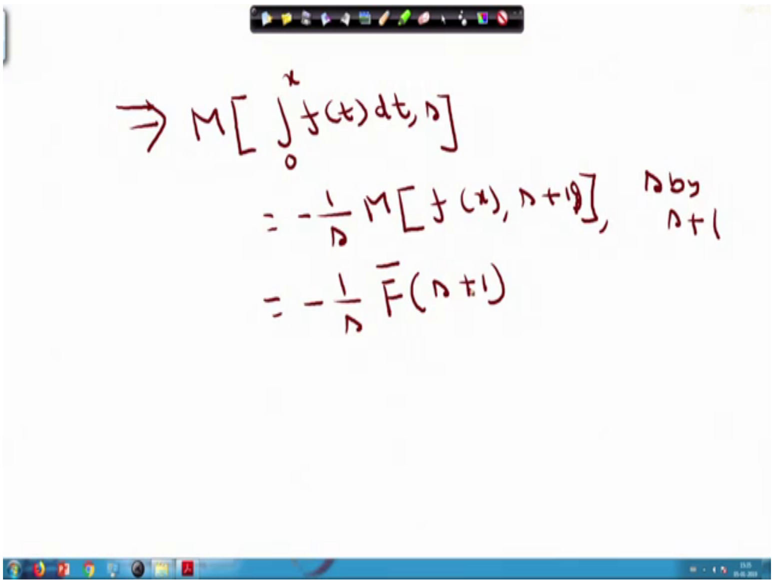

So, from here you can write down Mellin transform of 0 to x, 0 to x f t dt, where the variable is equals to s. This is equals minus s Mellin transform of f x s plus 1. What we are doing here basically we are replacing s by s plus 1. In the earlier one, you are replacing s by s plus 1. And if you replace s by s plus 1, then you will obtain Mellin transform of 0 to x f t dt variable is ds equals minus s Mellin transform of f x s plus 1. And this is nothing but minus 1 by s Mellin transform of f x s plus 1, this you can write down F bar s plus 1. So, therefore, Mellin transform of 0 to x f t dt is equals minus s F bar s plus 1 which completes the proof of this one

So, just let us quickly see this one. So, we are assuming that the phi x equals 0 to x f t dt, so that always you can write down phi dash x this is equals f x, where your phi 0 is equals to 0 from the integral itself you can find out.

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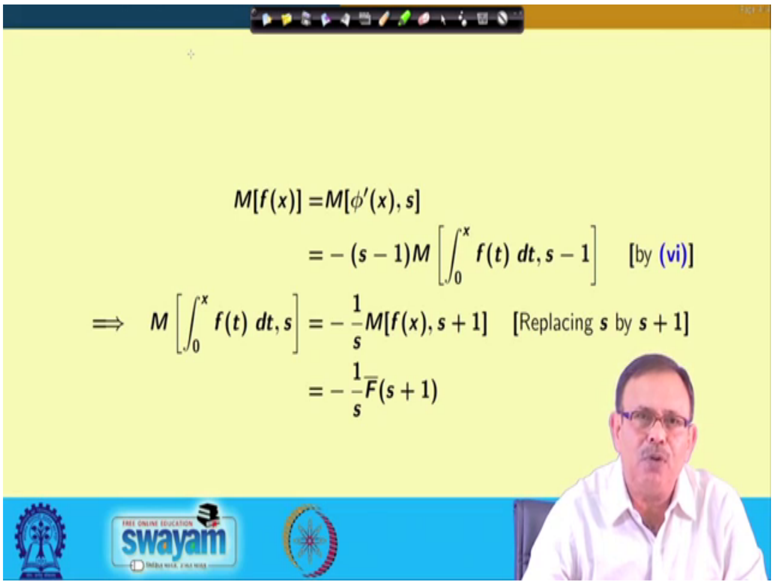

So, that Mellin transform of f x equals Mellin transform of phi dash x with the parameter as s, this is equals to using the property vi that is the derivative property which we have done earlier. Using that property you can write down this equals minus s minus 1 into Mellin transform of 0 to x f t dt, where the parameter is s minus 1. And this is again from here you can write down Mellin transform of 0 to x f t dt with the parameter as s equals minus 1 by s Mellin transform of f x with the parameter is s plus 1, so that directly you can write down minus 1 by s into the Mellin transform of s plus 1 that is F bar s plus 1 Mellin transform of f x over parameter s plus 1 which is nothing but F bar s plus 1 which completes the proof of this.

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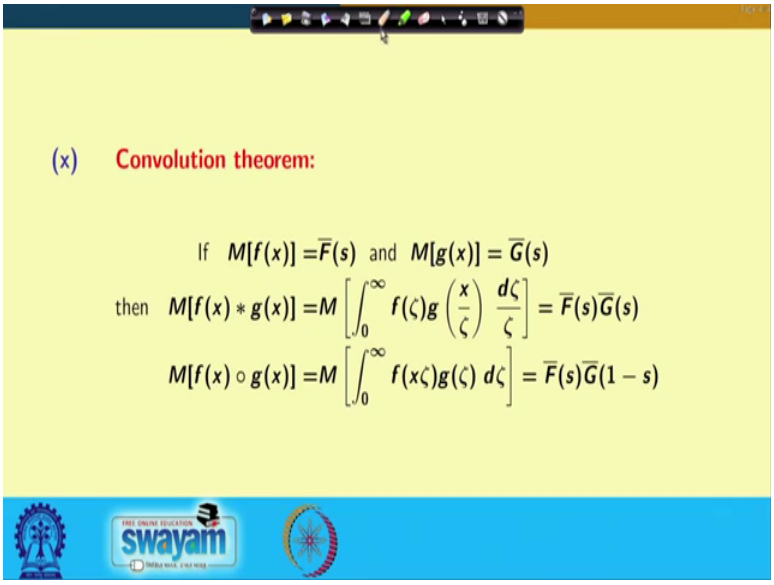

The next one is the convolution theorem. If the Mellin transform of f x is F bar s and Mellin transform of g x is G bar s because as you know for the convolution which we have done it for the Laplace transform, for the Fourier transform. We require two functions f x and g x, so we are assuming Mellin transform of f x is F bar s and Mellin transform of g x is G bar s. Then two ways we can define it Mellin transform of f x 1 operator is star operator g x equals Mellin transform of 0 to infinity f eta g of x by eta d eta by eta this equals as earlier we have seen equals F bar s into G bar s or in other sense the Mellin transform of f x into multiplied by Mellin transform of g x.

Or you can write down in this way also Mellin transform of f x o g x, this equals we can define it in the form of Mellin transform of 0 to infinity f x f of x eta into g eta d eta this equals it will be F bar s into G bar 1 minus s. So, this is the theorem, convolution theorem.

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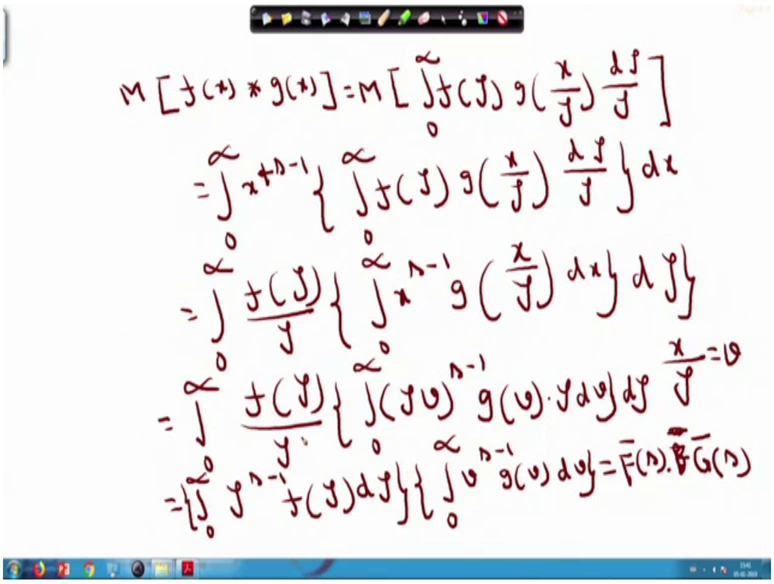

Let us see the proof of this particular theorem. The first one is Mellin transform of f x star g x. So, Mellin transform of f x star g x from the definition, you can write down 0 to infinity f eta milling transform of f eta into g of x by eta into d eta by eta. And this is equals to if I just put it here, the definition of Mellin transform, so that you will obtain x to the power s, sorry this is x to the power plus s minus 1 into your this integral will come that is 0 to infinity f eta into g of x by eta into d eta by eta into d x will come.

So, if I change the order of the integration that is d x and d eta, if I just change the order of this integration, I will obtain as f eta by eta I can bring outside and then inside the other integral 0 to infinity x to the power s minus 1 g of x by eta into dx and d eta. So, here if you substitute d eta, if you substitute x by eta, this is equal say v. If I substitute x by eta, this is equals to v, then you will obtain 0 to infinity f of eta by eta into 0 to infinity x to the power s minus 1. So, your x is the is eta into v. So, therefore, it will be eta into v to the power s minus 1 into g v into g of x by eta, this is nothing but v into d x will be eta into d v into d eta will come into; d eta will come.

So, this equals now I can break them two independent integrals as 0 to 1 will be 0 to infinity eta to the power s minus 1 into f of eta d eta, this is one integral independently I can break it. And the other one will be 0 to infinity v to the power s minus 1 from here I am getting g v into d v. So, you are getting these two integrals, one is 0 to infinity eta to the power s minus 1 f eta d eta which is nothing but capital F bar is that is the Mellin transform of f x.

And the next integral 0 to infinity v to the power s minus 1 g v into dv, this is nothing but the Mellin transform of the g x sorry this, so it will be F bar eta into G bar s. So, we are obtaining the proof of this one as Mellin transform of f x star g x this is equals F bar s into G bar s. So, I hope the proof is clear. Let us just see it once in the first part of the proof, because it has two parts.

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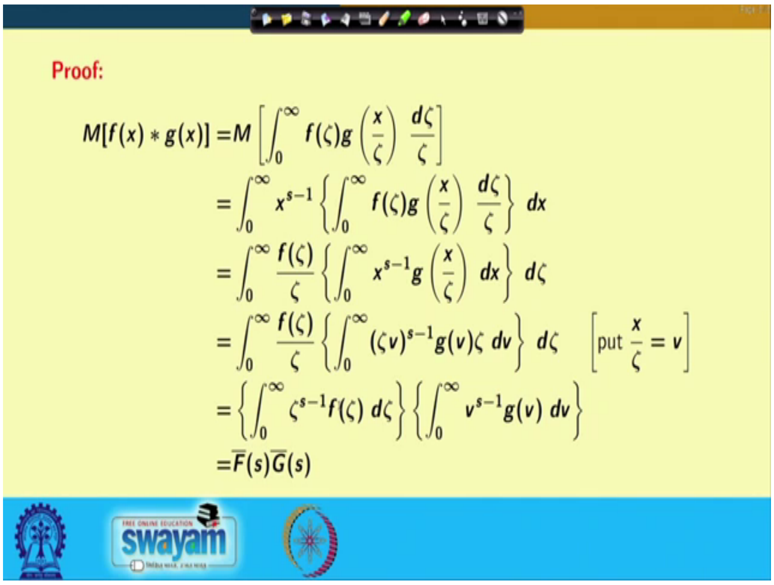

We have shown the first part of the proof we are starting from Mellin transform of f x convolution g x, this is equals from the definition you are writing. Mellin transform of 0 to infinity f eta into g of x by eta d eta by eta. Now, from the property definition of Mellin transform, this you can write as 0 to infinity x to the power s minus 1 0 to infinity f eta g of x by eta d eta by eta into dx. Now, if I substitute over here x by eta equals x by eta equals v, then first I am changing the order of the integration that is d eta and d x, and I am getting the third line.

From the third line, if I substitute x by eta equals v, I will obtain 0 to infinity f eta by eta 0 to infinity eta v to the power s minus 1 g v eta d v into d eta. So that now, the way to variables independently I can write down v and eta, this two I can write down independently. What we have written in the last line that is 0 to infinity eta to the power s minus 1 f eta d eta into 0 to infinity v to the power s minus 1 g v dv. This by this substitution x by eta equals v, I am simply independently I am making this these two parameters as independent under the integrations, so that the first integral 0 to infinity eta to the power s minus 1 f eta d eta is nothing but F bar s and the second integral 0 to infinity v to the power s minus 1 g b d b from the definition of Mellin transform it is G bar s. So, that the first from the according to the first convolution property we have proved that Mellin transform of f x convolution g x is equals to F bar s into G bar s.

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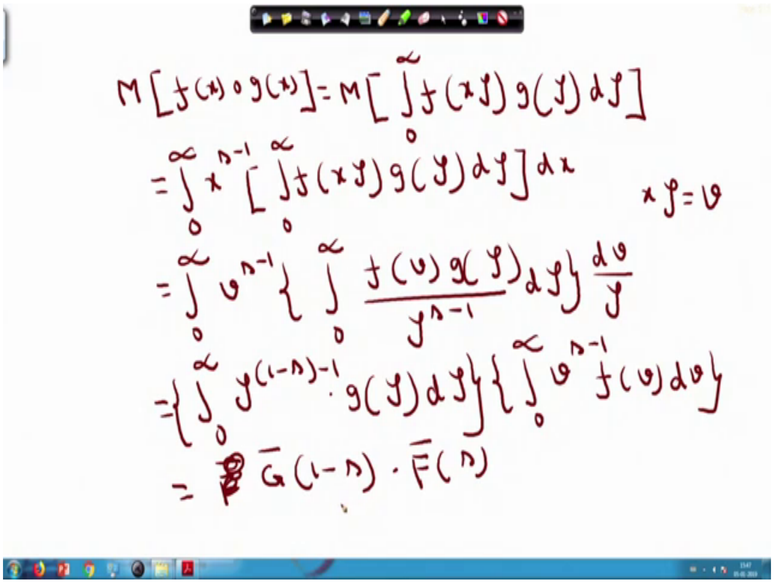

Now, let us see the next property that is using the next property how to prove this one. So, we have used two different convolution operators because the definitions we have used, two different definitions, anyone can use any one of these two. So, that now Mellin transform of f x convolution g x, this equals you can write down Mellin transform of 0 to infinity here it is f of x eta the definition has changed into g eta into d eta.

Now, again as usual from the last one what we have done that is from the definition of Mellin transform this equals you can write down 0 to infinity x to the power s minus 1 0 to infinity f of x eta g eta into d eta and dx will come here. So, that again if you put x eta equals v just like we have done earlier, here you substitute x eta equals v, the other case the substitution was other one x by eta equals v.

So, if I substitute this, this will be equals to 0 to infinity v to the power s minus 1 into 0 to infinity f v into g eta will be there divided by eta to the power s minus 1 d eta into dv by eta.

So, once I am writing this, this again now I can break it into two independent integrals, because the independent parameters v and eta I can separate it out and I can put it in two different integrals in the form of like this. One will be 0 to infinity eta to the power 1 minus s minus 1 eta to the power s was there into g eta d eta, eta to the power 1 minus s is there. So, I am writing this and by eta is there and the other integral will be 0 to infinity v to the power s minus 1 into function of v into d v.

So, now using this integral I am separating it out and I am getting two different integrals, so that the first one is nothing but this integral or let me write down the first one. The first integral 0 to infinity eta to the power 1 minus s minus 1 into g eta d eta, this is nothing but G bar 1 minus s that is the Mellin transform of G with respect to the parameter 1 minus s and the second integral is equals to F bar into s that is this is the value of the parameter here. F bar s is nothing but the Mellin transform of the function f x with respect to the parameter s. And this completes the proof that Mellin transform of f x convolution g x with respect to the definition that the definition of 0 to infinity f of x eta into g eta d eta, this equals F bar s into G bar 1 minus s. Let us see it once more here.

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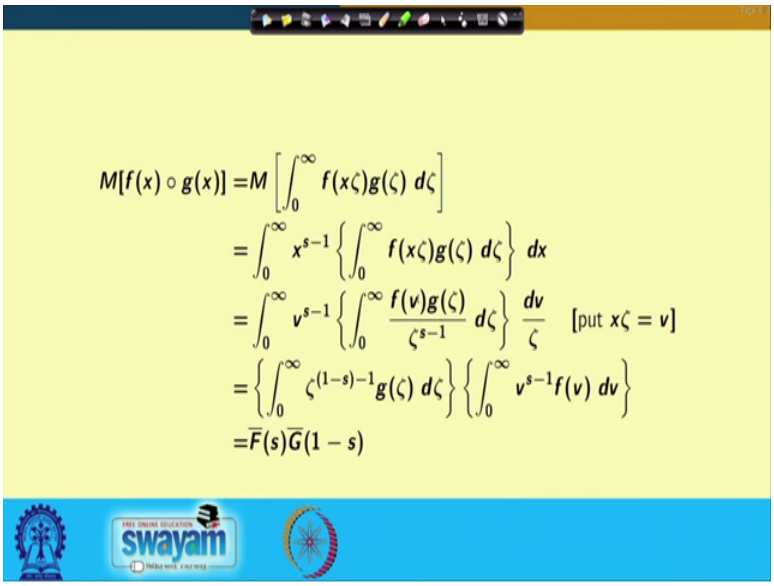

So, this we did it for the earlier case. Now, for the second one, we are starting with Mellin transform of f x convolution g x equals Mellin transform of the definition has changed little bit of convolution, I can use any one of these two definitions. So, using this definition 0 to infinity f of x eta into g eta d eta. Now from the definition of Mellin transform this equals, I can write down 0 to infinity x to the power s minus 1 into 0 to infinity f of x eta g eta d eta into dx. Using the substitution x eta equals v, please note that in the earlier case we use the substitution x by eta this is equals v. So, this equals 0 to infinity v to the power s minus 1 into 0 to infinity f v g eta by eta to the power s minus 1 d eta into dv by eta.

Now, I can separate the two independent variables v and eta separately and I can put them in two different integrals like this, 0 to infinity eta to the power 1 minus s minus 1 into g eta d eta into 0 to infinity v to the power s minus 1 f v into d v. So that the first integral is nothing but G bar into 1 minus s that is the first integral means 0 to infinity eta to the power 1 minus s minus 1 into g eta d eta is G bar 1 minus s. Whereas, the second integral 0 to infinity v to the power s minus 1 into f v dv is nothing but F bar s, so that the Mellin transform of convolution of two functions f x and g x equals the F bar s into G bar 1 minus s.

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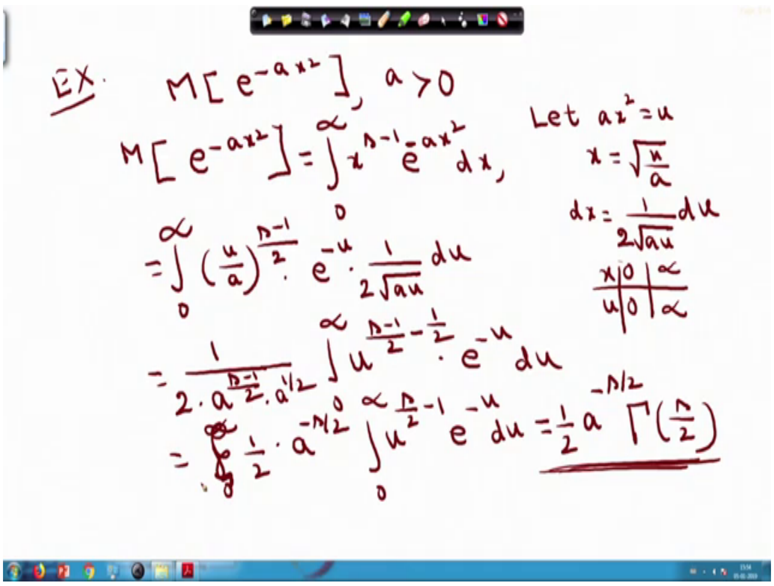

Now, using these properties, let us see; let us solve some example, so that you can understand it easily how actually they work. The first example is like this. We want to find out the Mellin transform of e power minus a x square, where it is given that a is greater than 0. So, just we have done it for the other transform; let us first find out the Mellin transform of some different type of functions. So, first one we are taking as Mellin transform of e power minus a x squared, so that the Mellin transform of e power minus a x squared, this is equals from definition you can write down 0 to infinity x to the power s minus 1 into e to the power minus a x square function is this into dx.

Now, let us substitute here say a x square equals u. So, that your x will be equals to root over u by a, we are taking the positive one. So, d x equals will be 1 by 2 into root over a u into du. And whenever your if you see, whenever your x is 0, your u is 0; whenever x is infinity, your u is also infinity, so that the limit of integration will remain the same.

So, by substitution a x square equals u, you can write down this integral as 0 to infinity limit will remain same, u by a x is root over this one. So, u by a to the power s minus 1 by two into e to the power minus u e power minus a x square, so that sorry, this is e power minus a x square 1 minus will come. So, e to the power minus u into your d x is 1 by root over twice au into du.

So, by substituting a x square equals u, where we are finding what is x is root over u by a d x you are getting limits if for changes of x what is happening on u that is the u limit of u remains the same 0 to infinity. So, this integral you are writing 0 to infinity u by a to the power s minus 1 by 2 into e to the power minus u into d x is this quantity 1 by 2 into root over au du. So, after simplification this equals you can write down 1 by 2 into a you can bring outside a to the power s minus 1 by 2 into 1 a is here a to the power half this a to the power half will be coming this is equals 0 to infinity. So, u power s minus 1 by 2 and minus half is there. So, u to the power s minus 1 by 2 minus this denominator u to the power half is there so, minus half into e to the power minus u into d u.

And once I am getting this, this equals you can write down 0 to infinity. So this equals before writing 0 to infinity, this will be equals to half into a to the power minus half plus half will be canceled, so that a to the power minus s by 2 only will come 0 to infinity u to the power this will be equals to s by 2 minus 1 minus half minus half. So, and this is equals e to the power minus u du.

And if you see this integral 0 to infinity u to the power s minus 1 by 2 into 2 to the power minus 1 this I can express it in terms of gamma function directly, so that I can write down half into a to the power minus s by 2 and value of this integral u to the power s by 2 minus 1 is there, so that it will be equals to gamma s by 2.

So, therefore, the Mellin transform of e power minus a x square is nothing but half into a to the power minus s by 2 into gamma s by 2. So, please note this one this result because afterwards also for certain other examples, we will use this particular result that Mellin transform of e power minus a x square equals half into a to the power minus s by 2 into gamma s by 2. So, I hope it is clear.

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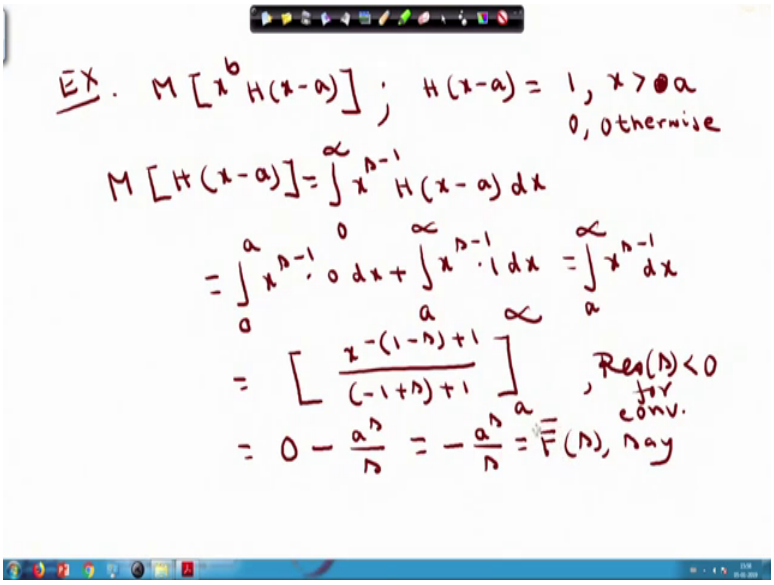

Let us go to the next example. We want to find out the Mellin transform of this thing x to the power b into H of x minus a say, where what is H, I hope you remember it, H is the Heaviside function which is defined as follows. H of x minus a this is equals to 1 for x greater than 0 and 0 otherwise. Sorry, if this is x minus a, h of x minus a, so it will not be 0, but this will be H of x minus a equals 1 for x greater than 0 and 0 otherwise. This we have discussed earlier that Heaviside function. So, we want to find out the value or Mellin transform of the function x to the power B into the H of x minus a, where H of x minus a is nothing but the Heaviside function which is defined as H of x minus a equals to 1 for x greater than a, and it is 0 otherwise.

Now, let us start with this one, what is the Mellin transform of the Heaviside function that we want to find out first. So, Mellin transform of this Heaviside function this is equals to 0 to infinity x to the power s minus 1 into h of x minus a into dx. So, as you know from this definition h of x minus a 1 for x greater than a and 0 otherwise.

So, this I can break it into two limits 0 to a x to the power s minus 1 in 0 to a value of the Heaviside function is 0 into 0 dx plus a to infinity x to the power s minus 1 into the value of the Heaviside function in this case is 1. So, this is equals to 1 d s. So, that the first integral will vanish you will have only this integral a to infinity x to the power s minus 1 into dx.

And if you evaluate this integral, this will be equals to x to the power minus 1 minus s plus 1 divided by minus 1 plus s plus 1 and the limiting value is this a to infinity. Whenever you evaluate the upper limit, the value will be 0 and the lower limit will be a minus a to the power s by s. And we are assuming that the real part of s is less than 0. This is always true for convergence. So, please note that real part of s is less than 0 from convergence we have used this one. So, value of this integral is a to the minus a to the power s by s and this is equal say F bar s we are assuming that.

Therefore, Mellin transform of the Heaviside function or unit step function H of x by a, this is equals minus a to the power s by s. So, for doing one problem, we also evaluated what is the Mellin transform of Heaviside function also. And this we are getting as minus a to the power s by s which is equals to F bar s we are assuming.

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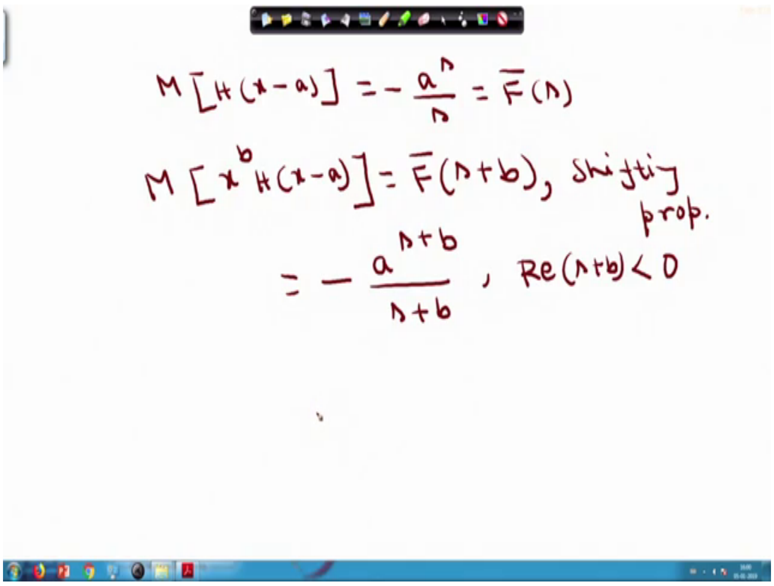

Therefore, once I have this particular property that H of x minus a this is equals to minus a to the power s by s, this is equal say F bar s. Therefore, Mellin transform of x to the power b H of x minus b, H of x minus a, this directly we can write down F bar s plus x to the power b is there s plus b, please note that we have used here the shifting property; we have used here the shifting property.

If I know that the Mellin transform of the Heaviside function H of x minus a is minus a to the power s by s which is equals F bar s say then Mellin transform of x to the power b H of x minus a, this directly we can write down using the shifting property as F bar s plus b, so that I know what is F bar s F bar s is minus e to the power is divided by s. And this equals therefore, I can write down minus e to the power s plus b by s plus b, so that s is replaced by s plus b.

And please note that the real part of s plus b should be less than 0. So, therefore, Mellin transform of the function x to the power b into H of x minus a, this is equals minus a to the power s plus b divided by s plus b, where H of x minus a is nothing but the Heaviside function. So, I hope you have understood how to find out the Mellin transform of sum functions. In the next class, we will go through some more examples, so that we can better understand how to find out the Mellin transform of some more well-known functions.