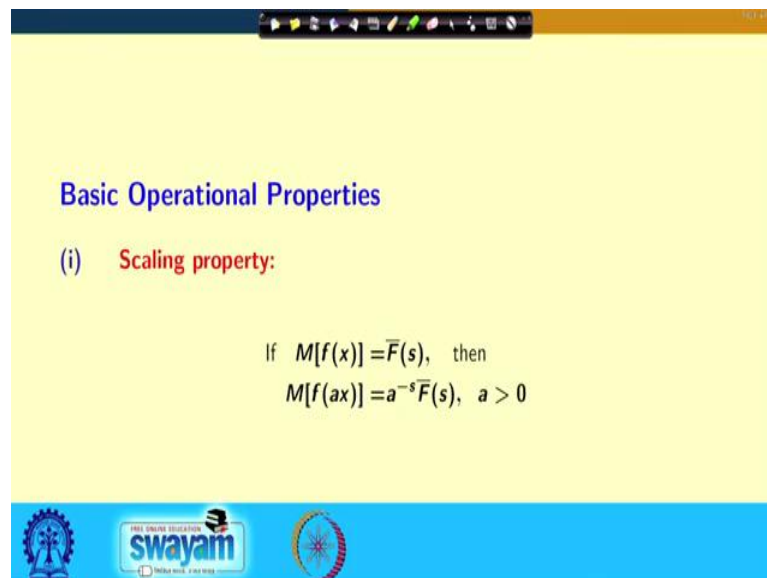


**Transform Calculus and its Applications in Differential Equations**  
**Prof. Adrijit Goswami**  
**Department of Mathematics**  
**Indian Institute of Technology, Kharagpur**

**Lecture – 55**  
**Properties of Mellin Transform**

In the last lecture, we started the Mellin transform. We had seen the definition of Mellin Transform and how to find out the Mellin transform of a function and also some of its applications. We have solved some examples also.

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**Basic Operational Properties**

(i) **Scaling property:**

If  $M[f(x)] = \bar{F}(s)$ , then  
 $M[f(ax)] = a^{-s} \bar{F}(s)$ ,  $a > 0$

In this particular lecture, what we are going to do is to study certain properties of Mellin transform. The first property is scaling property that is if  $M[f(x)] = \bar{F}(s)$ , then

$$M[f(ax)] = a^{-s} \bar{F}(s), \quad a > 0$$

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$$\begin{aligned} M[f(ax)] &= \int_0^{\infty} x^{n-1} f(ax) dx \\ &= \int_0^{\infty} \frac{v^{n-1}}{a^{n-1}} f(v) \cdot \frac{1}{a} dv \\ &= \frac{1}{a^n} \int_0^{\infty} v^{n-1} f(v) dv \\ &= \frac{1}{a^n} F(n) \end{aligned}$$

To prove the above result, let us start with the definition of Mellin transform.

$$M[f(ax)] = \int_0^{\infty} x^{s-1} f(ax) dx$$


If we put  $ax = v$  on the RHS, we will obtain,

$$\begin{aligned} M[f(ax)] &= \int_0^{\infty} \left(\frac{v}{a}\right)^{s-1} f(v) \frac{dv}{a} \\ &= a^{-s} \int_0^{\infty} v^{s-1} f(v) dv \\ &= a^{-s} \bar{F}(s) \end{aligned}$$

This completes the proof.

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**Proof:**

$$\begin{aligned} M[f(ax)] &= \int_0^{\infty} x^{s-1} f(ax) dx \\ &= \int_0^{\infty} \frac{v^{s-1}}{a^{s-1}} f(v) \frac{dv}{a} \quad [\text{put } ax = v] \\ &= \frac{1}{a^s} \int_0^{\infty} v^{s-1} f(v) dv \\ &= a^{-s} \bar{F}(s) \end{aligned}$$


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(ii) **Shifting property:**

$$M[x^a f(x)] = \bar{F}(s + a)$$

(iii)  $M[f(x^a)] = \frac{1}{a} \bar{F}\left(\frac{s}{a}\right)$

(iv)  $M\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = \bar{F}(1 - s)$

(v)  $M[\log x f(x)] = \frac{d}{ds} [\bar{F}(s)]$



The next property is the shifting property, that is,

$$M[x^a f(x)] = \bar{F}(s + a)$$

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$$\begin{aligned}M[x^a f(x)] &= \int_0^\infty x^{n-1} x^a f(x) dx \\&= \int_0^\infty x^{(n+a)-1} f(x) dx \\&= \bar{F}(n+a)\end{aligned}$$

Let us see the proof of this. From definition, we have,

$$\begin{aligned}M[x^a f(x)] &= \int_0^\infty x^{s-1} x^a f(x) dx \\&= \int_0^\infty x^{s+a-1} f(x) dx \\&= \bar{F}(s+a)\end{aligned}$$

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$$\begin{aligned}M[f(x^a)] &= \int_0^\infty x^{n-1} f(x^a) dx, \quad x^a = v \Rightarrow x = v^{1/a} \\&\quad dx = \frac{1}{a} v^{\frac{1}{a}-1} dv \\&= \int_0^\infty v^{\frac{n}{a}-1} f(v) \cdot \frac{1}{a} v^{\frac{1}{a}-1} dv \\&= \frac{1}{a} \int_0^\infty v^{\frac{n-1+1-a}{a}} \cdot f(v) dv \\&= \frac{1}{a} \int_0^\infty v^{\frac{n}{a}-1} f(v) dv \\&= \frac{1}{a} \bar{F}\left(\frac{n}{a}\right)\end{aligned}$$

Next, we will show that,

$$M[f(x^a)] = \frac{1}{a} \bar{F}\left(\frac{s}{a}\right)$$

Using the definition, we get,

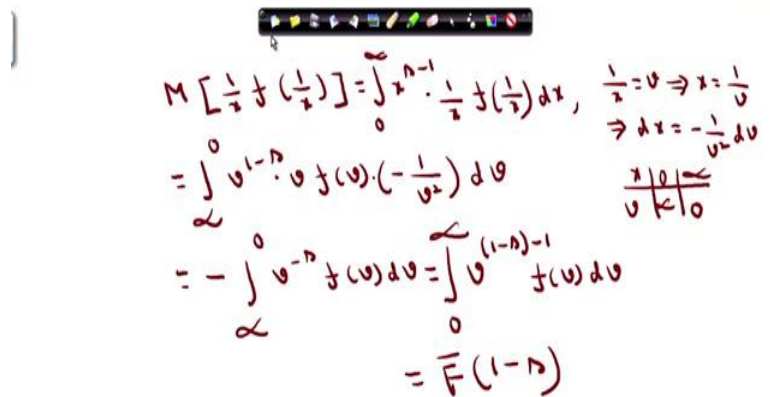
$$M[f(x^a)] = \int_0^{\infty} x^{s-1} f(x^a) dx$$

If we put  $x^a = v$  on the RHS of the above equation, then it will reduce to,

$$\begin{aligned} M[f(x^a)] &= \int_0^{\infty} v^{\frac{s-1}{a}} f(v) \frac{1}{a} v^{\frac{1}{a}-1} dv \\ &= \frac{1}{a} \int_0^{\infty} v^{\frac{s}{a}-1} f(v) dv \\ &= \frac{1}{a} \bar{F}\left(\frac{s}{a}\right) \end{aligned}$$

This completes the proof.

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$$\begin{aligned} M\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] &= \int_0^{\infty} x^{s-1} \cdot \frac{1}{x} f\left(\frac{1}{x}\right) dx, \quad \frac{1}{x} = v \Rightarrow x = \frac{1}{v} \\ &= \int_{\infty}^0 v^{1-s} \cdot v f(v) \cdot \left(-\frac{1}{v^2}\right) dv \quad \Rightarrow dx = -\frac{1}{v^2} dv \\ &= - \int_{\infty}^0 v^{-s} f(v) dv = \int_0^{\infty} v^{(1-s)-1} f(v) dv \\ &= \bar{F}(1-s) \end{aligned}$$

If  $M[f(x)] = \bar{F}(s)$ , let us find  $M\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right]$ .

Again from the definition of Mellin transform, we have,

$$M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] = \int_0^{\infty} x^{s-1} \frac{1}{x} f\left(\frac{1}{x}\right) dx$$

Let us use the substitution  $\frac{1}{x} = v$  on the RHS of the above equation. In this case  $v \rightarrow 0$  as  $x \rightarrow \infty$  and  $v \rightarrow \infty$  as  $x \rightarrow 0$ . Therefore, the above equation reduces to,

$$\begin{aligned} M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] &= \int_{\infty}^0 v^{1-s} v f(v) \left(-\frac{dv}{v^2}\right) \\ &= -\int_{\infty}^0 v^{-s} f(v) dv \end{aligned}$$

Now changing the limits of the integration, we will get,

$$M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] = \int_0^{\infty} v^{(1-s)-1} f(v) dv = \bar{F}(1-s)$$

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Handwritten derivation on a whiteboard:

$$M[(\log x)f(x)] = \int_0^{\infty} x^{s-1} (\log x) f(x) dx$$

$$\bar{F}(s) = \int_0^{\infty} x^{s-1} f(x) dx$$

$$\frac{d}{ds} \bar{F}(s) = \int_0^{\infty} \frac{d}{ds} (x^{s-1}) f(x) dx$$

$$= \int_0^{\infty} x^{s-1} (\log x) f(x) dx$$

$$= M[(\log x)f(x)]$$

Side note:  $\frac{d}{ds} (a^s) = (\log a) \cdot a^s$   
 Here  $a=x$  and  $s=s-1$

Let us move to the next problem where we will show that,

$$M[(\log x)f(x)] = \frac{d}{ds} (\bar{F}(s))$$

where,  $\bar{F}(s) = M[f(x)]$ .

From definition, we have,

$$\bar{F}(s) = \int_0^{\infty} x^{s-1} f(x) dx$$

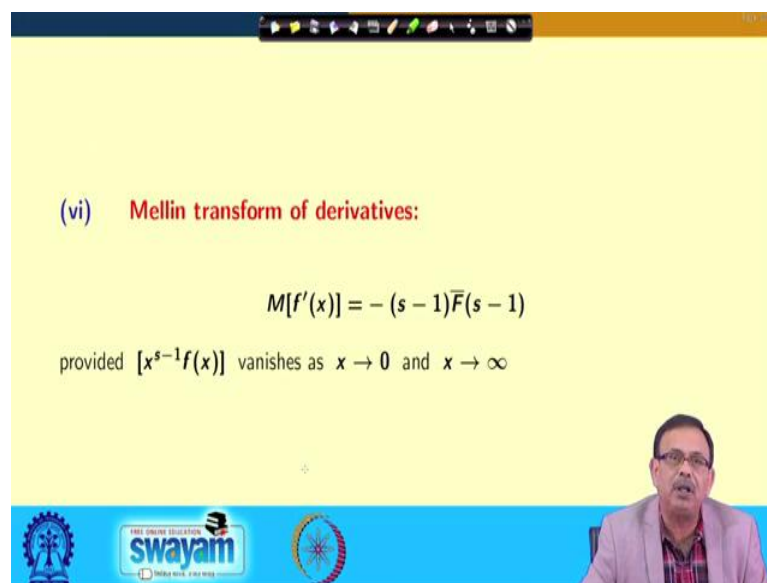
Now differentiating both sides of the above equation w.r.t.  $s$ , we will get,

$$\frac{d}{ds}(\bar{F}(s)) = \frac{d}{ds} \left( \int_0^{\infty} x^{s-1} f(x) dx \right)$$

Since the integration is with respect to  $x$ , we can take  $\frac{d}{ds}$  under integration sign i.e.,

$$\begin{aligned} \frac{d}{ds}(\bar{F}(s)) &= \int_0^{\infty} \frac{d}{ds} (x^{s-1}) f(x) dx \\ &= \int_0^{\infty} x^{s-1} \log x f(x) dx \\ &= M[(\log x) f(x)] \end{aligned}$$

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(vi) **Mellin transform of derivatives:**

$$M[f'(x)] = -(s-1)\bar{F}(s-1)$$

provided  $[x^{s-1}f(x)]$  vanishes as  $x \rightarrow 0$  and  $x \rightarrow \infty$

The next property is the Mellin transform of derivatives.

$$M[f'(x)] = -(s-1)\bar{F}(s-1)$$

provided,  $x^{s-1}f(x)$  vanishes as  $x \rightarrow 0$  and  $x \rightarrow \infty$ .

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$$\begin{aligned}
 M[f'(x)] &= \int_0^{\infty} x^{s-1} f'(x) dx \\
 &= [x^{s-1} f(x)]_0^{\infty} - (s-1) \int_0^{\infty} x^{s-2} f(x) dx \\
 &= \underbrace{[x^{s-1} f(x)]_0^{\infty}}_{\text{vanishes}} - (s-1) \int_0^{\infty} x^{s-1} \{x^{-1} f(x)\} dx \\
 &= -(s-1) \int_0^{\infty} x^{s-1} \{x^{-1} f(x)\} dx \\
 &= -(s-1) \bar{F}(s-1)
 \end{aligned}$$

From the definition, we can write,

$$M[f'(x)] = \int_0^{\infty} x^{s-1} f'(x) dx$$

Now, using integration by parts on the RHS of the above equation, we get,

$$M[f'(x)] = [x^{s-1} f(x)]_0^{\infty} - (s-1) \int_0^{\infty} x^{s-2} f(x) dx$$

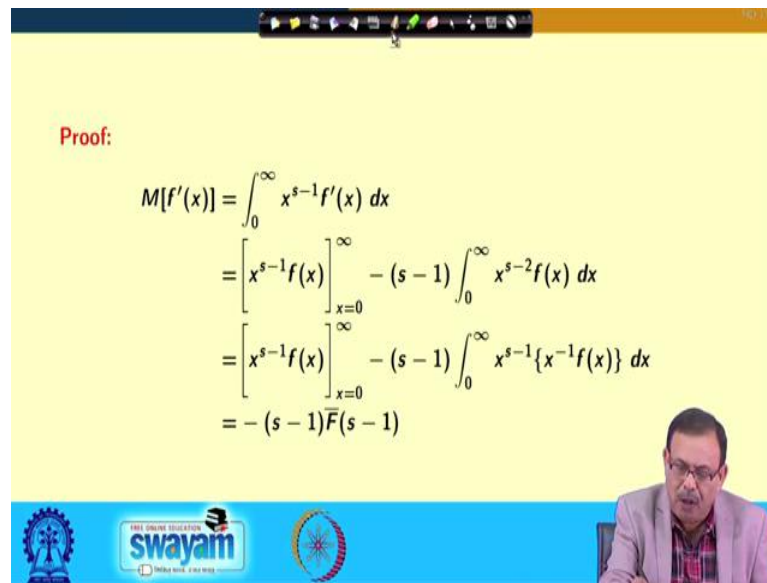
Since,  $x^{s-1} f(x)$  vanishes as  $x \rightarrow 0$  and  $x \rightarrow \infty$ , so we will get from the above equation,

$$\begin{aligned}
 M[f'(x)] &= -(s-1) \int_0^{\infty} x^{(s-1)-1} f(x) dx \\
 &= -(s-1) \bar{F}(s-1)
 \end{aligned}$$



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**Proof:**

$$\begin{aligned}
 M[f'(x)] &= \int_0^{\infty} x^{s-1} f'(x) dx \\
 &= \left[ x^{s-1} f(x) \right]_{x=0}^{\infty} - (s-1) \int_0^{\infty} x^{s-2} f(x) dx \\
 &= \left[ x^{s-1} f(x) \right]_{x=0}^{\infty} - (s-1) \int_0^{\infty} x^{s-1} \{x^{-1} f(x)\} dx \\
 &= -(s-1) \bar{F}(s-1)
 \end{aligned}$$


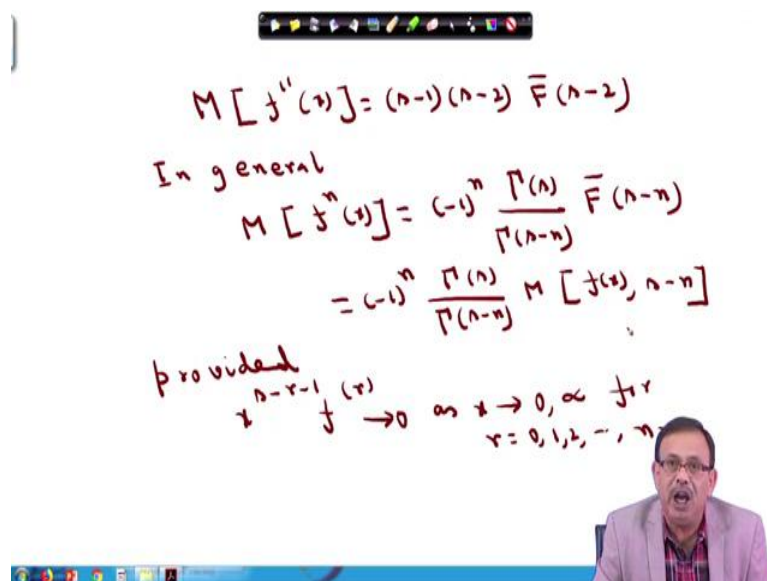
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$$M[f''(x)] = (s-1)(s-2) \bar{F}(s-2)$$

In general

$$\begin{aligned}
 M[f^n(x)] &= (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \bar{F}(s-n) \\
 &= (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} M[f(x), s-n]
 \end{aligned}$$

provided  $x^{s-r-1} f^r(x) \rightarrow 0$  as  $x \rightarrow 0, \infty$  for  $r = 0, 1, 2, \dots, n-1$



So, in the same way, we can find the Mellin transform of  $f''(x)$  as,

$$M[f''(x)] = (s-1)(s-2) \bar{F}(s-2)$$

In general,

$$M[f^n(x)] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \bar{F}(s-n)$$

provided,  $x^{s-r-1} f^r(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$  for  $r = 0, 1, 2, \dots, n-1$ .

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∴  $M[f''(x)] = (s-1)(s-2)\bar{F}(s-2)$

In general,  $M[f^{(n)}(x)] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \bar{F}(s-n)$

$$= (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} M[f(x), s-n]$$

provided  $x^{s-r-1}f^{(r)}(x) = 0$  as  $x \rightarrow 0$  for  $r = 0, 1, 2, \dots, n-1$

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(vii) If  $M[f(x)] = \bar{F}(s)$ , then  $M[xf'(x)] = -s\bar{F}(s)$  provided  $x^s f(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$

**Solution:**

$$M[xf'(x)] = \int_0^{\infty} x^s f'(x) dx$$

$$= \left[ x^s f(x) \right]_{x=0}^{\infty} - s \int_0^{\infty} x^{s-1} f(x) dx$$

$$= -s\bar{F}(s)$$

$$M[x^2 f''(x)] = (-1)^2 s(s+1)\bar{F}(s)$$

In general,  $M[x^n f^{(n)}(x)] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \bar{F}(s)$

The next property is, if  $M[f(x)] = \bar{F}(s)$ , then  $M[xf'(x)] = -s\bar{F}(s)$  provided  $x^s f(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$ .

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The image shows a handwritten derivation of the Laplace transform of  $x^n f'(x)$ . The steps are as follows:

$$\begin{aligned}M[x^n f'(x)] &= \int_0^{\infty} x^{n-1} \cdot x f'(x) dx = \int_0^{\infty} x^n f'(x) dx \\&= \left[ x^n f(x) \right]_0^{\infty} - n \int_0^{\infty} x^{n-1} f(x) dx \\&= \underbrace{0}_{\text{=0}} - n \int_0^{\infty} x^{n-1} f(x) dx \\&= -n \bar{F}(n) \\M[x^2 f'(x)] &= (-1)^2 n(n+1) \bar{F}(n) \\M[x^n f^{(n)}(x)] &= (-1)^n \frac{\Gamma(n+n)}{\Gamma(n)} \bar{F}(n)\end{aligned}$$

A man in a pink shirt is visible in the bottom right corner of the slide.

From the definition, we can write,

$$M[xf'(x)] = \int_0^{\infty} x^s f'(x) dx$$

Using integration by parts, we will get,

$$M[xf'(x)] = [x^s f(x)]_0^{\infty} - s \int_0^{\infty} x^{s-1} f(x) dx$$

Since  $x^s f(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$ , we will have,

$$M[xf'(x)] = -s \int_0^{\infty} x^{s-1} f(x) dx = -s \bar{F}(s)$$

Similarly, we can have,

$$M[x^2 f''(x)] = (-1)^2 s(s+1) \bar{F}(s)$$

In general,

$$M[x^n f^{(n)}(x)] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \bar{F}(s)$$

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(viii) Mellin transform of differential operators:

If  $M[f(x)] = \bar{F}(s)$ , then,

$$M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] = M[x^2 f''(x) + x f'(x)]$$
$$= (-1)^2 s^2 \bar{F}(s)$$

Next one is, if  $M[f(x)] = \bar{F}(s)$ , then

$$M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] = M[x^2 f''(x) + x f'(x)] = (-1)^2 s^2 \bar{F}(s)$$

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$$M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] = x \frac{d}{dx} (x f'(x))$$
$$= x [x f''(x) + f'(x)]$$
$$= M[x^2 f''(x) + x^2 f'(x)]$$

(Refer Slide Time: 27:25)

$$\begin{aligned}
 M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] &= M\left[x^2 f''(x) + 2x f'(x)\right] \\
 &= M\left[x^2 f''(x)\right] + M\left[2x f'(x)\right] \\
 &= -s \bar{F}(s) + 2(s+1) \bar{F}(s) \quad \text{(VII)} \\
 &= (-1)^2 s^2 \bar{F}(s) \\
 M\left[\left(x \frac{d}{dx}\right)^n f(x)\right] &= (-1)^n s^n \bar{F}(s)
 \end{aligned}$$

Let us see the proof.

$$\begin{aligned}
 M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] &= M\left[x^2 f''(x) + 2x f'(x)\right] \\
 &= M\left[x^2 f''(x)\right] + M\left[2x f'(x)\right]
 \end{aligned}$$

Again we know that,

$$\begin{aligned}
 M\left[2x f'(x)\right] &= -s \bar{F}(s) \\
 M\left[x^2 f''(x)\right] &= (-1)^2 s(s+1) \bar{F}(s)
 \end{aligned}$$

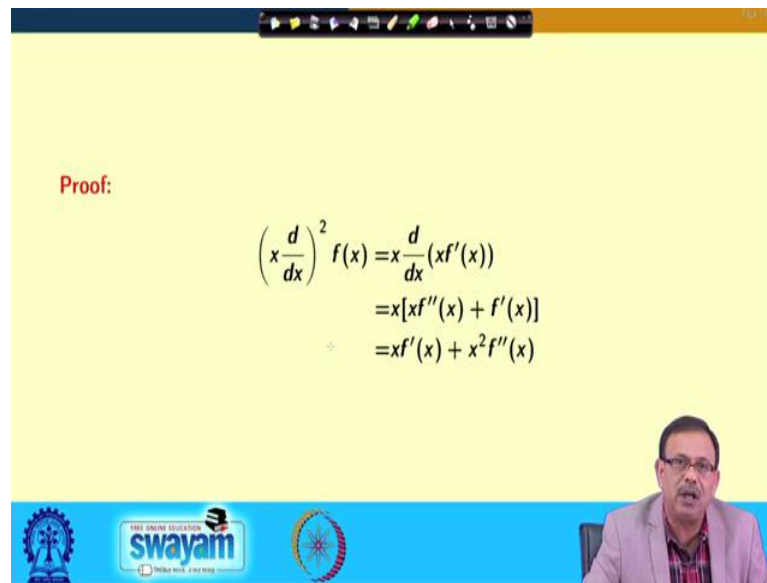
Putting these values in the above equation, we get,

$$\begin{aligned}
 M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] &= (-1)^2 s(s+1) \bar{F}(s) - s \bar{F}(s) \\
 &= (s^2 + s - s) \bar{F}(s) \\
 &= (-1)^2 s^2 \bar{F}(s)
 \end{aligned}$$

In general, we can say that,

$$M\left[\left(x \frac{d}{dx}\right)^n f(x)\right] = (-1)^n s^n \bar{F}(s)$$

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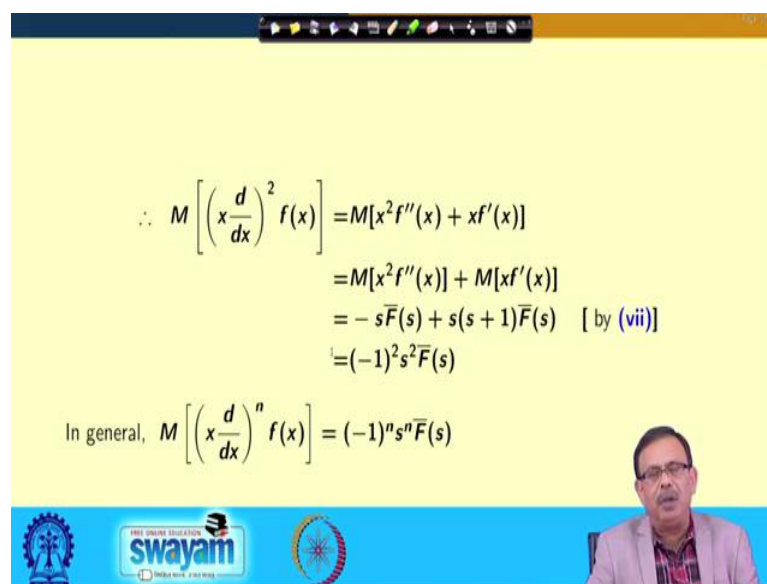


Proof:

$$\begin{aligned}\left(x \frac{d}{dx}\right)^2 f(x) &= x \frac{d}{dx}(xf'(x)) \\ &= x[xf''(x) + f'(x)] \\ &= xf'(x) + x^2 f''(x)\end{aligned}$$

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$$\begin{aligned}\therefore M\left[\left(x \frac{d}{dx}\right)^2 f(x)\right] &= M[x^2 f''(x) + xf'(x)] \\ &= M[x^2 f''(x)] + M[xf'(x)] \\ &= -s\bar{F}(s) + s(s+1)\bar{F}(s) \quad [\text{by (vii)}] \\ &= (-1)^2 s^2 \bar{F}(s)\end{aligned}$$

In general,  $M\left[\left(x \frac{d}{dx}\right)^n f(x)\right] = (-1)^n s^n \bar{F}(s)$

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So, in the next lecture, we will see the Mellin transform of integration of a function and the convolution of the Mellin transform of two functions as well as some applications of Mellin transform. Thank you.