Transform Calculus and its Applications in Differential Equations
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Lecture – 55
Properties of Mellin Transform

In the last lecture, we started the Mellin transform. We had seen the definition of Mellin Transform and how to find out the Mellin transform of a function and also some of its applications. We have solved some examples also.

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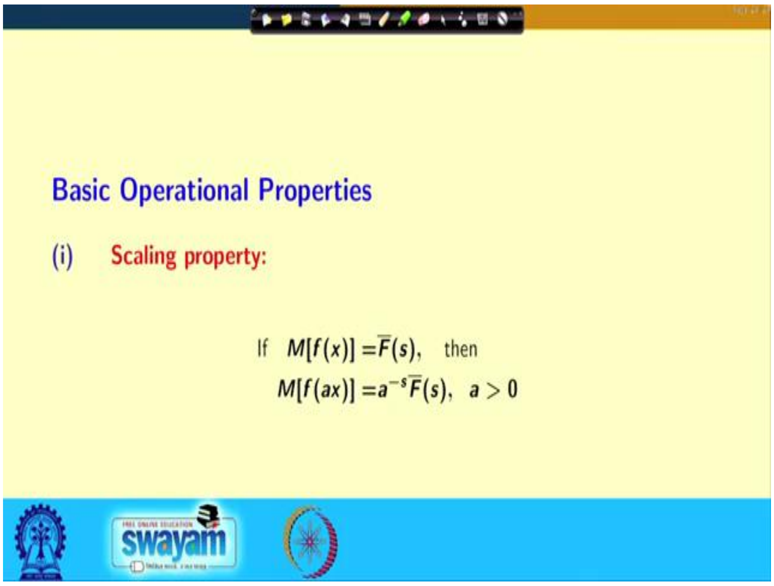

In this particular lecture, what we are going to do is to study certain properties of Mellin transform. The first property is scaling property that is if $M[f(x)] = \bar{F}(s)$, then

$$M[f(ax)] = a^{-s}\bar{F}(s), \qquad a > 0$$

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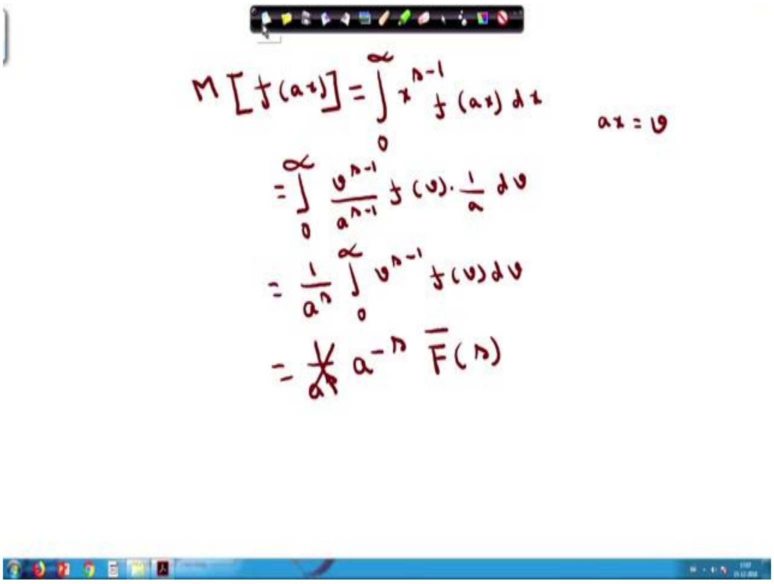

To prove the above result, let us start with the definition of Mellin transform.

$$M[f(ax)] = \int_0^\infty x^{s-1} f(ax) dx$$

If we put $ax = v$ on the RHS, we will obtain,

$$M[f(ax)] = \int_0^\infty \left(\frac{v}{a}\right)^{s-1} f(v) \frac{dv}{a}$$

$$= a^{-s} \int_0^\infty v^{s-1} f(v) dv$$

$$= a^{-s} \bar{F}(s)$$

This completes the proof.

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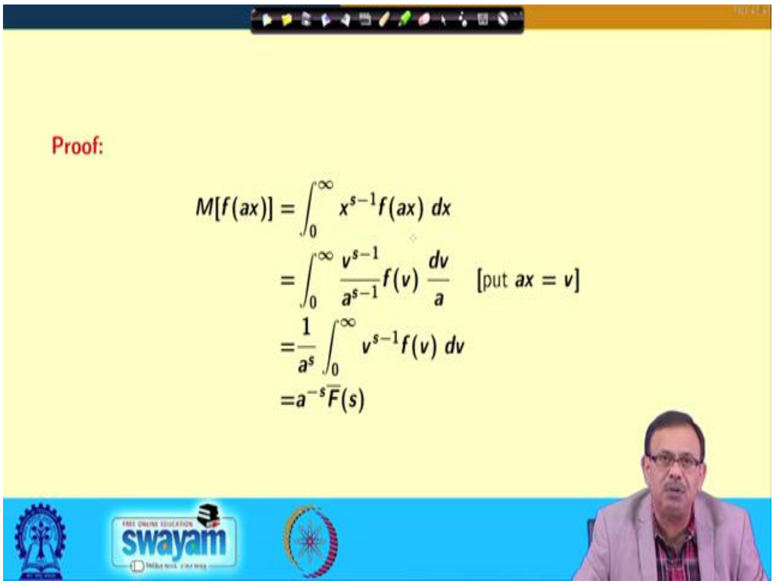

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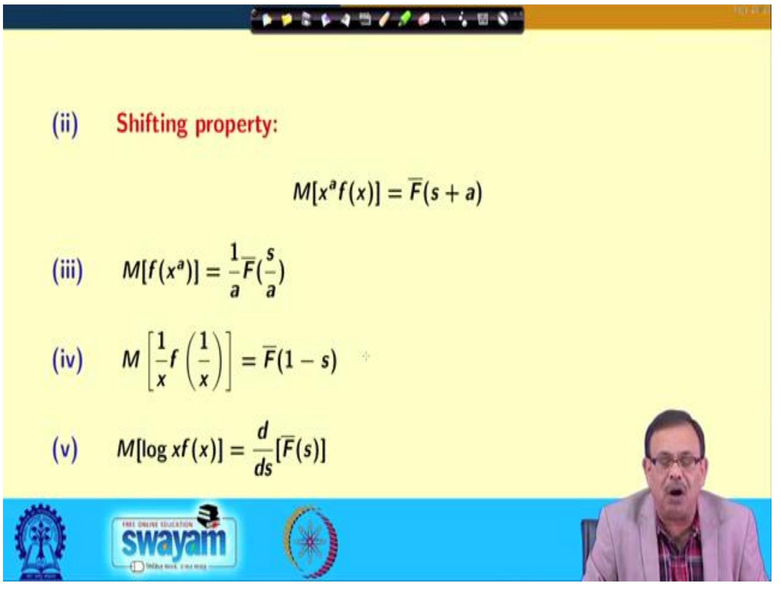

The next property is the shifting property, that is,

$$M[x^a f(x)] = \bar{F}(s+a)$$

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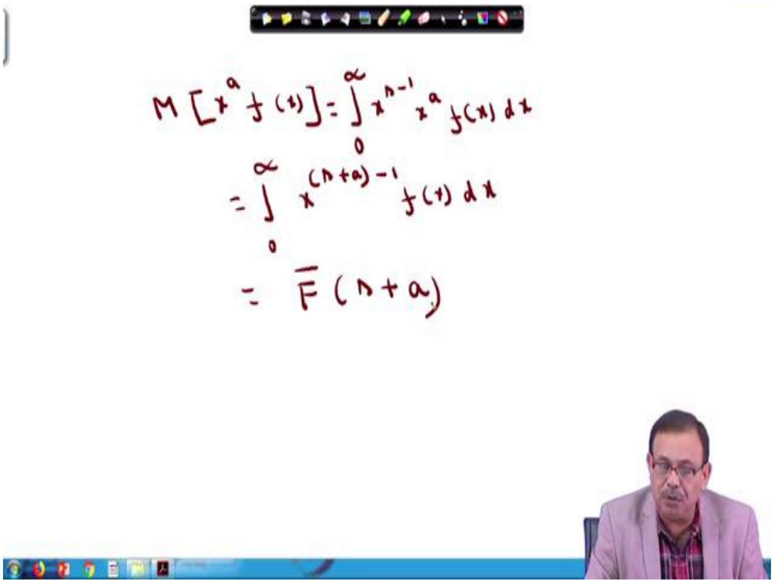

Let us see the proof of this. From definition, we have,

$$M[x^a f(x)] = \int_0^\infty x^{s-1} x^a f(x) dx$$
$$= \int_0^\infty x^{s+a-1} f(x) dx$$
$$= \bar{F}(s+a)$$

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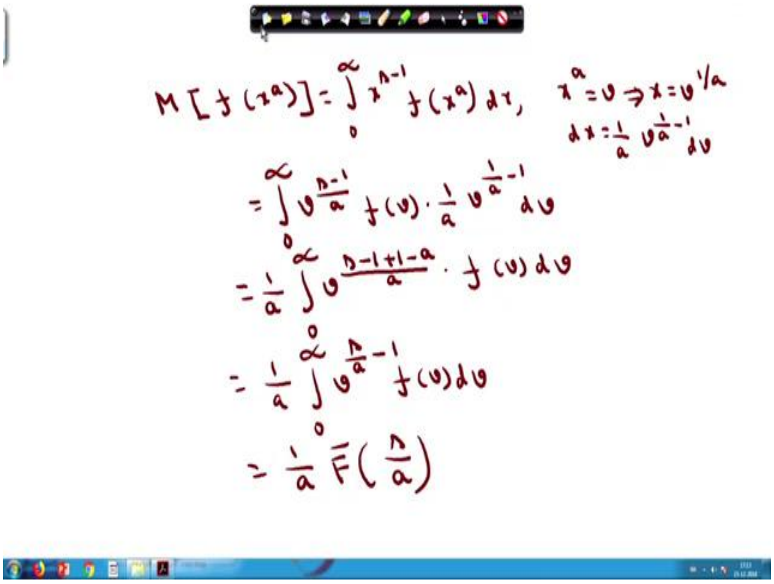

Next, we will show that,

$$M[f(x^a)] = \frac{1}{a}\bar{F}\left(\frac{s}{a}\right)$$

Using the definition, we get,

$$M[f(x^a)] = \int_0^\infty x^{s-1} f(x^a) dx$$

If we put $x^a = v$ on the RHS of the above equation, then it will reduce to,

$$\begin{aligned} M[f(x^a)] &= \int_0^\infty v^{\frac{s-1}{a}} f(v) \frac{1}{a}\, v^{\frac{1}{a}-1}\, dv \\ &= \frac{1}{a}\int_0^\infty v^{\frac{s}{a}-1} f(v) dv \\ &= \frac{1}{a}\bar{F}\left(\frac{s}{a}\right) \end{aligned}$$

This completes the proof.

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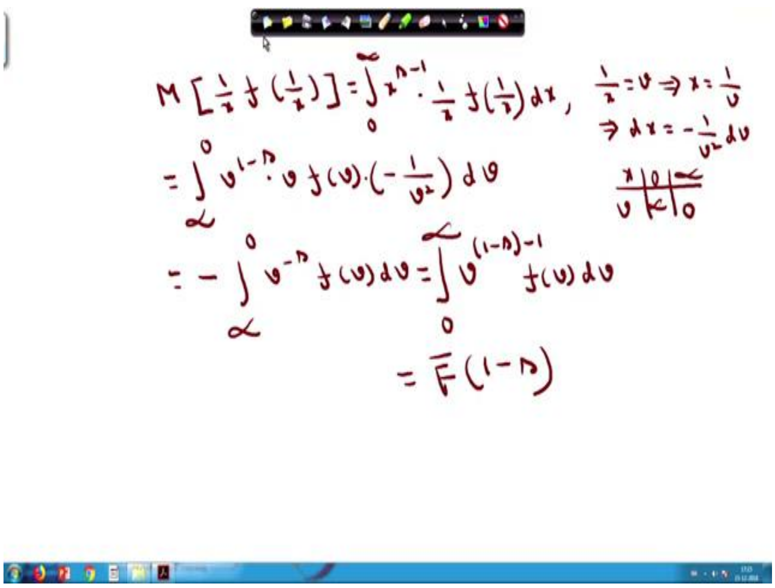

If $M[f(x)] = \bar{F}(s)$, let us find $M\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right]$.

Again from the definition of Mellin transform, we have,

$$M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=\int_0^\infty x^{s-1}\frac{1}{x}f\left(\frac{1}{x}\right)dx$$

Let us use the substitution $\frac{1}{x}=v$ on the RHS of the above equation. In this case $v\to 0$ as $x\to\infty$ and $v\to\infty$ as $x\to 0$. Therefore, the above equation reduces to,

$$M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=\int_\infty^0 v^{1-s}v\,f(v)\left(-\frac{dv}{v^2}\right)$$

$$=-\int_\infty^0 v^{-s}\,f(v)dv$$

Now changing the limits of the integration, we will get,

$$M\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=\int_0^\infty v^{(1-s)-1}\,f(v)dv=\bar{F}(1-s)$$

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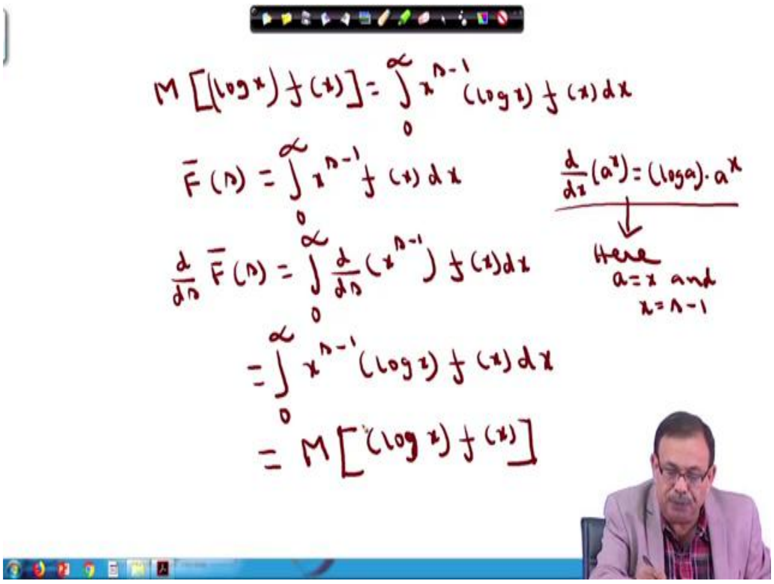

Let us move to the next problem where we will show that,

$$M[(\log x)f(x)]=\frac{d}{ds}\left(\bar{F}(s)\right)$$

where, $\bar{F}(s)=M[f(x)]$.

From definition, we have,

$$\bar{F}(s) = \int_0^\infty x^{s-1} f(x) dx$$

Now differentiating both sides of the above equation w.r.t. s, we will get,

$$\frac{d}{ds}\left(\bar{F}(s)\right) = \frac{d}{ds}\left(\int_0^\infty x^{s-1} f(x) dx\right)$$

Since the integration is with respect to s, we can take $\frac{d}{ds}$ under integration sign i.e.,

$$\begin{aligned} \frac{d}{ds}\left(\bar{F}(s)\right) &= \int_0^\infty \frac{d}{ds}(x^{s-1}) f(x) dx \\ &= \int_0^\infty x^{s-1} \log x \ f(x) dx \\ &= M[(\log x) f(x)] \end{aligned}$$

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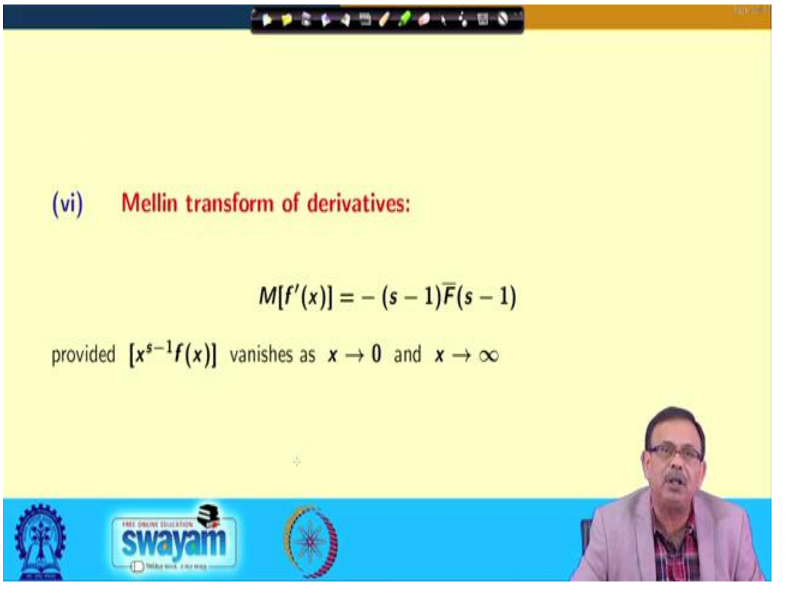

The next property is the Mellin transform of derivatives.

$$M[f'(x)] = -(s-1)\bar{F}(s-1)$$

provided, $x^{s-1} f(x)$ vanishes as $x \to 0$ and $x \to \infty$.

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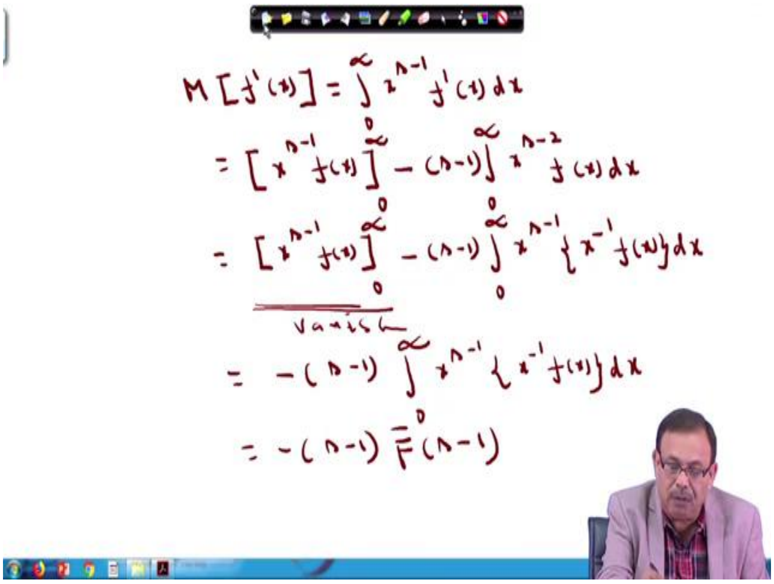

From the definition, we can write,

$$M[f'(x)] = \int_0^\infty x^{s-1} f'(x) dx$$

Now, using integration by parts on the RHS of the above equation, we get,

$$M[f'(x)] = [x^{s-1} f(x)]_0^\infty - (s-1)\int_0^\infty x^{s-2} f(x) dx$$

Since, $x^{s-1}f(x)$ vanishes as $x \to 0$ and $x \to \infty$, so we will get from the above equation,

$$M[f'(x)] = -(s-1)\int_0^\infty x^{(s-1)-1} f(x) dx$$

$$= -(s-1)\bar{F}(s-1)$$

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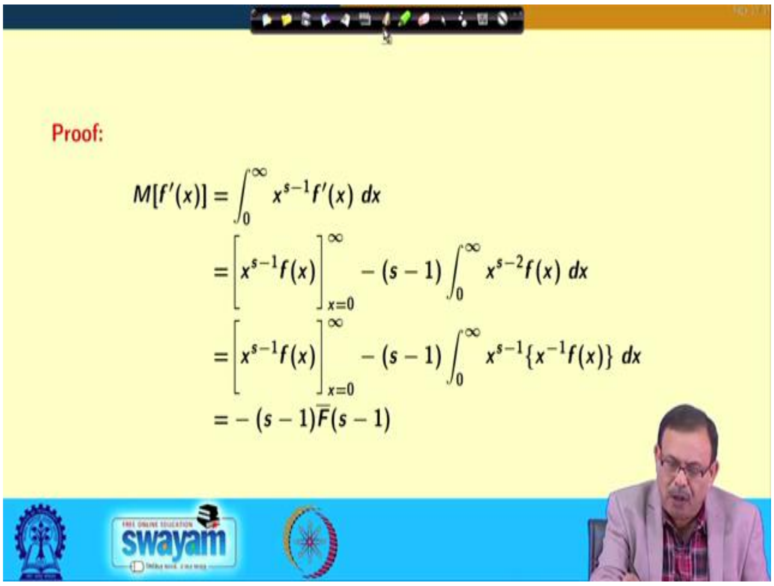

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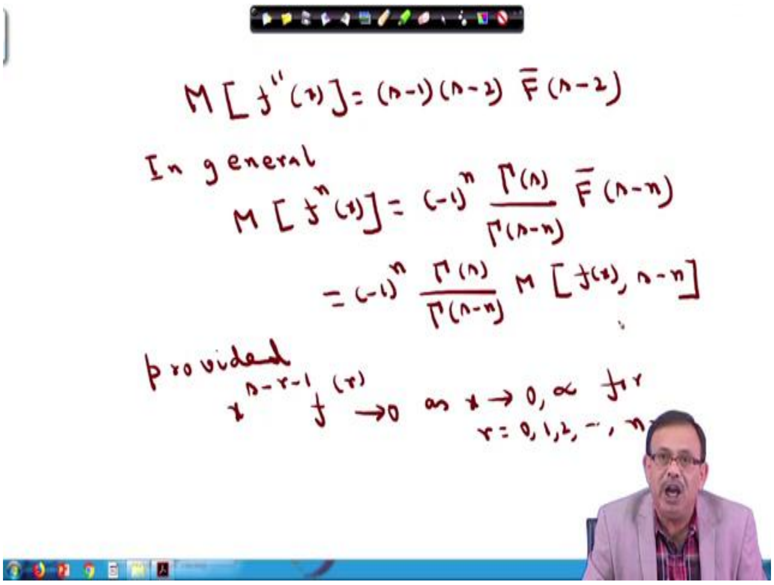

So, in the same way, we can find the Mellin transform of $f''(x)$ as,

$$M[f''(x)] = (s-1)(s-2)\bar{F}(s-2)$$

In general,

$$M[f^n(x)] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \bar{F}(s-n)$$

provided, $x^{s-r-1} f^r(x) \to 0$ as $x \to 0$ and $x \to \infty$ for $r = 0,1,2,\cdots,n-1$.

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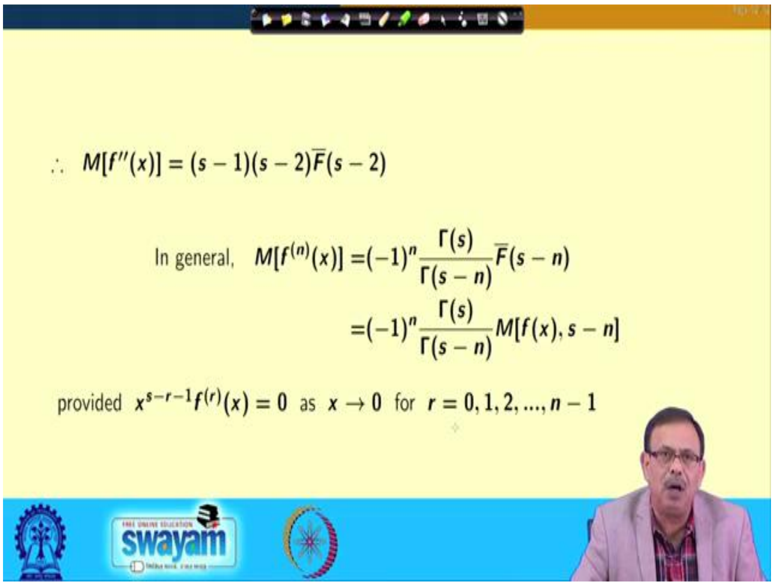

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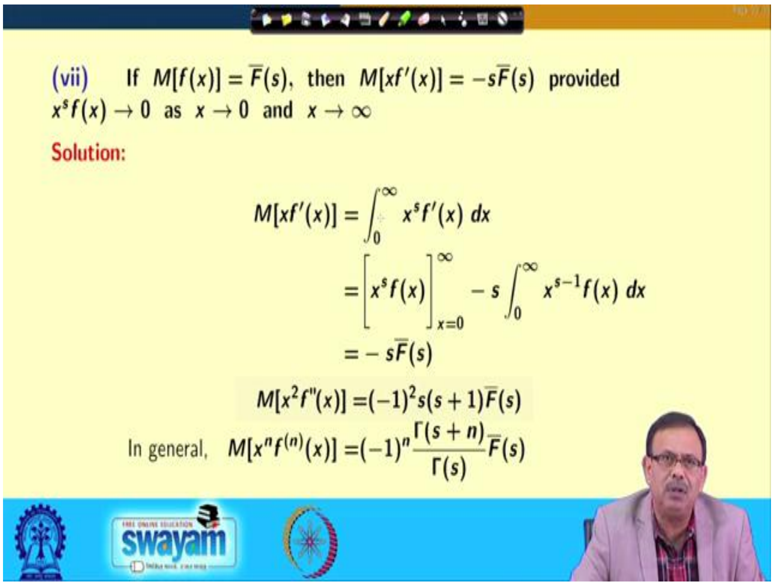

The next property is, if $M[f(x)] = \bar{F}(s)$, then $M[xf'(x)] = -s\bar{F}(s)$ provided $x^s f(x) \to 0$ as $x \to 0$ and $x \to \infty$.

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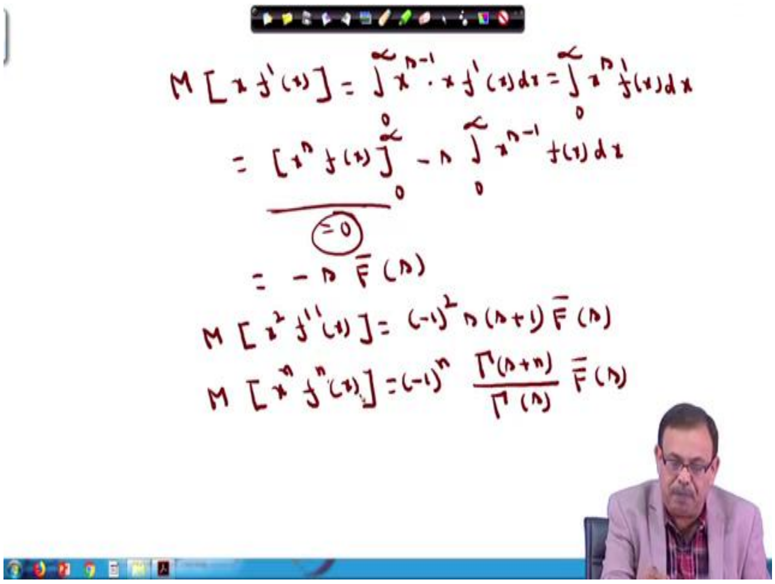

From the definition, we can write,

$$M[xf'(x)] = \int_0^\infty x^s f'(x)dx$$

Using integration by parts, we will get,

$$M[xf'(x)] = [x^s f(x)]_0^\infty - s\int_0^\infty x^{s-1} f(x)dx$$

Since $x^s f(x) \to 0$ as $x \to 0$ and $x \to \infty$, we will have,

$$M[xf'(x)] = -s\int_0^\infty x^{s-1} f(x)dx = -s\bar{F}(s)$$

Similarly, we can have,

$$M[x^2 f''(x)] = (-1)^2 s(s+1)\bar{F}(s)$$

In general,

$$M\left[x^n f^{(n)}(x)\right] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)}\bar{F}(s)$$

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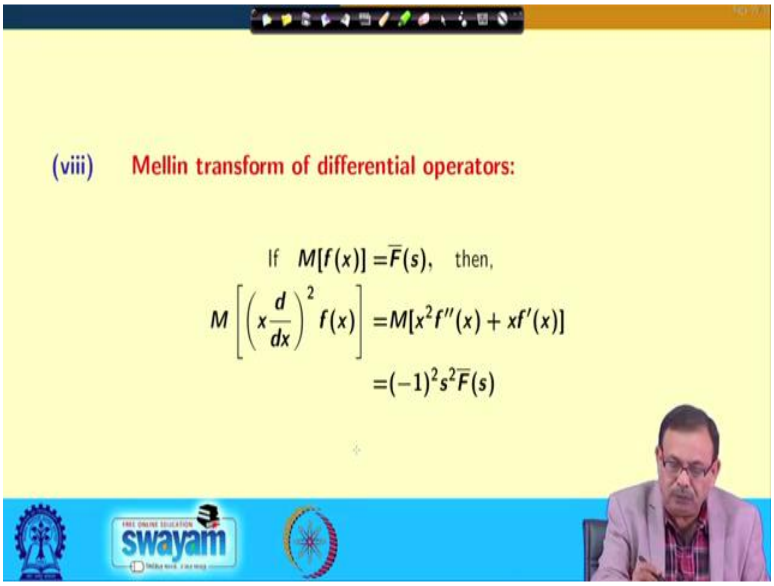

Next one is, if $M[f(x)] = \bar{F}(s)$, then

$$M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M[x^2 f''(x) + xf'(x)] = (-1)^2 s^2 \bar{F}(s)$$

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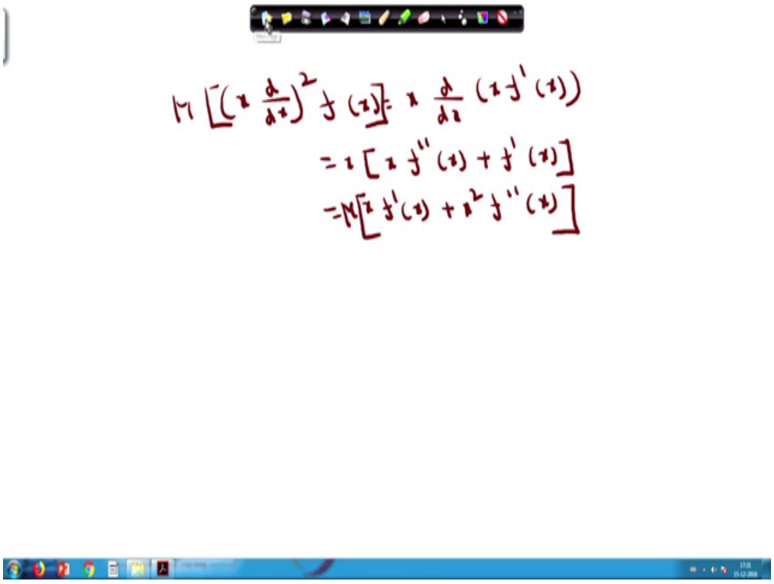

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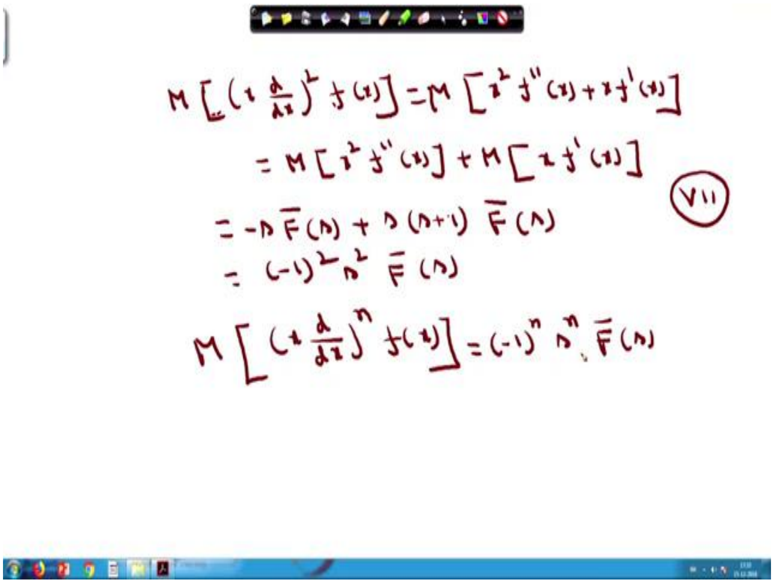

Let us see the proof.

$$M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M[x^2 f''(x) + xf'(x)]$$

$$= M[x^2 f''(x)] + M[xf'(x)]$$

Again we know that,

$$M[xf'(x)] = -s\bar{F}(s)$$

$$M[x^2 f''(x)] = (-1)^2 s(s+1)\bar{F}(s)$$

Putting these values in the above equation, we get,

$$M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = (-1)^2 s(s+1)\bar{F}(s) - s\bar{F}(s)$$

$$= (s^2 + s - s)\bar{F}(s)$$

$$= (-1)^2 s^2 \bar{F}(s)$$

In general, we can say that,

$$M\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n s^n \bar{F}(s)$$

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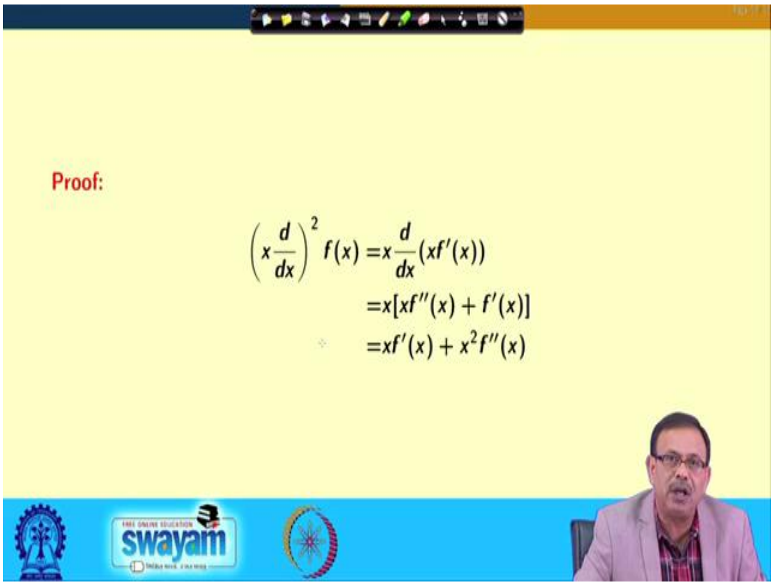

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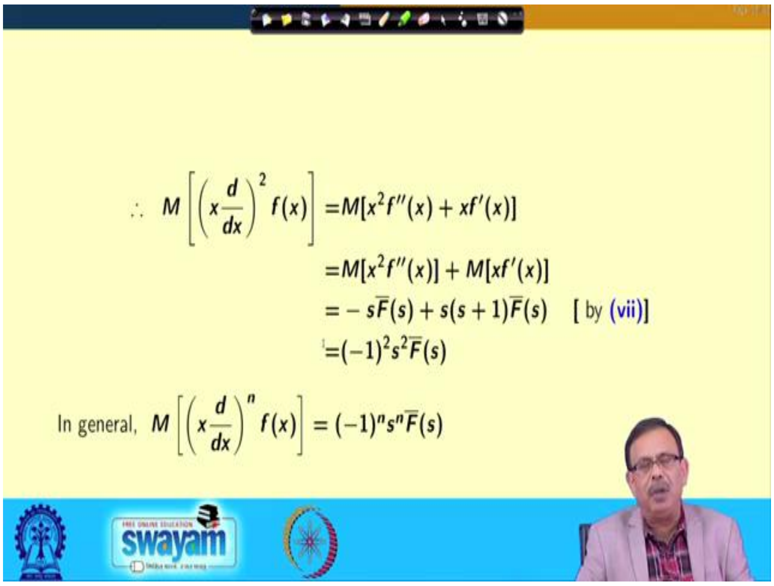

So, in the next lecture, we will see the Mellin transform of integration of a function and the convolution of the Mellin transform of two functions as well as some applications of Mellin transform. Thank you.