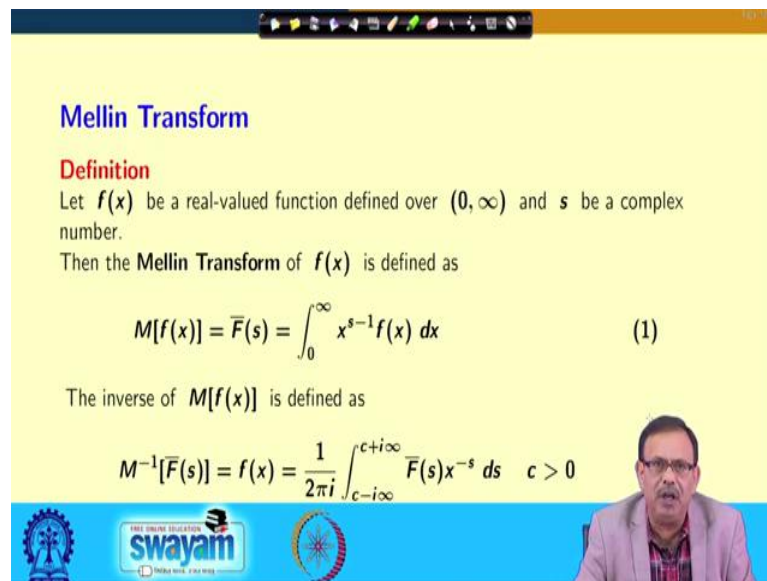


Transform Calculus and Its Applications in Differential Equations
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Indian Institute of Technology, Kharagpur

Lecture – 54
Introduction to Mellin Transform

So, till now what we have covered are the Laplace transform, Fourier transform and finite Fourier sine and cosine transform and we have also studied how to solve ordinary differential equation, partial differential equation using these transforms. Now let us come to a different transform, that is Mellin Transform. This particular transform is also used in various real life problems of physics and electronics. So, we will just see how this transform is being used.

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Mellin Transform

Definition
Let $f(x)$ be a real-valued function defined over $(0, \infty)$ and s be a complex number.
Then the Mellin Transform of $f(x)$ is defined as

$$M[f(x)] = \bar{F}(s) = \int_0^{\infty} x^{s-1} f(x) dx \quad (1)$$

The inverse of $M[f(x)]$ is defined as

$$M^{-1}[\bar{F}(s)] = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{F}(s) x^{-s} ds \quad c > 0$$

The slide also features logos for IIT Kharagpur and the Swamyam initiative at the bottom.

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The image shows handwritten mathematical formulas for the Mellin transform and its inverse. The first formula is $M[f(x)] = \bar{F}(s) = \int_0^{\infty} x^{s-1} f(x) dx$, with a circled x^{s-1} and a circled $s-1$. The second formula is $M^{-1}[\bar{F}(s)] = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{F}(s) x^{-s} ds$, with $c+i\infty$ and $c-i\infty$ written above and below the integral respectively. The background shows a presentation slide with a toolbar at the top and a taskbar at the bottom.

So, first let us go to the definition of Mellin transform. Let $f(x)$ be a real valued function defined over $(0, \infty)$ and s be a complex number. Then the Mellin transform of $f(x)$ is defined as

$$M[f(x)] = \bar{F}(s) = \int_0^{\infty} x^{s-1} f(x) dx \quad (1)$$

So, basically for Mellin transform, the kernel is x^{s-1} . So, whenever we are changing the kernel, then we are getting the other transform.

The inverse Mellin transform is defined as

$$M^{-1}[\bar{F}(s)] = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{F}(s) x^{-s} ds \quad (2)$$

where, $c > 0$. Due to the limitation of the number of lectures, we are not going to show the detailed calculations. We will directly move to the properties of the Mellin transform.

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Mellin Transform of some well-known functions

1.

$$f(x) = e^{-ax}, \quad a > 0$$
$$M[e^{-ax}] = \int_0^{\infty} x^{s-1} e^{-ax} dx$$
$$= \frac{1}{a^s} \int_0^{\infty} v^{s-1} e^{-v} dv \quad [\text{put } ax = v]$$
$$= \frac{\Gamma(s)}{a^s}$$

The slide includes a video inset of a man speaking and logos for Swamyam and other educational institutions.

Now let us see how to find the Mellin transform of some well-known functions.

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$$f(x) = e^{-ax}, \quad a > 0$$
$$M[e^{-ax}] = \int_0^{\infty} x^{s-1} e^{-ax} dx$$
$$= \frac{1}{a^s} \int_0^{\infty} v^{s-1} e^{-v} dv$$
$$= \frac{\Gamma(s)}{a^s}$$

The handwritten derivation includes annotations: $ax = v$ and $x = v/a$ with arrows pointing to the substitution steps.

Let, $f(x) = e^{-ax}, a > 0$. From the definition of Mellin transform, we have,

$$M[e^{-ax}] = \int_0^{\infty} x^{s-1} e^{-ax} dx$$

Let us substitute $ax = v$ in the integration. Then the above equation is reduced to,

$$\begin{aligned} M[e^{-ax}] &= \int_0^{\infty} \left(\frac{v}{a}\right)^{s-1} e^{-v} \frac{dv}{a} \\ &= \frac{1}{a^s} \int_0^{\infty} v^{s-1} e^{-v} dv \end{aligned}$$

The integral on the RHS is nothing but the Gamma function.

$$\therefore M[e^{-ax}] = \frac{\Gamma(s)}{a^s}$$

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2.

$$\begin{aligned} f(x) &= \frac{1}{1+x} \\ M\left[\frac{1}{1+x}\right] &= \int_0^{\infty} x^{s-1} \frac{1}{1+x} dx \\ &= \int_0^1 \frac{v^{s-1}}{(1-v)^{s-1}} (1-v) \frac{dv}{(1-v)^2} \left[\text{put } x = \frac{v}{1-v} \right] \\ &= \int_0^1 v^{s-1} (1-v)^{(1-s)-1} dv \\ &= B(s, 1-s) \quad [\text{where } B(s, 1-s) \text{ is the Beta function}] \\ &= \Gamma(s)\Gamma(1-s) \end{aligned}$$

Now we will try to find out the Mellin Transform of $\frac{1}{1+x}$

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$$\begin{aligned}
 f(x) &= \frac{1}{1+x} \\
 M\left[\frac{1}{1+x}\right] &= \int_0^{\infty} x^{n-1} \cdot \frac{1}{1+x} dx, \\
 &= \int_0^1 \frac{v^{n-1} (1-v)}{(1-v)^{n-1} (1-v)^2} dv \\
 &= \int_0^1 v^{n-1} (1-v)^{(1-n)-1} dv \\
 &= B(n, 1-n) = \Gamma(n) \Gamma(1-n)
 \end{aligned}$$

$$\begin{aligned}
 x &= \frac{v}{1-v} \\
 v &= \frac{x}{1+x} \\
 1-v &= 1 - \frac{x}{1+x} \\
 &= \frac{1}{1+x}
 \end{aligned}$$

From the definition of Mellin transform, we have,

$$M\left[\frac{1}{1+x}\right] = \int_0^{\infty} x^{s-1} \frac{1}{1+x} dx$$

Let us substitute $x = \frac{v}{1-v}$ in the integration. Then the above equation is reduced to,

$$\begin{aligned}
 M\left[\frac{1}{1+x}\right] &= \int_0^1 \left(\frac{v}{1-v}\right)^{s-1} (1-v) \frac{dv}{(1-v)^2} \\
 &= \int_0^1 v^{s-1} (1-v)^{(1-s)-1} dv
 \end{aligned}$$

The integral on the RHS is Beta function.

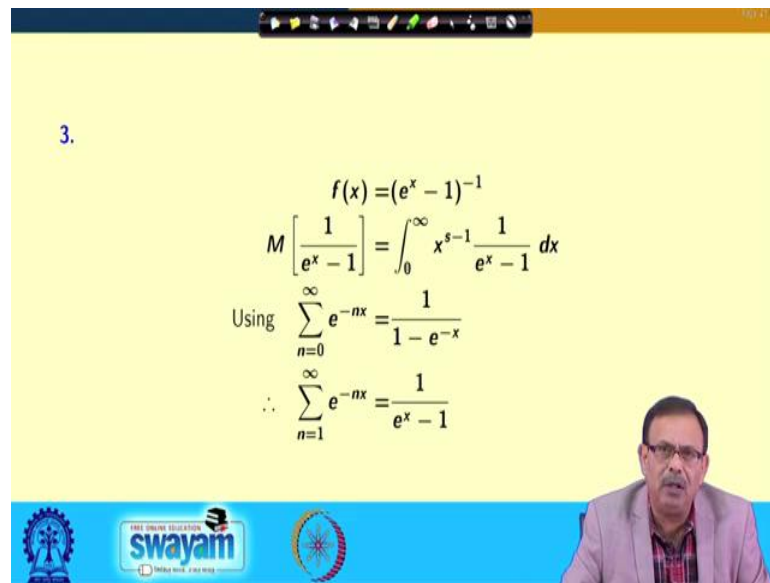
$$\begin{aligned}
 \therefore M\left[\frac{1}{1+x}\right] &= B(s, 1-s) \\
 &= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s+1-s)} \\
 &= \Gamma(s)\Gamma(1-s) \quad [\because \Gamma(1) = 1]
 \end{aligned}$$

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3.

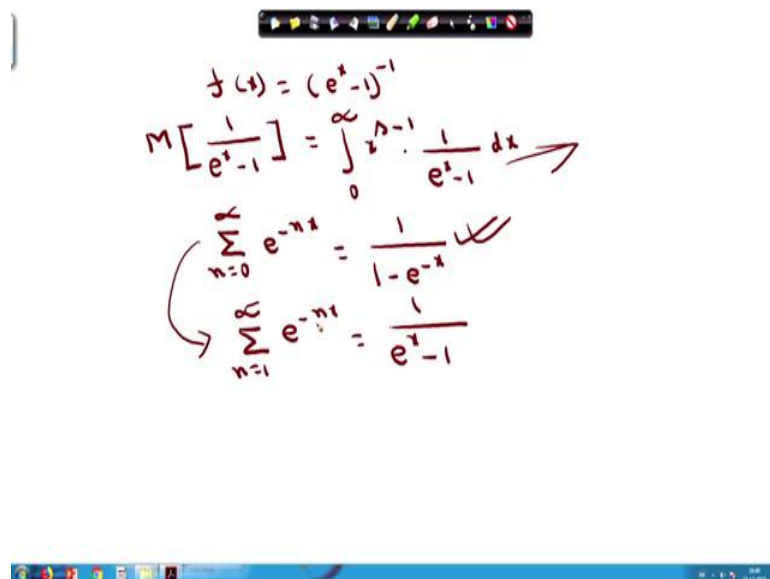
$$f(x) = (e^x - 1)^{-1}$$
$$M\left[\frac{1}{e^x - 1}\right] = \int_0^{\infty} x^{s-1} \frac{1}{e^x - 1} dx$$

Using $\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1 - e^{-x}}$

$$\therefore \sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x - 1}$$


Next we will find the Mellin transform of $f(x) = \frac{1}{e^x - 1}$

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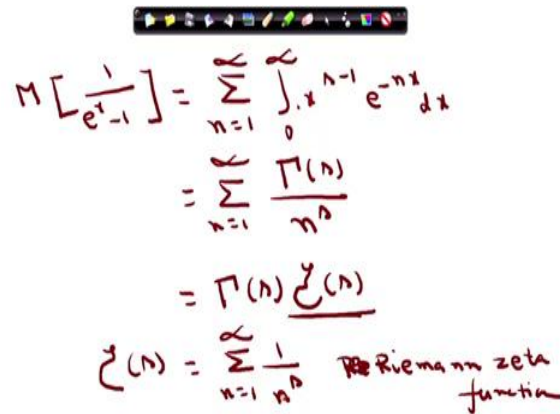
From the definition of Mellin transform, we have,

$$M\left[\frac{1}{e^x - 1}\right] = \int_0^{\infty} x^{s-1} \frac{1}{e^x - 1} dx$$

Using the infinite series expansion of $\frac{1}{1-x}$, we have,

$$\begin{aligned}\frac{1}{1-e^{-x}} &= 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots \\ \Rightarrow \frac{e^x}{e^x - 1} &= 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots \\ \Rightarrow \frac{1}{e^x - 1} &= e^{-x} + e^{-2x} + e^{-3x} + \dots \\ \Rightarrow \frac{1}{e^x - 1} &= \sum_{n=1}^{\infty} e^{-nx}\end{aligned}$$

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Handwritten derivation:

$$\begin{aligned}M\left[\frac{1}{e^x - 1}\right] &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{n-1} e^{-nx} dx \\ &= \sum_{n=1}^{\infty} \frac{\Gamma(n)}{n^n} \\ &= \Gamma(n) \zeta(n) \\ \zeta(n) &= \sum_{n=1}^{\infty} \frac{1}{n^n} \quad \text{Riemann zeta function}\end{aligned}$$

Using above relation, we will have,

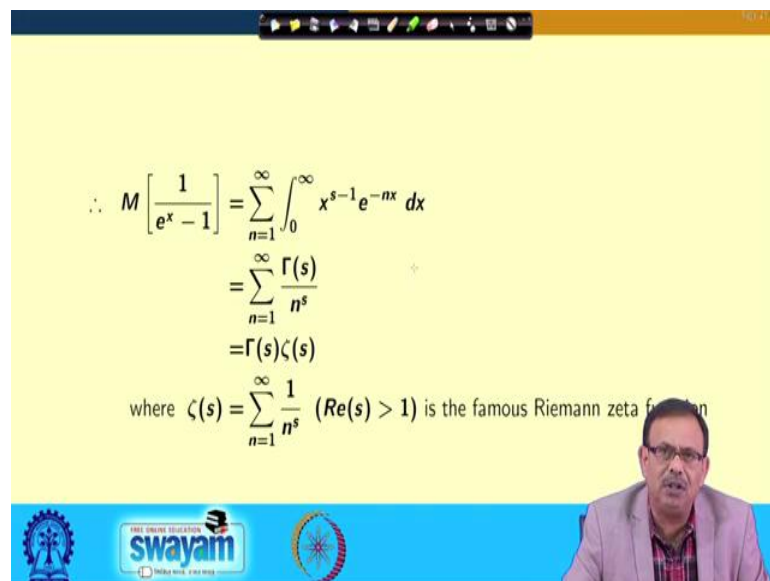
$$M\left[\frac{1}{e^x - 1}\right] = \int_0^{\infty} x^{s-1} \left(\sum_{n=1}^{\infty} e^{-nx} \right) dx$$

Now, taking the summation outside the integral sign, we get,

$$\begin{aligned}M\left[\frac{1}{e^x - 1}\right] &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx \\&= \sum_{n=1}^{\infty} M[e^{-nx}] \\&= \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} \\&= \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} \\&= \Gamma(s)\zeta(s)\end{aligned}$$

where, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is Riemann zeta function provided $Re(s) > 1$.

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The screenshot shows a slide with a yellow background. At the top, there is a navigation bar with various icons. The main content of the slide is the following derivation:

$$\begin{aligned}\therefore M\left[\frac{1}{e^x - 1}\right] &= \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx \\&= \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} \\&= \Gamma(s)\zeta(s)\end{aligned}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ($Re(s) > 1$) is the famous Riemann zeta function

In the bottom right corner, there is a small video inset showing a man with glasses and a beard, wearing a light-colored jacket, speaking. At the bottom of the slide, there are logos for 'swayam' and 'All India Institute of Space Sciences'.

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4.

$$f(x) = \frac{1}{(1+x)^n}$$

$$M\left[\frac{1}{(1+x)^n}\right] = \int_0^{\infty} x^{s-1}(1+x)^{-n} dx$$

$$= \int_0^1 v^{s-1}(1-v)^{n-s-1} dv \quad \left[\text{put } x = \frac{v}{1-v} \right]$$

$$= B(s, n-s)$$

$$= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}$$

Our next problem is say, $f(x) = \frac{1}{(1+x)^n}$

(Refer Slide Time: 19:50)

$$f(x) = \frac{1}{(1+x)^n}$$

$$M\left[\frac{1}{(1+x)^n}\right] = \int_0^{\infty} x^{s-1}(1+x)^{-n} dx,$$

$$= \int_0^1 v^{s-1}(1-v)^{n-s-1} dv$$

$$= B(s, n-s)$$

$$= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}$$

$$x = \frac{v}{1-v}$$

$$v = \frac{x}{1+x}$$

$$1-v = \frac{1}{1+x}$$

Using the definition of Mellin transform, we have,

$$M\left[\frac{1}{(1+x)^n}\right] = \int_0^{\infty} x^{s-1} \frac{1}{(1+x)^n} dx$$

Let us substitute $x = \frac{v}{1-v}$ in the integration. Then the above equation is reduced to,

$$\begin{aligned} M\left[\frac{1}{(1+x)^n}\right] &= \int_0^1 \left(\frac{v}{1-v}\right)^{s-1} (1-v)^n \frac{dv}{(1-v)^2} \\ &= \int_0^1 v^{s-1} (1-v)^{(n-s)-1} dv \end{aligned}$$

The integral on the RHS is well-known Beta function.

$$\begin{aligned} \therefore M\left[\frac{1}{(1+x)^n}\right] &= B(s, n-s) \\ &= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(s+n-s)} \\ &= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)} \end{aligned}$$

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5. Mellin transform of $\sin kx$ and $\cos kx$

$$\begin{aligned} M[e^{-ikx}] &= \frac{\Gamma(s)}{(ik)^s} \quad [\text{using 1.}] \\ &= \frac{\Gamma(s)}{k^s} \left[\cos \frac{s\pi}{2} - i \sin \frac{s\pi}{2} \right] \end{aligned}$$

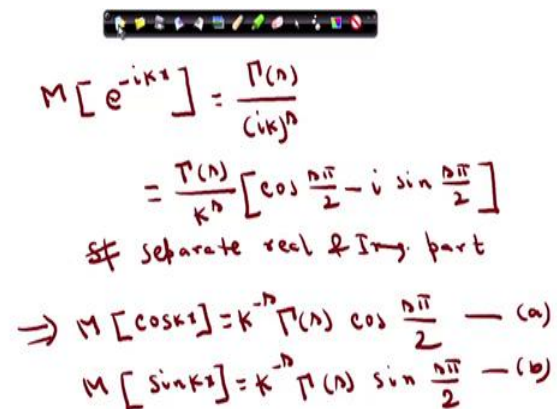
Separating real and imaginary parts

$$\Rightarrow M[\cos kx] = k^{-s} \Gamma(s) \cos \frac{s\pi}{2} \quad (a)$$

$$M[\sin kx] = k^{-s} \Gamma(s) \sin \frac{s\pi}{2} \quad (b)$$

Now, we want to find out the Mellin transform of $\sin kx$ and $\cos kx$.

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$$\begin{aligned}M[e^{-ikx}] &= \frac{\Gamma(n)}{(ik)^n} \\&= \frac{\Gamma(n)}{k^n} \left[\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right] \\&\text{separate real \& Imag. part} \\ \Rightarrow M[\cos kx] &= k^{-n} \Gamma(n) \cos \frac{n\pi}{2} \quad \text{--- (a)} \\ M[\sin kx] &= k^{-n} \Gamma(n) \sin \frac{n\pi}{2} \quad \text{--- (b)}\end{aligned}$$

From the first problem, we can write down,

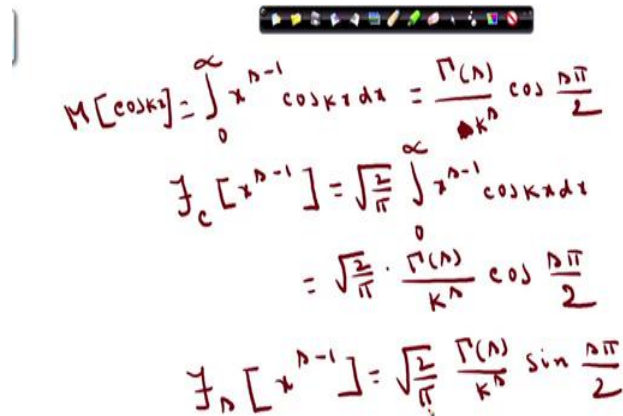
$$\begin{aligned}M[e^{-ikx}] &= \frac{\Gamma(s)}{(ik)^s} \\&= \frac{\Gamma(s)}{k^s} (-i)^s \\&= \frac{\Gamma(s)}{k^s} \left(\cos \frac{s\pi}{2} - i \sin \frac{s\pi}{2} \right)\end{aligned}$$

Now comparing the real and imaginary parts from the above equation, we have,

$$M[\cos kx] = \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2} \quad (a)$$

$$M[\sin kx] = \frac{\Gamma(s)}{k^s} \sin \frac{s\pi}{2} \quad (b)$$

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The image shows handwritten mathematical derivations. At the top, there is a toolbar with various drawing tools. The first equation is the Mellin transform of $\cos kx$: $M[\cos kx] = \int_0^{\infty} x^{n-1} \cos kx dx = \frac{\Gamma(n)}{k^n} \cos \frac{n\pi}{2}$. Below it, the Fourier cosine transform of x^{n-1} is derived: $\mathcal{F}_c[x^{n-1}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{n-1} \cos kx dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\Gamma(n)}{k^n} \cos \frac{n\pi}{2}$. The final equation shows the Fourier sine transform of x^{n-1} : $\mathcal{F}_s[x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n)}{k^n} \sin \frac{n\pi}{2}$.

From the definition of Mellin transform and the result obtained in (a), we can have,

$$\int_0^{\infty} x^{s-1} \cos kx dx = \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2}$$

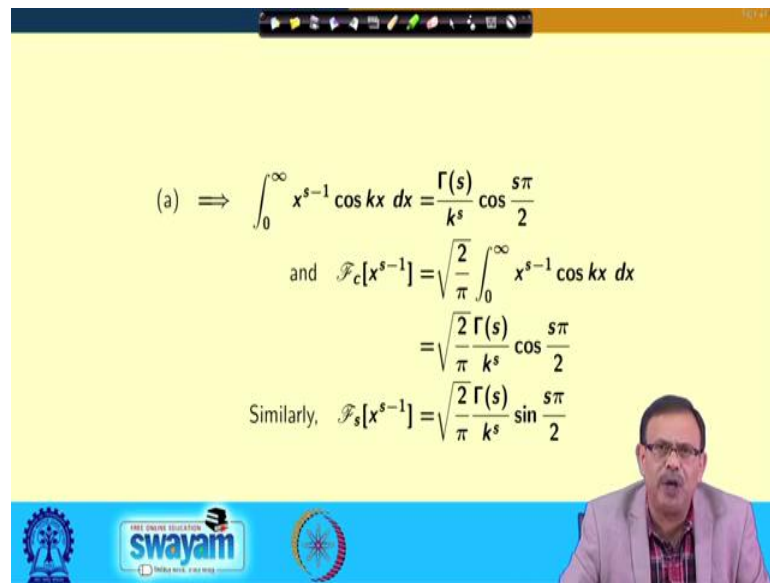
Now from the definition of Fourier cosine transform, we have,

$$\begin{aligned} \mathcal{F}_c[x^{s-1}] &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{s-1} \cos kx dx \\ &= \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2} \end{aligned}$$

Similarly,

$$\mathcal{F}_s[x^{s-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \sin \frac{s\pi}{2}$$

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$$(a) \Rightarrow \int_0^{\infty} x^{s-1} \cos kx \, dx = \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2}$$
$$\text{and } \mathcal{F}_c[x^{s-1}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{s-1} \cos kx \, dx$$
$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2}$$
$$\text{Similarly, } \mathcal{F}_s[x^{s-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \sin \frac{s\pi}{2}$$

The slide features a yellow background with mathematical derivations. At the bottom, there is a blue banner with logos for Swamyam (Free Online Education) and other institutions. A small video inset of a man is visible in the bottom right corner.

Thank you.