Transform Calculus and Its Applications in Differential Equations
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Lecture – 54
Introduction to Mellin Transform

So, till now what we have covered are the Laplace transform, Fourier transform and finite Fourier sine and cosine transform and we have also studied how to solve ordinary differential equation, partial differential equation using these transforms. Now let us come to a different transform, that is Mellin Transform. This particular transform is also used in various real life problems of physics and electronics. So, we will just see how this transform is being used.

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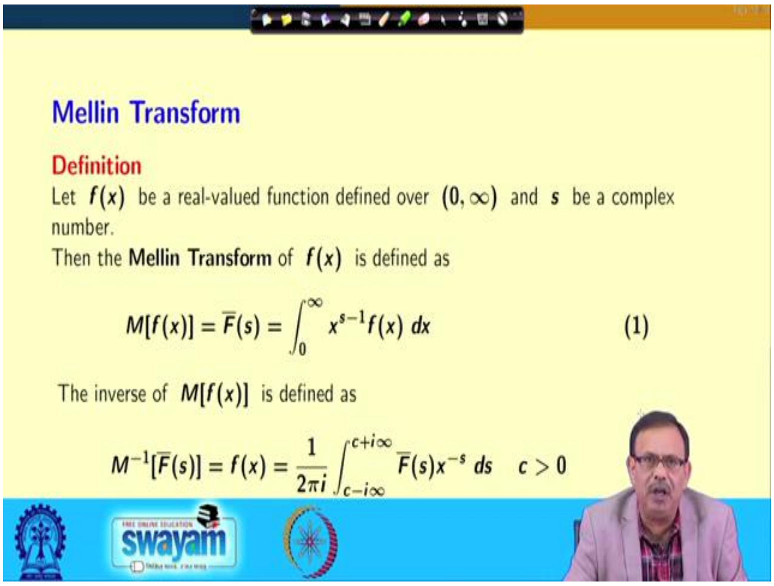

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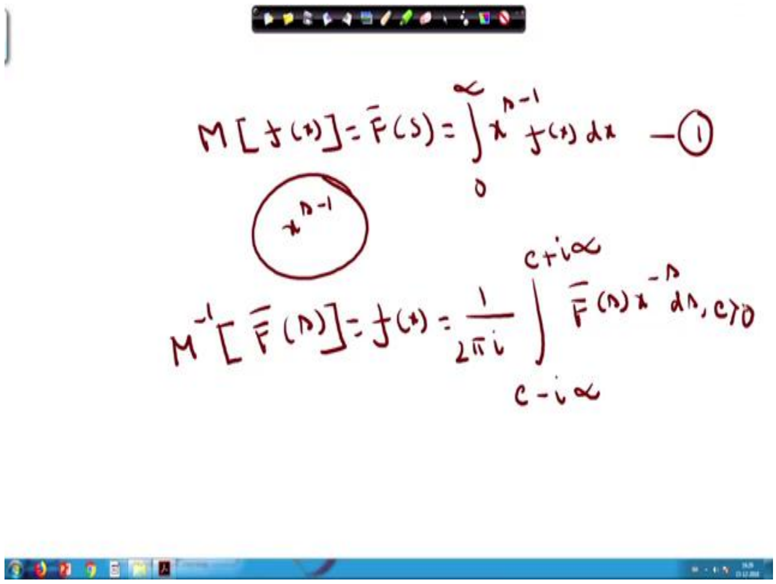

So, first let us go to the definition of Mellin transform. Let $f(x)$ be a real valued function defined over $(0, \infty)$ and s be a complex number. Then the Mellin transform of $f(x)$ is defined as

$$M[f(x)] = \bar{F}(s) = \int_0^\infty x^{s-1} f(x) dx \qquad (1)$$

So, basically for Mellin transform, the kernel is x^{s-1}. So, whenever we are changing the kernel, then we are getting the other transform.

The inverse Mellin transform is defined as

$$M^{-1}[\bar{F}(s)] = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{F}(s)\, x^{-s} ds \qquad (2)$$

where, $c > 0$. Due to the limitation of the number of lectures, we are not going to show the detailed calculations. We will directly move to the properties of the Mellin transform.

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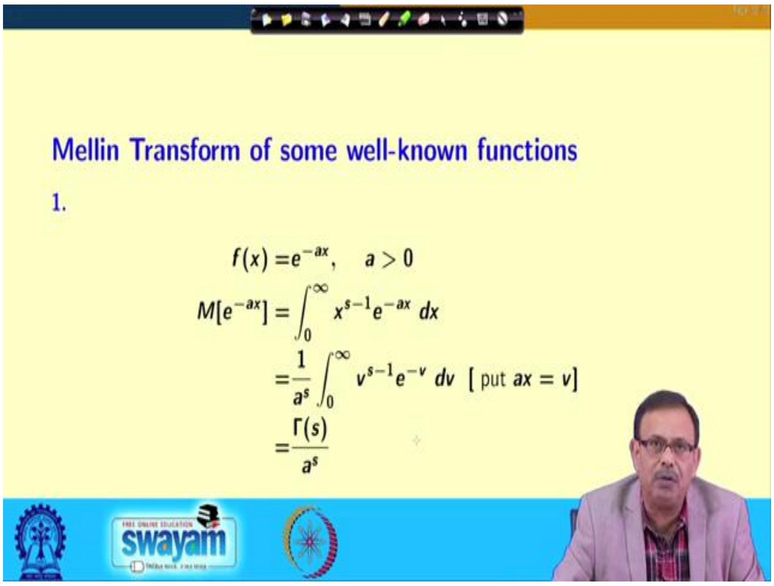

Now let us see how to find the Mellin transform of some well-known functions.

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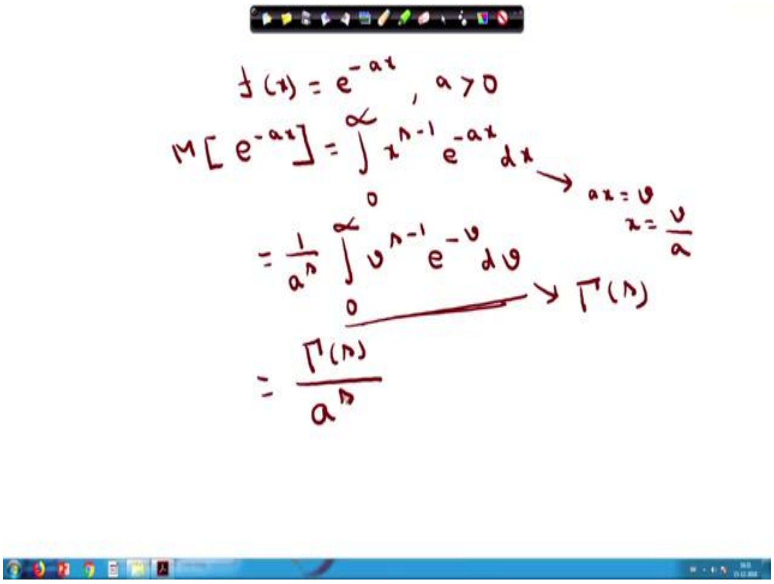

Let, $f(x) = e^{-ax}, a > 0$. From the definition of Mellin transform, we have,

$$M[e^{-ax}] = \int_0^\infty x^{s-1} e^{-ax} dx$$

Let us substitute $ax = v$ in the integration. Then the above equation is reduced to,

$$M[e^{-ax}] = \int_0^{\infty} \left(\frac{v}{a}\right)^{s-1} e^{-v} \frac{dv}{a}$$
$$= \frac{1}{a^s} \int_0^{\infty} v^{s-1} e^{-v} dv$$

The integral on the RHS is nothing but the Gamma function.

$$\therefore \; M[e^{-ax}] = \frac{\Gamma(s)}{a^s}$$

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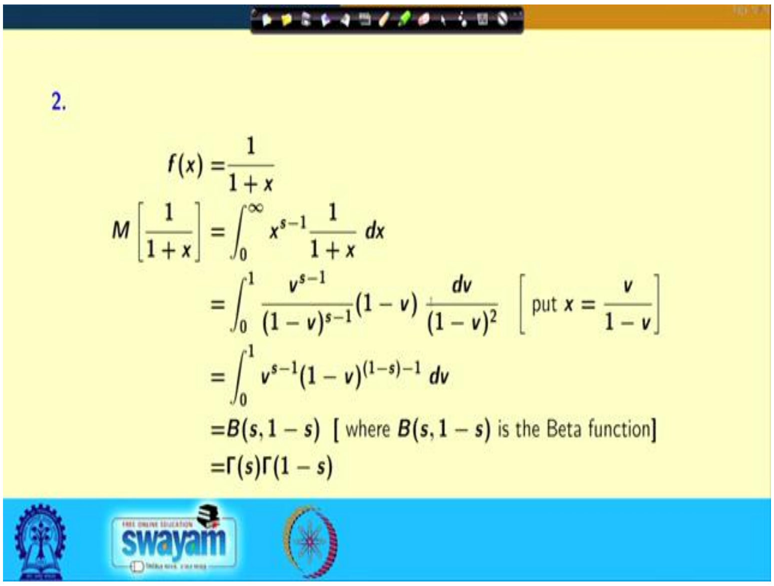

Now we will try to find out the Mellin Transform of $\frac{1}{1+x}$

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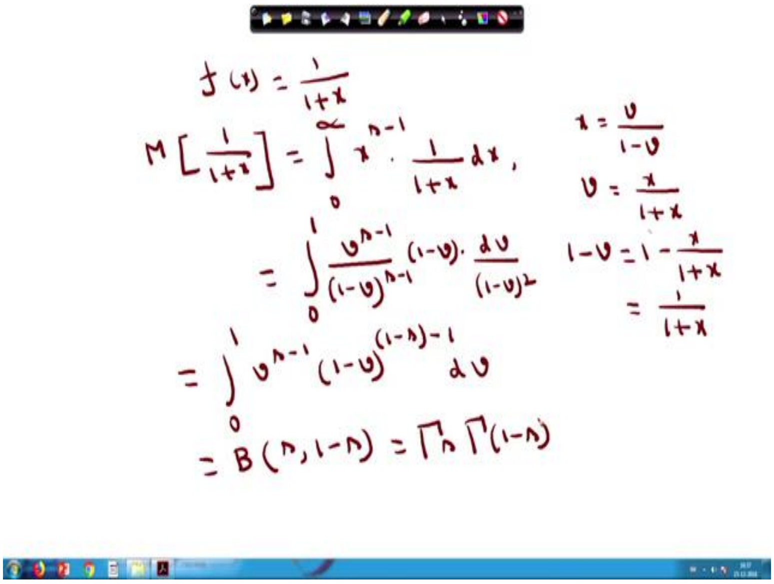

From the definition of Mellin transform, we have,

$$M\left[\frac{1}{1+x}\right] = \int_0^\infty x^{s-1}\frac{1}{1+x}\,dx$$

Let us substitute $x = \frac{v}{1-v}$ in the integration. Then the above equation is reduced to,

$$M\left[\frac{1}{1+x}\right] = \int_0^1 \left(\frac{v}{1-v}\right)^{s-1}(1-v)\frac{dv}{(1-v)^2}$$
$$= \int_0^1 v^{s-1}(1-v)^{(1-s)-1}\,dv$$

The integral on the RHS is Beta function.

$$\therefore M\left[\frac{1}{1+x}\right] = B(s, 1-s)$$
$$= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s+1-s)}$$
$$= \Gamma(s)\Gamma(1-s) \quad [\because\ \Gamma(1) = 1]$$

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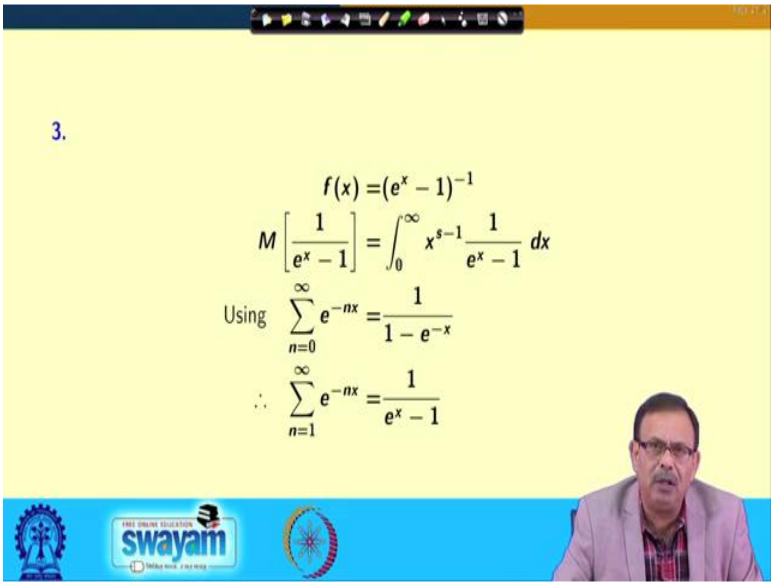

Next we will find the Mellin transform of $f(x) = \frac{1}{e^x - 1}$

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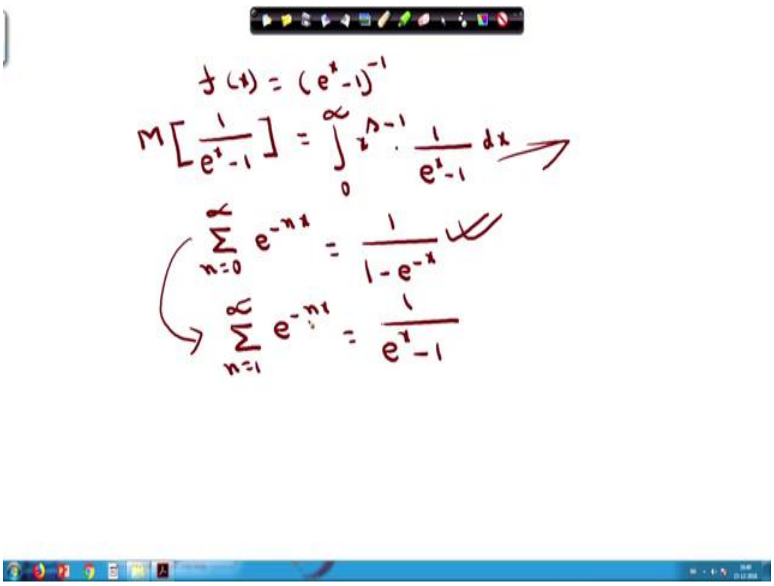

From the definition of Mellin transform, we have,

$$M\left[\frac{1}{e^x - 1}\right] = \int_0^\infty x^{s-1}\frac{1}{e^x - 1}\,dx$$

Using the infinite series expansion of $\frac{1}{1-x}$, we have,

$$\frac{1}{1-e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots$$
$$\Rightarrow \frac{e^{x}}{e^{x}-1} = 1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots$$
$$\Rightarrow \frac{1}{e^{x}-1} = e^{-x} + e^{-2x} + e^{-3x} + \cdots$$
$$\Rightarrow \frac{1}{e^{x}-1} = \sum_{n=1}^{\infty} e^{-nx}$$

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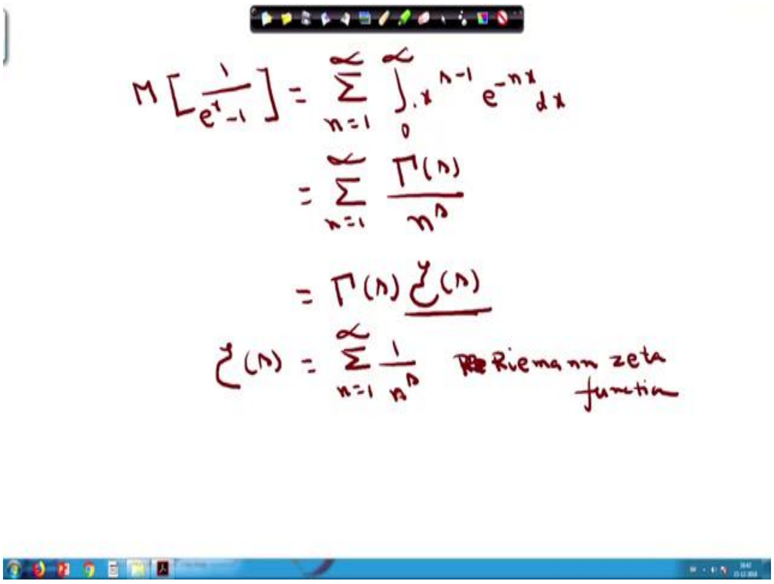

Using above relation, we will have,

$$M\left[\frac{1}{e^{x}-1}\right] = \int_{0}^{\infty} x^{s-1}\left(\sum_{n=1}^{\infty} e^{-nx}\right) dx$$

Now, taking the summation outside the integral sign, we get,

$$M\left[\frac{1}{e^x - 1}\right] = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx$$
$$= \sum_{n=1}^{\infty} M[e^{-nx}]$$
$$= \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s}$$
$$= \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= \Gamma(s)\zeta(s)$$

where, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is Riemann zeta function provided $Re(s) > 1$.

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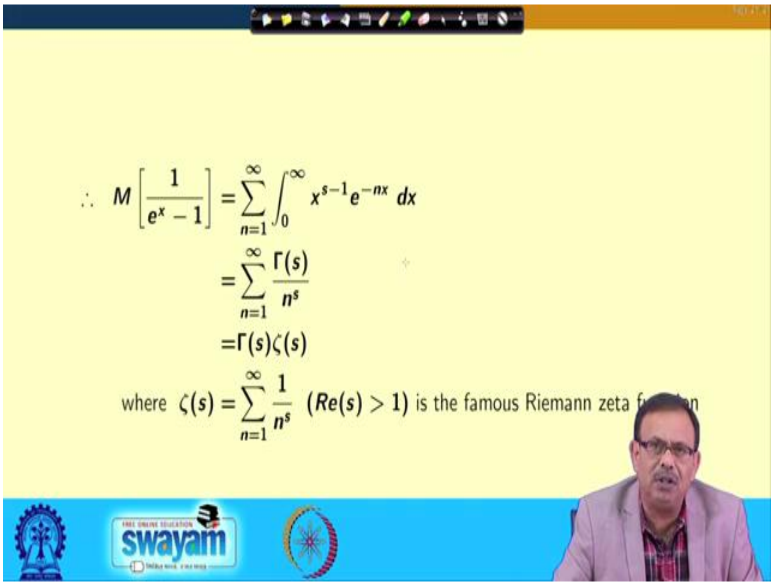

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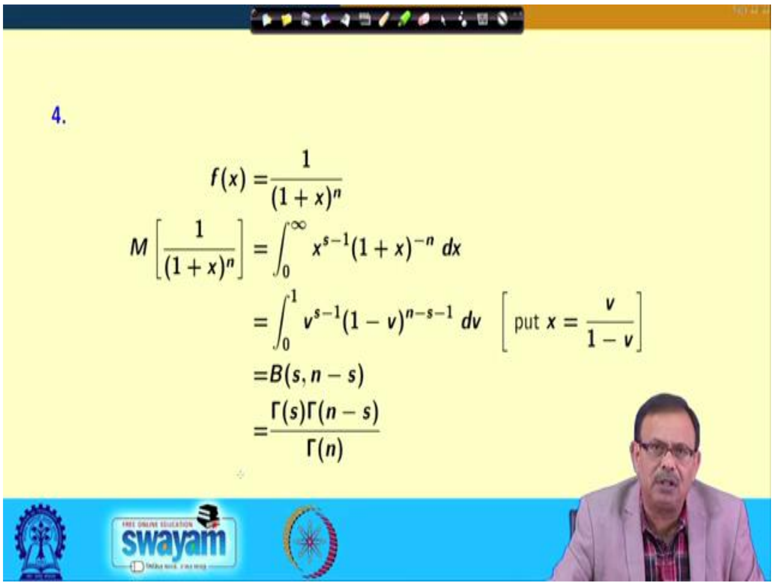

Our next problem is say, $f(x) = \frac{1}{(1+x)^n}$

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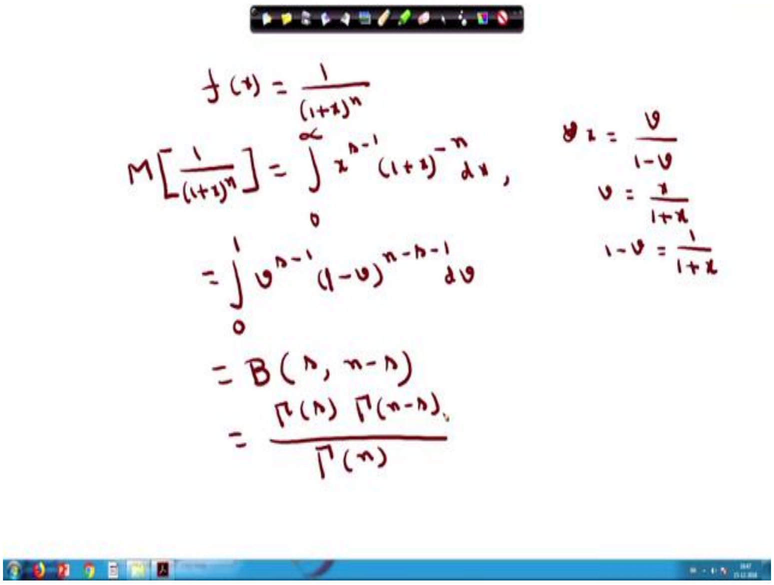

Using the definition of Mellin transform, we have,

$$M\left[\frac{1}{(1+x)^n}\right] = \int_0^\infty x^{s-1}\frac{1}{(1+x)^n}dx$$

Let us substitute $x = \frac{v}{1-v}$ in the integration. Then the above equation is reduced to,

$$M\left[\frac{1}{(1+x)^n}\right] = \int_0^1 \left(\frac{v}{1-v}\right)^{s-1} (1-v)^n \frac{dv}{(1-v)^2}$$
$$= \int_0^1 v^{s-1}(1-v)^{(n-s)-1}\,dv$$

The integral on the RHS is well-known Beta function.

$$\therefore M\left[\frac{1}{(1+x)^n}\right] = B(s, n-s)$$
$$= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(s+n-s)}$$
$$= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}$$

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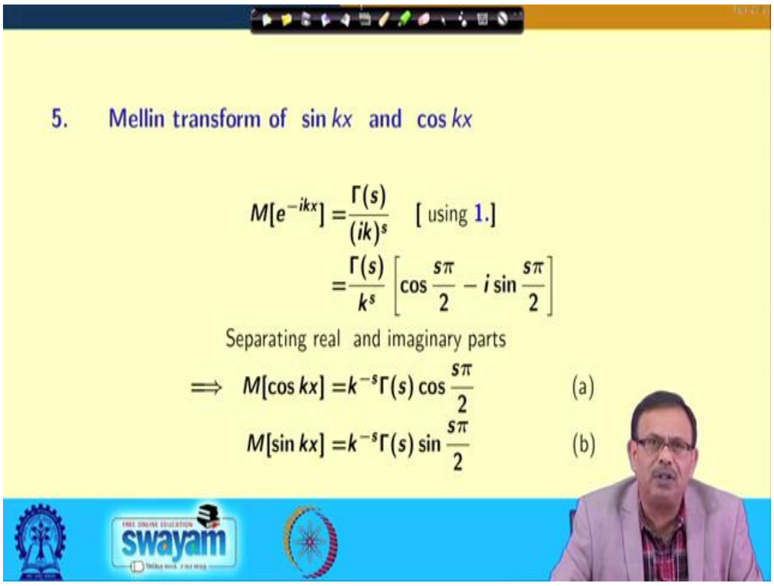

Now, we want to find out the Mellin transform of $\sin kx$ and $\cos kx$.

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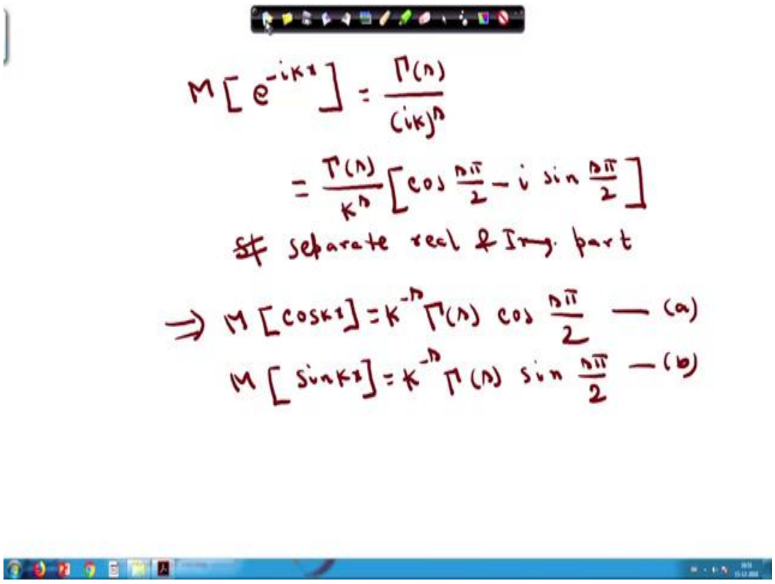

From the first problem, we can write down,

$$M\left[e^{-ikx}\right] = \frac{\Gamma(s)}{(ik)^s}$$

$$= \frac{\Gamma(s)}{k^s}\ (-i)^s$$

$$= \frac{\Gamma(s)}{k^s}\left(\cos\frac{s\pi}{2} - i\sin\frac{s\pi}{2}\right)$$

Now comparing the real and imaginary parts from the above equation, we have,

$$M[\cos kx] = \frac{\Gamma(s)}{k^s}\cos\frac{s\pi}{2} \qquad (a)$$

$$M[\sin kx] = \frac{\Gamma(s)}{k^s}\sin\frac{s\pi}{2} \qquad (b)$$

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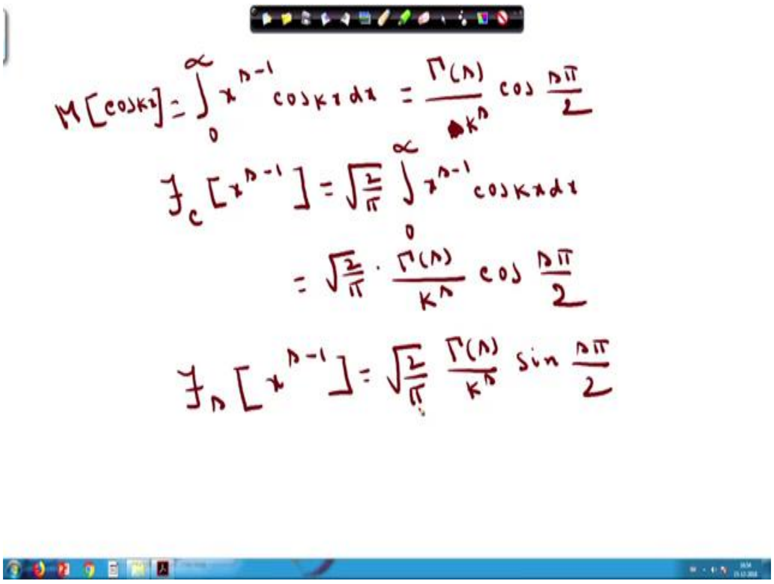

From the definition of Mellin transform and the result obtained in (a), we can have,

$$\int_0^\infty x^{s-1} \cos kx \ dx = \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2}$$

Now from the definition of Fourier cosine transform, we have,

$$\mathcal{F}_c[x^{s-1}] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{s-1} \cos kx \ dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \cos \frac{s\pi}{2}$$

Similarly,

$$\mathcal{F}_s[x^{s-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(s)}{k^s} \sin \frac{s\pi}{2}$$

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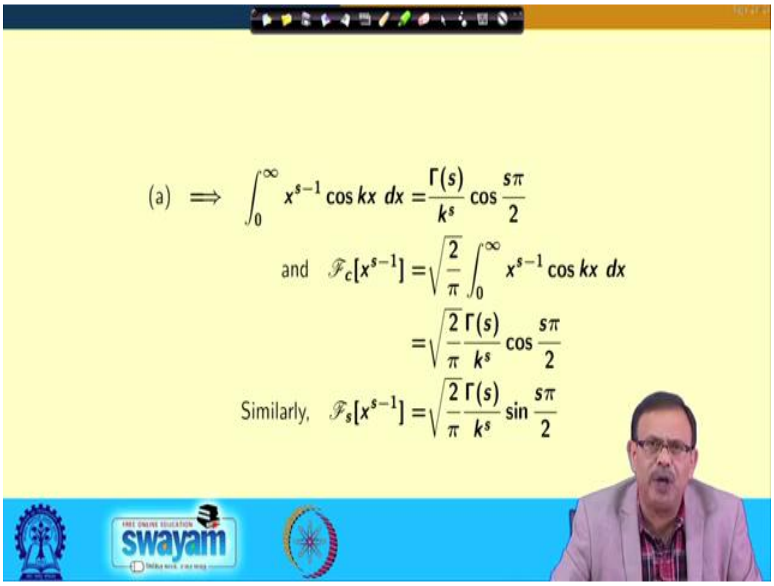

Thank you.