

**Transform Calculus and its Applications in Differential Equations**  
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**Lecture - 53**  
**Solution of Boundary Value Problems using Finite Fourier Transform - II**

In the last lecture, we had started the solution of boundary value problems using finite Fourier sine or cosine transform. We had stated one example, but due to the shortage of time, we could not start it. So, let us see the problem now.

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**Example**

Use finite transform to solve

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}, \quad 0 \leq x \leq \pi, \quad t > 0$$

with  $\frac{\partial v}{\partial x} = 0$ , when  $x = 0$  and  $x = \pi$ ,  $t > 0$

$v = f(x)$  when  $t = 0$ ,  $0 \leq x \leq \pi$

Since  $\frac{\partial v}{\partial x} = 0$  at  $x = 0$  and  $x = \pi$  are given, we will use finite Fourier cosine transform with respect to  $x$ .

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$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}, \quad 0 \leq x \leq \pi, \quad t > 0$$

Apply Finite F.C.T. w.r.t.  $x$

$$\int_0^{\pi} \frac{\partial v}{\partial t} \cos nx \, dx = \kappa \int_0^{\pi} \frac{\partial^2 v}{\partial x^2} \cos nx \, dx$$
$$\frac{d\bar{v}_c}{dt} = \kappa \left[ -n^2 \bar{v}_c - \left\{ \overset{=0}{v_x(0,t)} - \overset{=0}{v_x(\pi,t)} \cos n\pi \right\} \right]$$
$$\frac{d\bar{v}_c}{dt} = -\kappa n^2 \bar{v}_c$$

Now applying finite Fourier cosine transform with respect to  $x$  on the given equation, we obtain,

$$\int_0^{\pi} \frac{\partial v}{\partial t} \cos nx \, dx = \kappa \int_0^{\pi} \frac{\partial^2 v}{\partial x^2} \cos nx \, dx$$
$$\Rightarrow \frac{d\bar{v}_c}{dt} = \kappa \left[ -n^2 \bar{v}_c - \{v_x(0,t) - v_x(\pi,t) \cos n\pi\} \right]$$

putting the values  $v_x(0,t) = v_x(\pi,t) = 0$  in the above equation, we will obtain a first order ODE as,

$$\frac{d\bar{v}_c}{dt} = -\kappa n^2 \bar{v}_c$$

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The image shows a handwritten derivation on a whiteboard. At the top, there is a toolbar with various drawing tools. The main text consists of several lines of mathematical work:

$$\bar{v}_c = A e^{-kn^2 t} \quad \text{--- (1)} \Rightarrow \text{At } t=0, \bar{v}_c = A$$
$$v = f(x), t=0$$
$$\text{At } t=0, \bar{v}_c = \int_0^\pi v(y,0) \cos ny \, dy$$
$$A = \int_0^\pi f(y) \cos ny \, dy \Rightarrow \bar{v}_c(0) = \int_0^\pi f(y) \, dy$$
$$\bar{v}_c = e^{-kn^2 t} \int_0^\pi f(y) \cos ny \, dy$$

The final equation is underlined.

The general solution of the above equation will be given as,

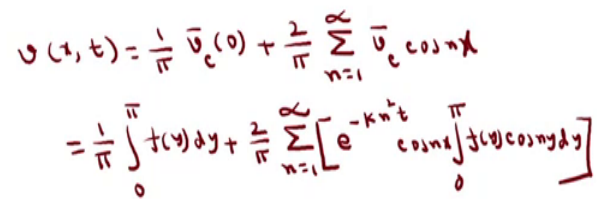
$$\bar{v}_c(n, t) = A e^{-kn^2 t} \quad (1)$$

So, we have to now find out the value of the constant  $A$ . (1) implies,  $\bar{v}_c = A$  at  $t = 0$

From the given conditions, we have,

$$\begin{aligned} \text{at } t = 0, A = \bar{v}_c &= \int_0^\pi v(y,0) \cos ny \, dy \\ &= \int_0^\pi f(y) \cos ny \, dy \\ \therefore \bar{v}_c(n, t) &= e^{-kn^2 t} \int_0^\pi f(y) \cos ny \, dy \end{aligned}$$

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$$\begin{aligned} v(x, t) &= \frac{1}{\pi} \bar{v}_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} \bar{v}_c \cos nx \\ &= \frac{1}{\pi} \int_0^{\pi} f(y) dy + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ e^{-kn^2 t} \cos nx \int_0^{\pi} f(y) \cos ny dy \right] \end{aligned}$$



Now using the inverse finite Fourier cosine transform, we will obtain  $v(x, t)$  as,

$$\begin{aligned} v(x, t) &= \frac{1}{\pi} \bar{v}_c(0, t) + \frac{2}{\pi} \sum_{n=1}^{\infty} \bar{v}_c(n, t) \cos nx \\ &= \frac{1}{\pi} \int_0^{\pi} f(y) dy + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ e^{-kn^2 t} \cos nx \int_0^{\pi} f(y) \cos ny dy \right] \end{aligned}$$



Since we do not know the value of  $f(x)$ , so we cannot evaluate the integral in both cases, but if we know  $f(x)$ , we can evaluate the integral. So, like this way we can find out the solution.

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**Solution:**

$$\int_0^{\pi} \frac{\partial v}{\partial t} \cos nx \, dx = k \int_0^{\pi} \frac{\partial^2 v}{\partial x^2} \cos nx \, dx$$
$$\Rightarrow \frac{d\bar{v}_c}{dt} = k \left[ -n^2 \bar{v}_c - \{v_x(0, t) - v_x(\pi, t) \cos n\pi\} \right]$$
$$\Rightarrow \frac{d\bar{v}_c}{dt} = -kn^2 \bar{v}_c$$
$$\therefore \bar{v}_c = Ae^{-kn^2 t} \quad (1)$$


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$$\text{At } t = 0, \bar{v}_c = \int_0^{\pi} f(y) \cos ny \, dy$$
$$\therefore A = \int_0^{\pi} f(y) \cos ny \, dy$$
$$\therefore \bar{v}_c = e^{-kn^2 t} \int_0^{\pi} f(y) \cos ny \, dy$$


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Taking the inverse finite Fourier cosine transform

$$v(x, t) = \frac{1}{\pi} \bar{v}_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} \bar{v}_c(n) \cos nx$$
$$= \frac{1}{\pi} \int_0^{\pi} f(y) dy + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ e^{-kn^2 t} \cos nx \int_0^{\pi} f(y) \cos ny dy \right]$$

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**Example**  
Use finite Fourier sine transform to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < l, \quad t > 0$$

with  $u(0, t) = 0, \quad u(l, t) = 0$

$$u(x, 0) = \begin{cases} 2u_0 \left(\frac{x}{l}\right) & , \quad 0 \leq x \leq \frac{l}{2} \\ 2u_0 \left(1 - \frac{x}{l}\right) & , \quad \frac{l}{2} \leq x \leq l \end{cases}$$

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Let us take one more example as shown in the above slide, so that it becomes very clear, how to find out the solutions to boundary value problems using the finite Fourier sine or cosine transform.

Here,  $t$  is given as  $> 0$  but  $x$  has a finite range that is  $(0, l)$  and also the values of  $u(0, t)$  and  $u(l, t)$  are given. Therefore, to solve this particular problem, we will use the finite Fourier sine transform.

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < l, \quad t > 0, \quad u(0,t) = u(l,t) = 0$$

Apply Finite F.S.T. w.r. to  $x$

$$\int_0^l \frac{\partial u}{\partial t} \sin\left(\frac{n\pi}{l}x\right) dx = \int_0^l \frac{\partial^2 u}{\partial x^2} \sin\left(\frac{n\pi}{l}x\right) dx$$

$$\frac{d\bar{u}_s}{dt} = \left[ \left[ \frac{\partial u}{\partial x} \sin\frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l \frac{\partial u}{\partial x} \cos\frac{n\pi x}{l} dx \right]$$

$$= \left[ \frac{\partial u}{\partial x} \sin\frac{n\pi x}{l} - \frac{n\pi}{l} u \cos\frac{n\pi x}{l} \right]_{x=0}^l - \frac{n^2\pi^2}{l^2} \int_0^l u(x,t) \sin\frac{n\pi x}{l} dx$$

So, applying finite Fourier sine transform with respect to  $x$  on both sides of the given partial differential equation, we obtain,

$$\int_0^l \frac{\partial u}{\partial t} \sin\frac{n\pi x}{l} dx = \int_0^l \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{l} dx$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -\frac{n^2\pi^2}{l^2} \bar{u}_s(n,t) + \frac{n\pi}{l} [u(0,t) - u(l,t) \cos n\pi]$$

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$$\frac{d\bar{u}_s}{dt} = -\frac{n^2\pi^2}{l^2} \bar{u}_s(n,t) + \frac{n\pi}{l} [u(0,t) - u(l,t) \cos n\pi]$$

$$\frac{d\bar{u}_s}{dt} + \frac{n^2\pi^2}{l^2} \bar{u}_s = 0$$

$$\bar{u}_s = \bar{u}_s(n,t) = e^{-\frac{n^2\pi^2}{l^2}t}$$

$u(x,0) = \begin{cases} 0 \leq x \leq \frac{l}{2} \\ \frac{l}{2} \leq x \leq l \end{cases}$

$$\bar{u}_s(x,0) = \int_0^l u(x,0) \sin\frac{n\pi x}{l} dx$$

Putting the values  $u(0, t) = 0$  and  $u(l, t) = 0$  in the above equation, we will get a first order ODE as,

$$\Rightarrow \frac{d\bar{u}_s}{dt} + \frac{n^2\pi^2}{l^2}\bar{u}_s = 0$$

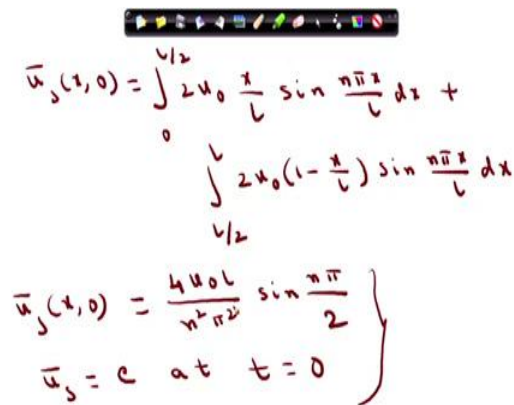
So, directly we can write down the general solution of the ODE as

$$\bar{u}_s(n, t) = c e^{-\frac{n^2\pi^2 t}{l^2}}$$

Now, our next job is to find out the value of this constant  $c$ .

$$\text{at } t = 0, \bar{u}_s = c$$

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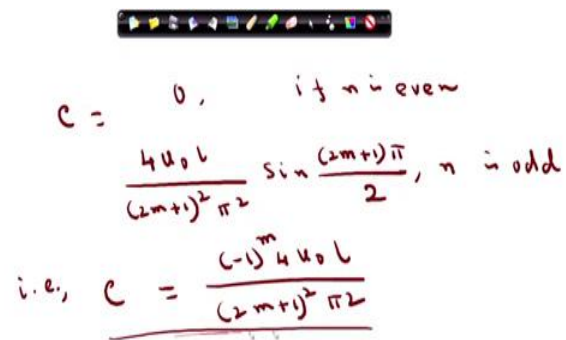
$$\begin{aligned} \bar{u}_s(x, 0) &= \int_0^{l/2} 2u_0 \frac{x}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^l 2u_0 \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} dx \\ \bar{u}_s(x, 0) &= \frac{4u_0 l}{n^2 \pi^2} \sin \frac{n\pi}{2} \\ \bar{u}_s &= c \text{ at } t = 0 \end{aligned}$$

From the given conditions, we have,

$$\begin{aligned} \text{at } t = 0, c = \bar{u}_s(n, 0) &= \int_0^l u(x, 0) \sin \frac{n\pi x}{l} dx \\ &= \int_0^{l/2} 2u_0 \frac{x}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^l 2u_0 \left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l} dx \\ &= \frac{4u_0 l}{n^2 \pi^2} \sin \frac{n\pi}{2} \end{aligned}$$



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$$c = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4u_0 l}{(2m+1)^2 \pi^2} \sin \frac{(2m+1)\pi}{2}, & \text{if } n \text{ is odd} \end{cases}$$

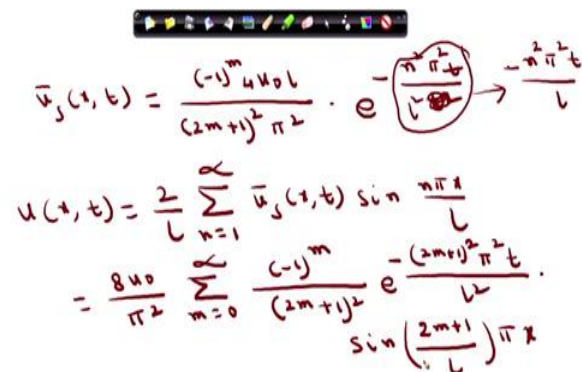
i.e.,  $c = \frac{(-1)^m 4u_0 l}{(2m+1)^2 \pi^2}$

Therefore  $c$  can be written as,

$$c = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{(-1)^m 4u_0 l}{(2m+1)^2 \pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

where,  $n = 2m + 1$ .

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$$u_s(x,t) = \frac{(-1)^m 4u_0 l}{(2m+1)^2 \pi^2} \cdot e^{-\frac{n^2 \pi^2 t}{l^2}} \rightarrow \frac{(-1)^m 4u_0 l}{(2m+1)^2 \pi^2} \cdot e^{-\frac{n^2 \pi^2 t}{l^2}} \sin \frac{n \pi x}{l}$$
$$u(x,t) = \frac{2}{l} \sum_{n=1}^{\infty} u_s(x,t) \sin \frac{n \pi x}{l}$$
$$= \frac{8u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} e^{-\frac{(2m+1)^2 \pi^2 t}{l^2}} \sin \left( \frac{2m+1}{l} \pi x \right)$$

$$\therefore \bar{u}_s(n, t) = c e^{-\frac{n^2 \pi^2 t}{l^2}}$$

where  $c$  is defined in the last page. Now using the inverse finite Fourier sine transform, we will get  $u(x, t)$  as,

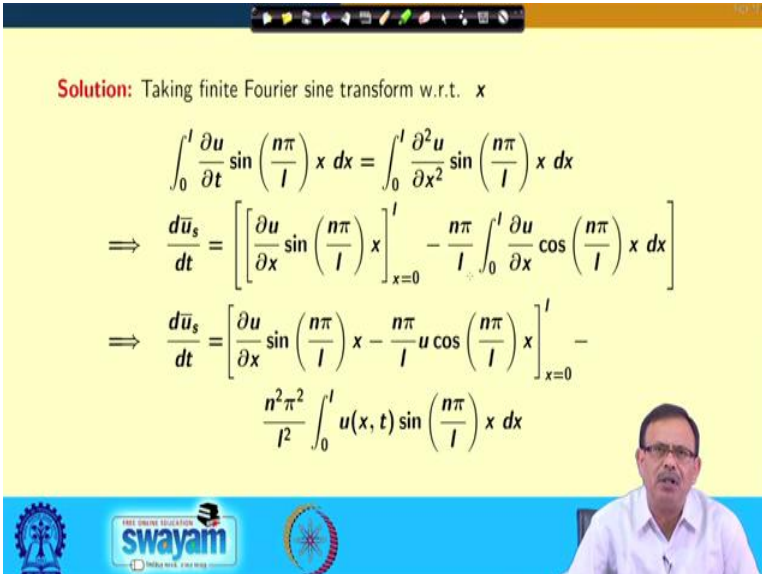
$$\begin{aligned} u(x, t) &= \frac{2}{l} \sum_{n=1}^{\infty} \bar{u}_s(n, t) \sin \frac{n\pi x}{l} \\ &= \frac{8u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} e^{-\frac{(2m+1)^2 \pi^2 t}{l^2}} \sin \frac{(2m+1)\pi x}{l} \end{aligned}$$

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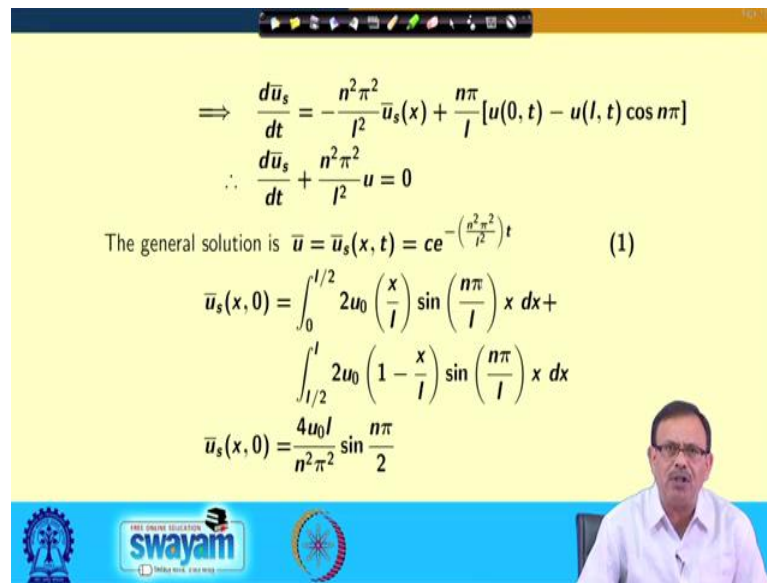
**Solution:** Taking finite Fourier sine transform w.r.t.  $x$

$$\int_0^l \frac{\partial u}{\partial t} \sin \left( \frac{n\pi}{l} \right) x \, dx = \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \left( \frac{n\pi}{l} \right) x \, dx$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = \left[ \left[ \frac{\partial u}{\partial x} \sin \left( \frac{n\pi}{l} \right) x \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l \frac{\partial u}{\partial x} \cos \left( \frac{n\pi}{l} \right) x \, dx \right]$$

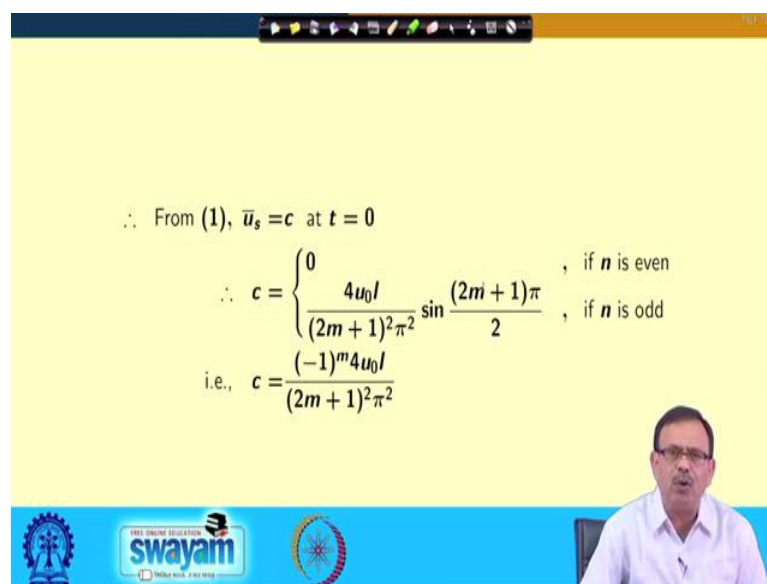
$$\Rightarrow \frac{d\bar{u}_s}{dt} = \left[ \frac{\partial u}{\partial x} \sin \left( \frac{n\pi}{l} \right) x - \frac{n\pi}{l} u \cos \left( \frac{n\pi}{l} \right) x \right]_{x=0}^l - \frac{n^2 \pi^2}{l^2} \int_0^l u(x, t) \sin \left( \frac{n\pi}{l} \right) x \, dx$$


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The slide shows a differential equation and its solution. The equation is  $\frac{d\bar{u}_s}{dt} = -\frac{n^2\pi^2}{l^2}\bar{u}_s(x) + \frac{n\pi}{l}[u(0,t) - u(l,t)\cos n\pi]$ . This is simplified to  $\frac{d\bar{u}_s}{dt} + \frac{n^2\pi^2}{l^2}\bar{u}_s = 0$ . The general solution is given as  $\bar{u} = \bar{u}_s(x, t) = ce^{-\left(\frac{n^2\pi^2}{l^2}\right)t}$  (1). The initial condition  $\bar{u}_s(x, 0)$  is expressed as the sum of two integrals:  $\int_0^{l/2} 2u_0 \left(\frac{x}{l}\right) \sin\left(\frac{n\pi}{l}\right)x dx + \int_{l/2}^l 2u_0 \left(1 - \frac{x}{l}\right) \sin\left(\frac{n\pi}{l}\right)x dx$ . The final result is  $\bar{u}_s(x, 0) = \frac{4u_0l}{n^2\pi^2} \sin \frac{n\pi}{2}$ . A small video inset of the presenter is visible in the bottom right corner.

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The slide shows the determination of the constant  $c$  from equation (1) at  $t = 0$ . It states  $\therefore$  From (1),  $\bar{u}_s = c$  at  $t = 0$ . The constant  $c$  is defined as  $c = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \frac{4u_0l}{(2m+1)^2\pi^2} \sin \frac{(2m+1)\pi}{2} & , \text{ if } n \text{ is odd} \end{cases}$ . This is further simplified to  $\text{i.e., } c = \frac{(-1)^m 4u_0l}{(2m+1)^2\pi^2}$ . A small video inset of the presenter is visible in the bottom right corner.

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$$\therefore \bar{u}_s(x, t) = \frac{(-1)^m 4u_0 l}{(2m+1)^2 \pi^2} \left[ e^{-\left(\frac{n^2 \pi^2}{l^2}\right) t} \right]$$
$$\therefore u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} \bar{u}_s(x, t) \sin\left(\frac{n\pi}{l}\right) x$$
$$= \frac{8u_0}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} e^{-\frac{(2m+1)^2 \pi^2}{l^2} t} \sin\left(\frac{2m+1}{l}\right) \pi x$$

So, by this way, we can use the finite Fourier sine or cosine transform for solving the boundary value problems, where the variables are provided in a finite range. Thank you.