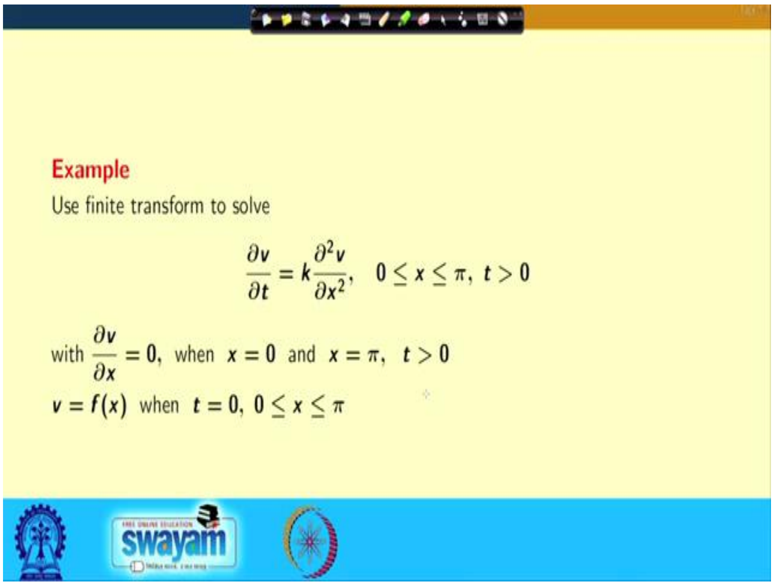
Transform Calculus and its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 53
Solution of Boundary Value Problems using Finite Fourier Transform - II

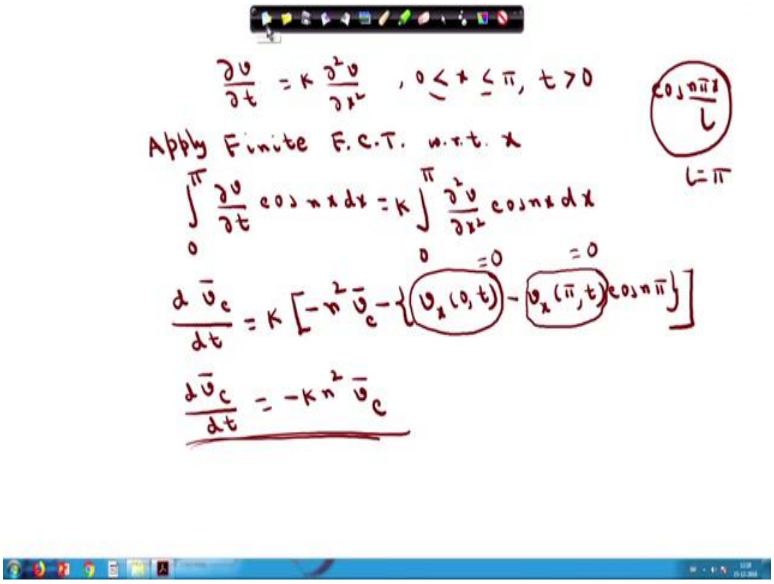
In the last lecture, we had started the solution of boundary value problems using finite Fourier sine or cosine transform. We had stated one example, but due to the shortage of time, we could not start it. So, let us see the problem now.

(Refer Slide Time: 00:37)



Since $\frac{\partial v}{\partial x} = 0$ at $x = 0$ and $x = \pi$ are given, we will use finite Fourier cosine transform with respect to x.

(Refer Slide Time: 02:28)



Now applying finite Fourier cosine transform with respect to x on the given equation, we obtain,

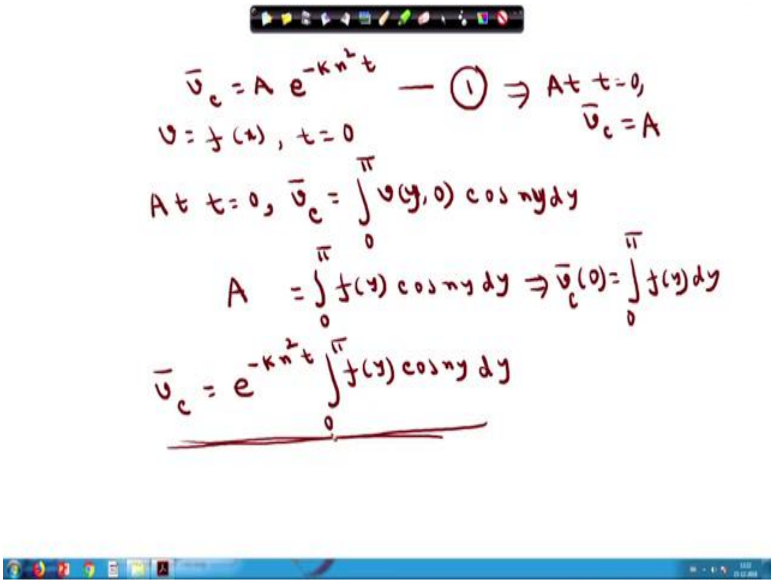
$$\int_0^{\pi} \frac{\partial v}{\partial t} \cos nx \, dx = k \int_0^{\pi} \frac{\partial^2 v}{\partial x^2} \cos nx \, dx$$

$$\Rightarrow \frac{d\overline{v_c}}{dt} = k[-n^2 \, \overline{v_c} - \{v_x(0,t) - v_x(\pi,t) \cos n\pi\}]$$

putting the values $v_x(0,t) = v_x(\pi,t) = 0$ in the above equation, we will obtain a first order ODE as,

$$\frac{d\overline{v_c}}{dt} = -kn^2 \, \overline{v_c}$$

(Refer Slide Time: 05:58)



The general solution of the above equation will be given as,

$$\overline{v_c}(n,t) = A\, e^{-kn^2 t} \tag{1}$$

So, we have to now find out the value of the constant A. (1) implies, $\overline{v_c} = A$ at $t = 0$

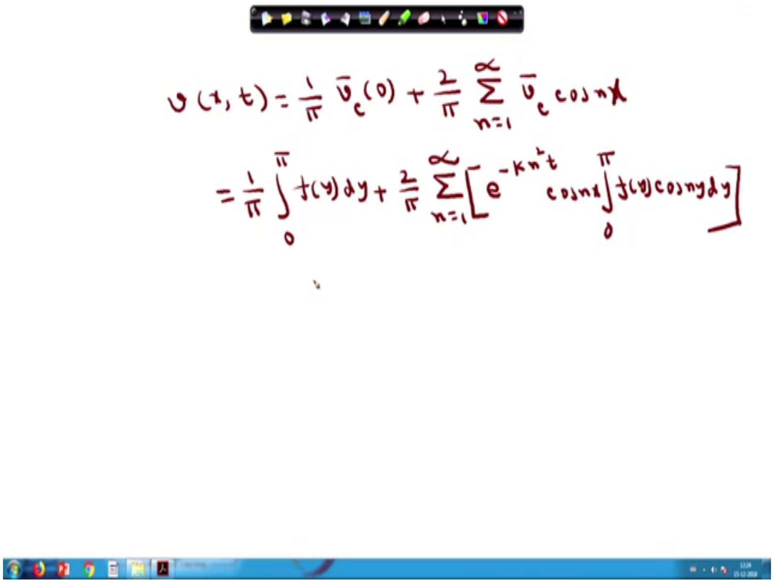
From the given conditions, we have,

$$\text{at } t = 0,\ \ A = \overline{v_c} = \int_0^{\pi} v(y,0)\cos ny\ dy$$

$$= \int_0^{\pi} f(y)\cos ny\ dy$$

$$\therefore\ \overline{v_c}(n,t) = e^{-kn^2 t}\int_0^{\pi} f(y)\cos ny\ dy$$

(Refer Slide Time: 09:33)



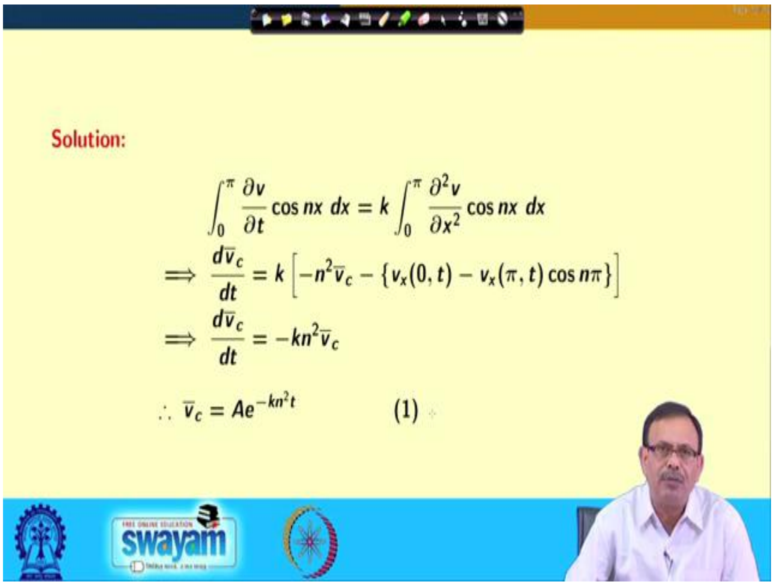
Now using the inverse finite Fourier cosine transform, we will obtain $v(x,t)$ as,

$$v(x,t) = \frac{1}{\pi}\overline{v_c}(0,t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\overline{v_c}(n,t)\ \cos nx$$

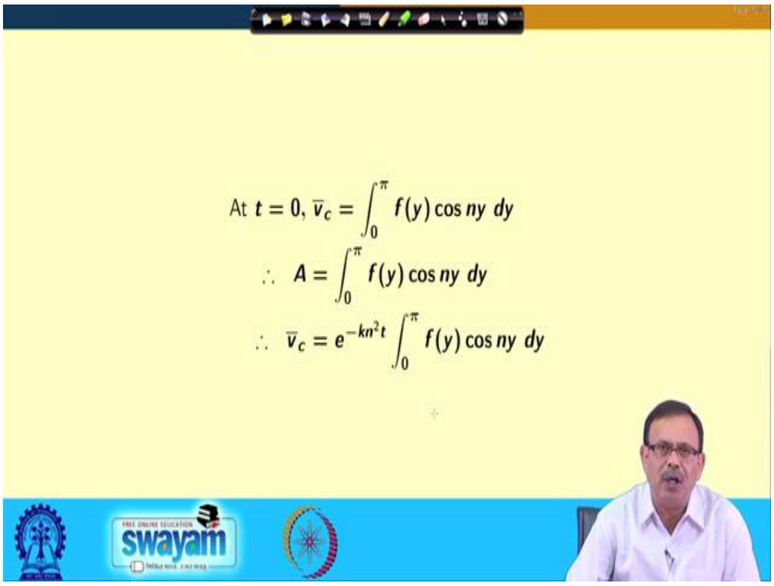
$$= \frac{1}{\pi}\int_0^{\pi} f(y)dy + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[e^{-kn^2t}\cos nx \int_0^{\pi} f(y)\cos ny\ dy\right]$$

Since we do not know the value of $f(x)$, so we cannot evaluate the integral in both cases, but if we know $f(x)$, we can evaluate the integral. So, like this way we can find out the solution.

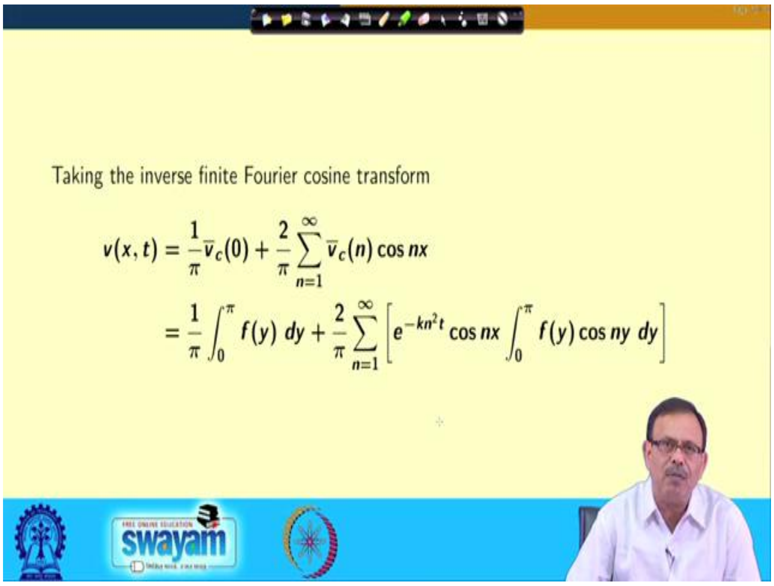
(Refer Slide Time: 11:50)



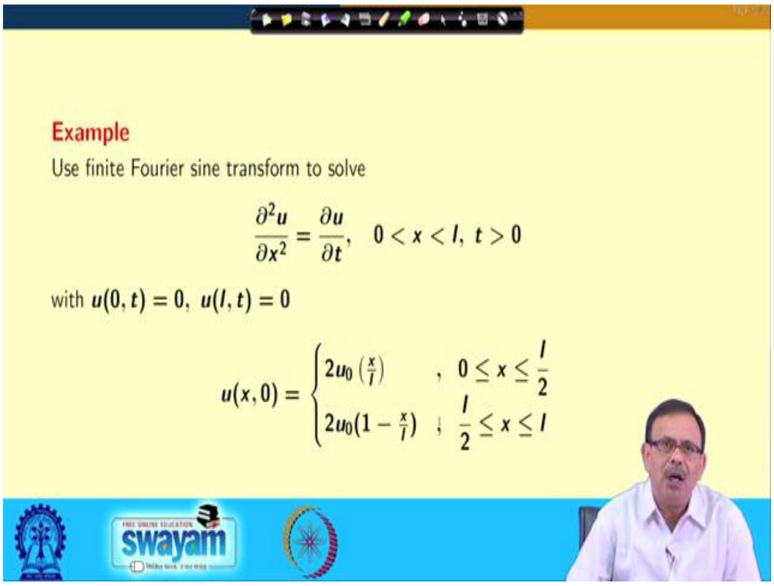
(Refer Slide Time: 14:21)



(Refer Slide Time: 15:10)



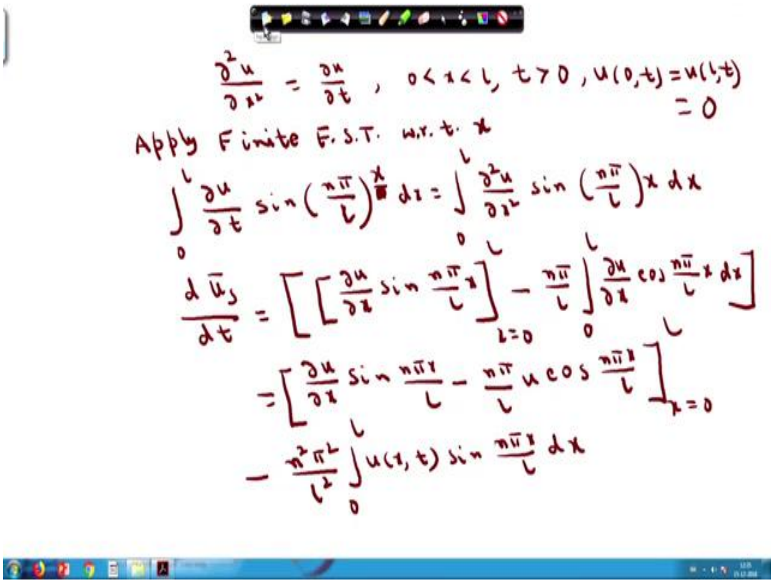
(Refer Slide Time: 16:04)



Let us take one more example as shown in the above slide, so that it becomes very clear, how to find out the solutions to boundary value problems using the finite Fourier sine or cosine transform.

Here, t is given as > 0 but x has a finite range that is $(0, l)$ and also the values of $u(0, t)$ and $u(l, t)$ are given. Therefore, to solve this particular problem, we will use the finite Fourier sine transform.

(Refer Slide Time: 17:42)

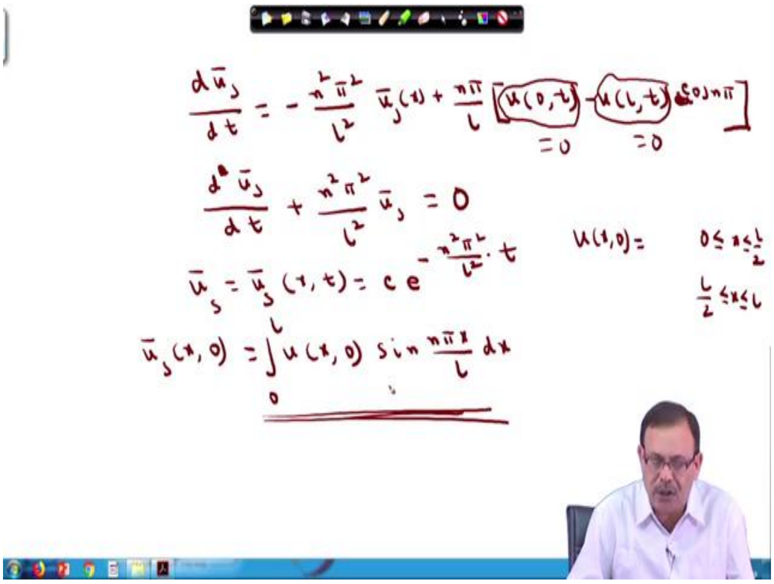


So, applying finite Fourier sine transform with respect to x on both sides of the given partial differential equation, we obtain,

$$\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi x}{l}dx \ = \int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{l}dx$$

$$\Rightarrow \frac{d\overline{u_s}}{dt} = -\frac{n^2\pi^2}{l^2}\ \overline{u_s}(n,t) + \frac{n\pi}{l}\,[u(0,t) - u(l,t)\cos n\pi]$$

(Refer Slide Time: 22:07)



Putting the values $u(0,t) = 0$ and $u(l,t) = 0$ in the above equation, we will get a first order ODE as,

$$\Rightarrow \frac{d\overline{u_s}}{dt} + \frac{n^2\pi^2}{l^2}\overline{u_s} = 0$$

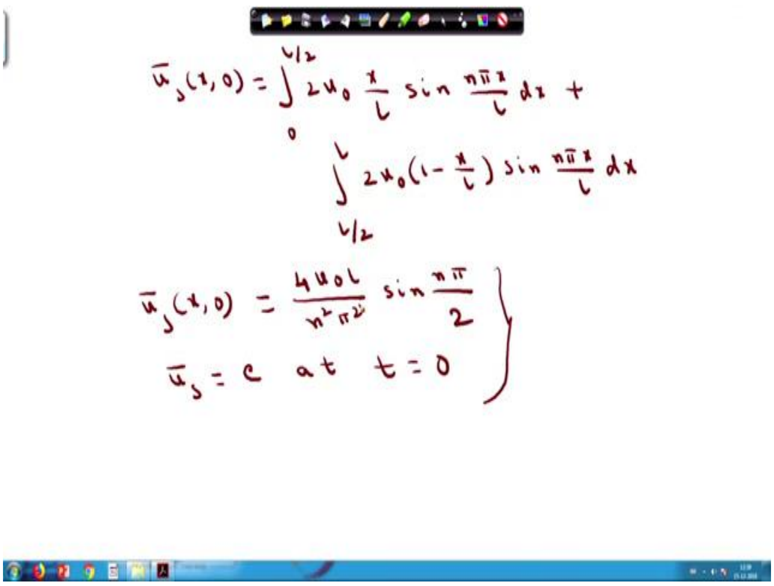
So, directly we can write down the general solution of the ODE as

$$\overline{u_s}(n,t) = c\, e^{-\frac{n^2\pi^2 t}{l^2}}$$

Now, our next job is to find out the value of this constant c.

$$\text{at } t = 0,\ \ \overline{u_s} = c$$

(Refer Slide Time: 25:34)



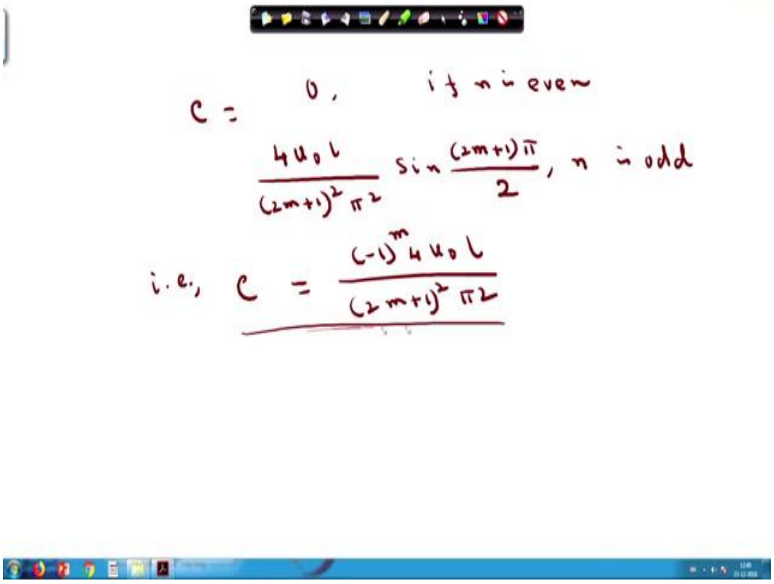
From the given conditions, we have,

$$\text{at } t = 0,\ \ c = \overline{u_s}(n,0) = \int_0^l u(x,0)\sin\frac{n\pi x}{l}\, dx$$

$$= \int_0^{\frac{l}{2}} 2u_0 \frac{x}{l}\sin\frac{n\pi x}{l}\, dx + \int_{\frac{l}{2}}^{l} 2u_0\left(1 - \frac{x}{l}\right)\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{4u_0 l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

(Refer Slide Time: 26:53)

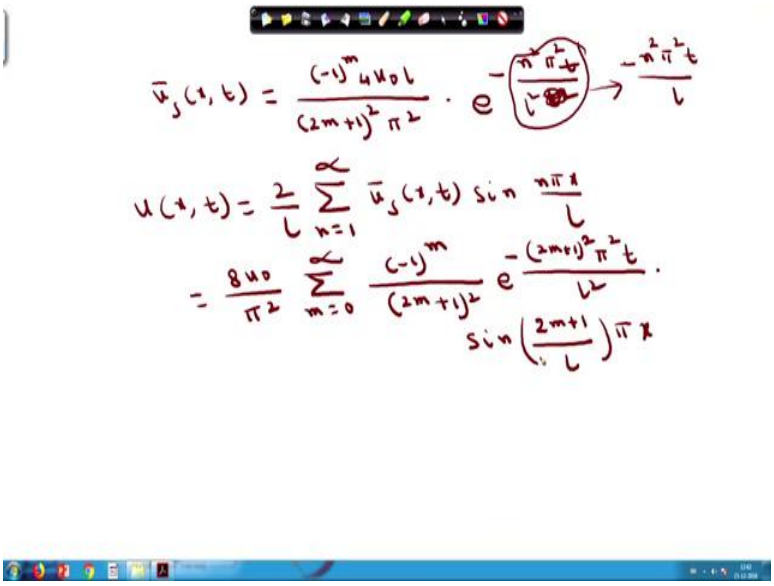


Therefore c can be written as,

$$c = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \dfrac{(-1)^m 4u_0 l}{(2m+1)^2\pi^2} & , \text{ if } n \text{ is odd} \end{cases}$$

where, $n = 2m + 1$.

(Refer Slide Time: 27:47)



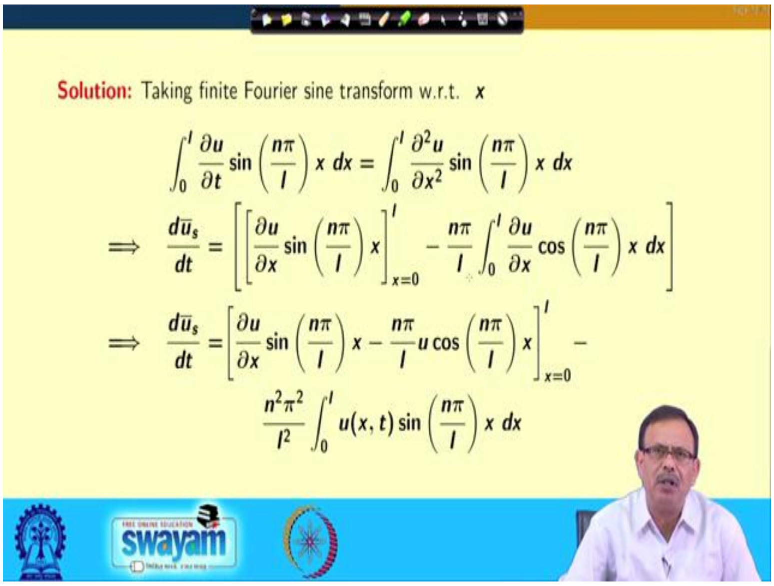
$$\therefore \overline{u_s}(n,t) = c\, e^{-\frac{n^2\pi^2 t}{l^2}}$$

where c is defined in the last page. Now using the inverse finite Fourier sine transform, we will get $u(x,t)$ as,

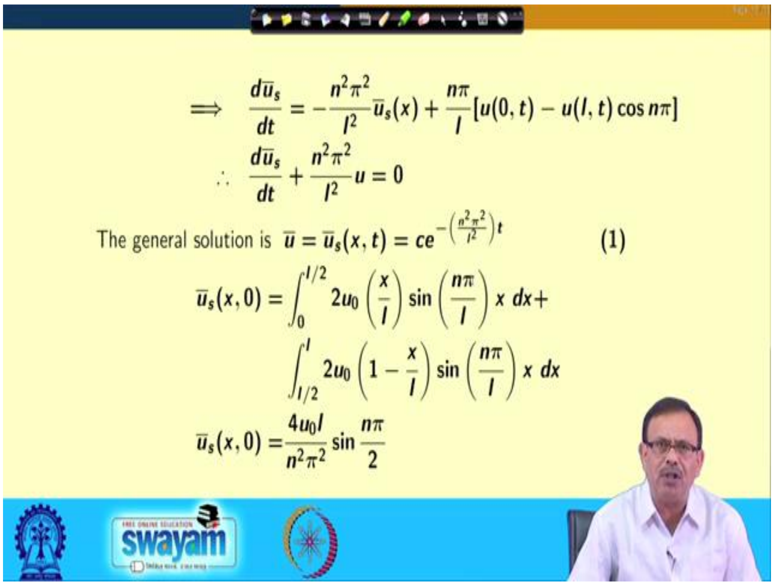
$$u(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \overline{u_s}(n,t)\ \sin\frac{n\pi x}{l}$$

$$= \frac{8\mathrm{u}_0}{\pi^2}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}\, e^{-\frac{(2m+1)^2\pi^2 t}{l^2}} \sin\frac{(2m+1)\pi x}{l}$$

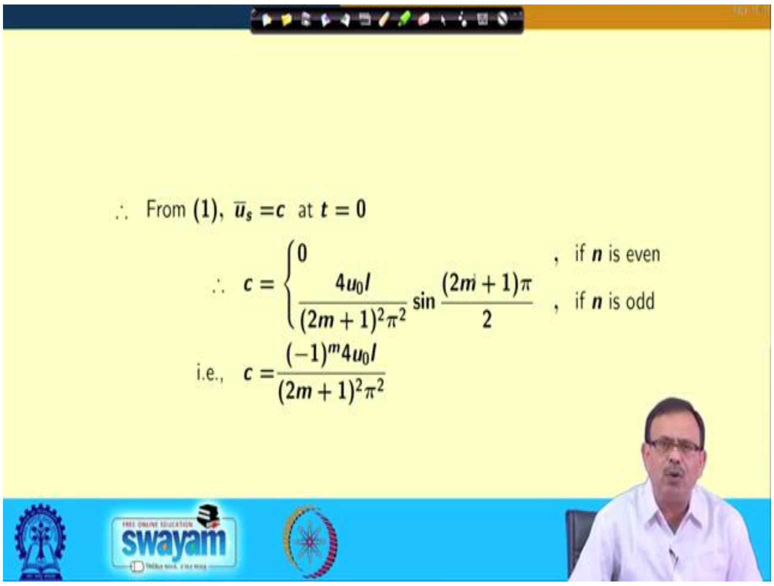
(Refer Slide Time: 30:10)



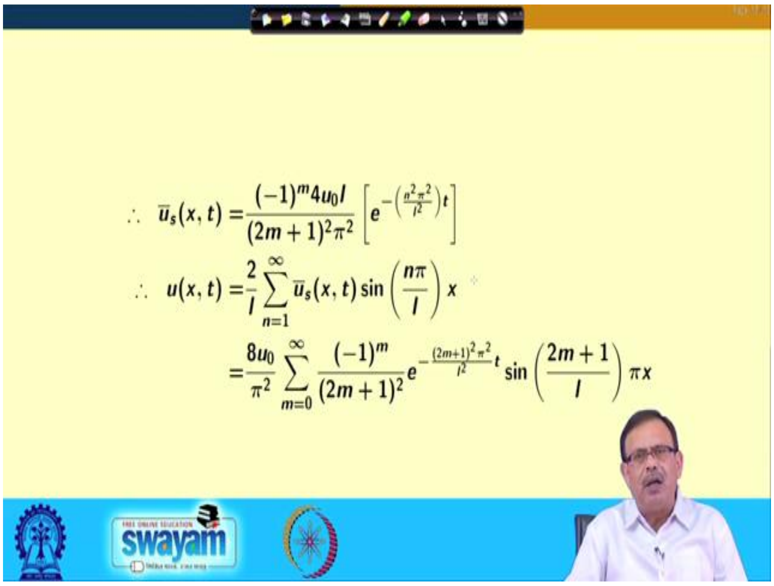
(Refer Slide Time: 30:39)



(Refer Slide Time: 31:48)



(Refer Slide Time: 32:12)



So, by this way, we can use the finite Fourier sine or cosine transform for solving the boundary value problems, where the variables are provided in a finite range. Thank you.