Transform Calculus and its Applications in Differential Equations
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Lecture - 52
Solution of Boundary Value Problems using Finite Fourier Transform - I

In the last lecture, we have discussed what is the finite Fourier transform and what are the properties of finite Fourier transform along with how to find out the Finite Fourier sine and cosine transforms of $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. In this lecture, we will solve few problems, then we will take some finite boundary value problems and we will see how using the finite transforms, we can find out the solution of those boundary value problems.

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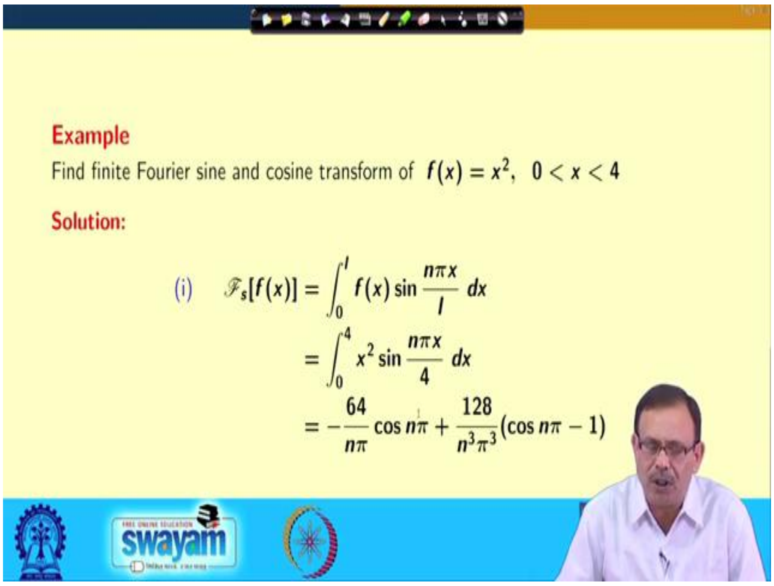

So, let us take the first example. We want to find out the Fourier sine and cosine transform of $f(x) = x^2, 0 < x < 4$. So, here we see if we have to use the Fourier cosine and sine transform, x should lie in between 0 to ∞, but the function is defined here in 0 to 4. So, we are forced to use the finite Fourier cosine or sine transform.

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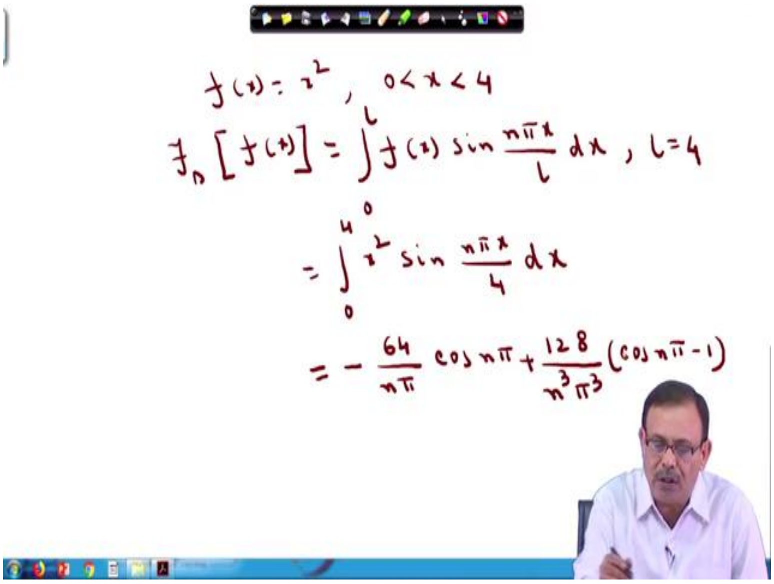

From definition of finite Fourier sine transform, we have,

$$\mathcal{F}_s[f(x)] = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^4 x^2 \sin\frac{n\pi x}{4}\,dx$$

$$= -\frac{64}{n\pi}\cos n\pi + \frac{128}{n^3\pi^3}(\cos n\pi - 1)$$

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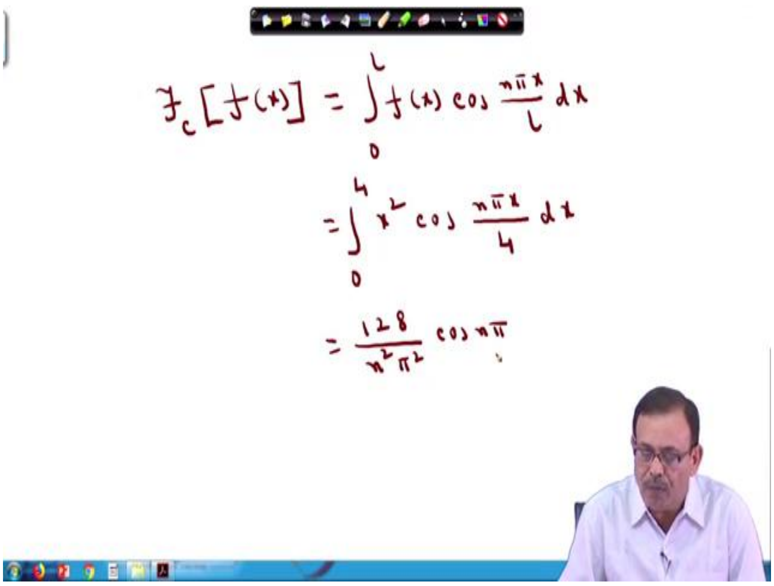

From definition of finite Fourier cosine transform, we have,

$$\mathcal{F}_c[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$
$$= \int_0^4 x^2\cos\frac{n\pi x}{4}\,dx$$
$$= \frac{128}{n^2\pi^2}\cos n\pi$$

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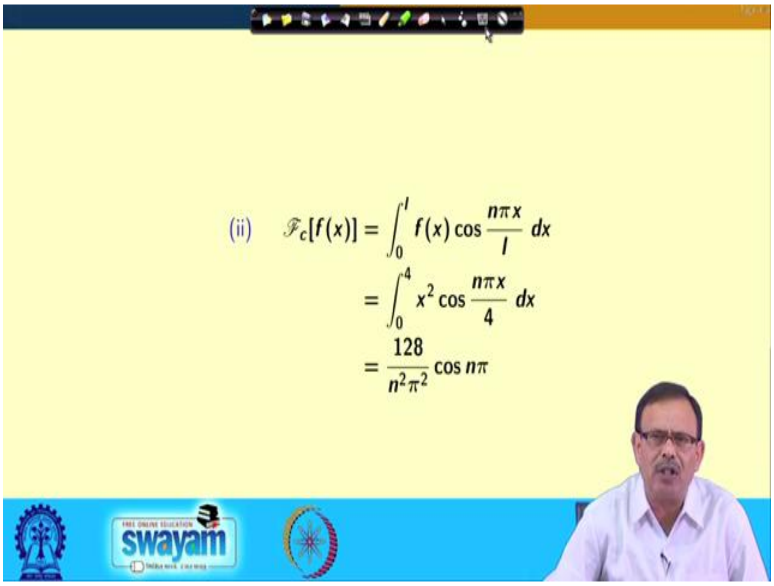

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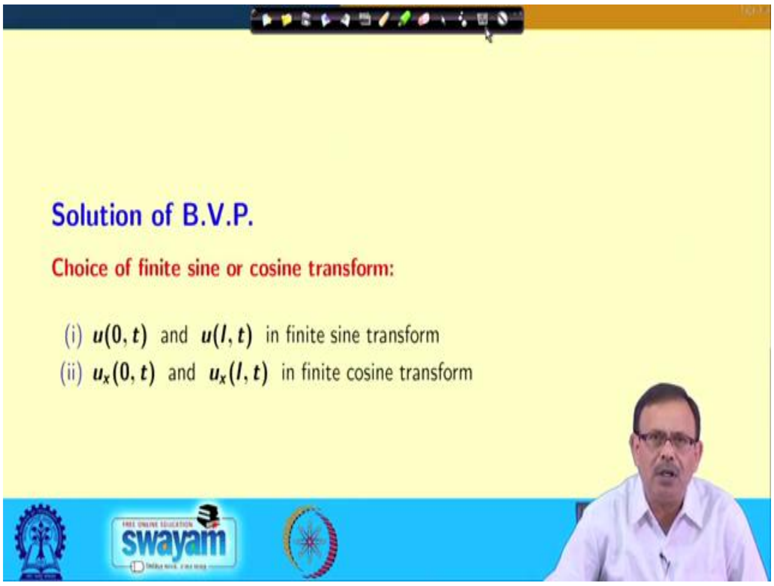

To solve boundary value problem using finite Fourier sine or cosine transform, which one we have to choose it depends upon the given conditions. If $u(0,t)$ and $u(l,t)$ are given, then we will use finite Fourier sine transform and if $u_x(0,t)$ and $u_x(l,t)$ are given, then we will use finite Fourier cosine transform.

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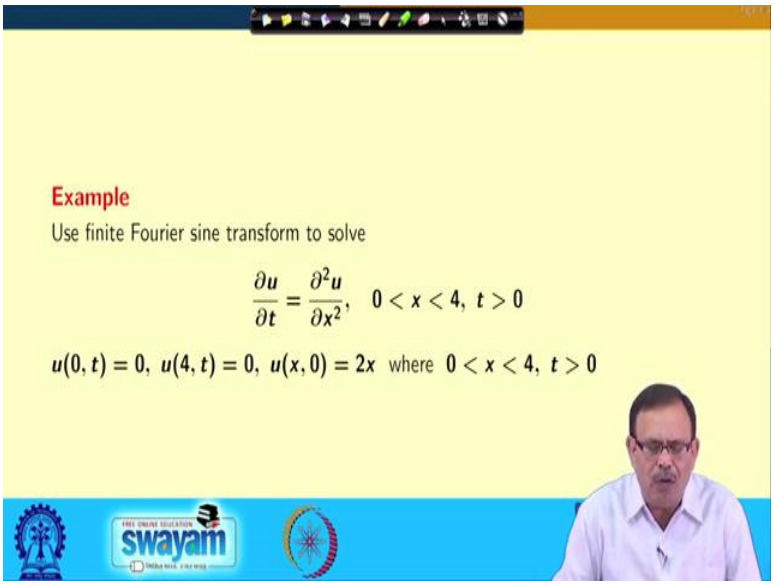

Let us consider the following problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 4, \ \ t > 0$$

with $u(0,t) = 0, \ u(4,t) = 0, \ u(x,0) = 2x$ where $0 < x < 4, \ t > 0$.

Since $u(0,t)$ and $u(4,t)$ are known to us, we can use the finite Fourier sine transform with respect to x.

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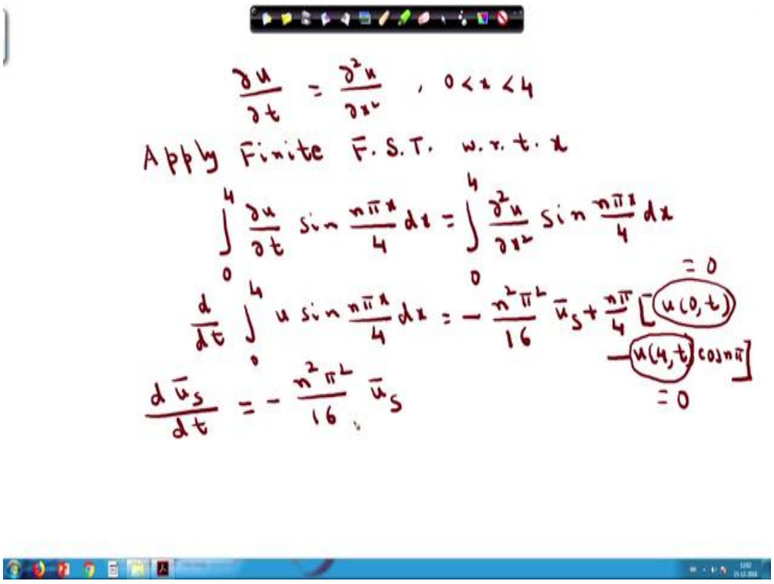

Let us see the solution process. Applying finite Fourier sine transform with respect to x on both sides of the given differential equation, we will get,

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi x}{4} dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{4} dx$$

$$\Rightarrow \frac{d\overline{u_s}}{dt} = -\frac{n^2\pi^2}{16} \overline{u_s} + \frac{n\pi}{4} [u(0,t) - u(4,t) \cos n\pi]$$

Here $\overline{u_s}$ denotes finite Fourier sine transform of $u(x,t)$. Putting the values of $u(0,t) = 0$ and $u(4,t) = 0$ in the above equation, we will get,

$$\Rightarrow \frac{d\overline{u_s}}{dt} = -\frac{n^2\pi^2}{16} \overline{u_s}$$

So, this is a simple first order ODE.

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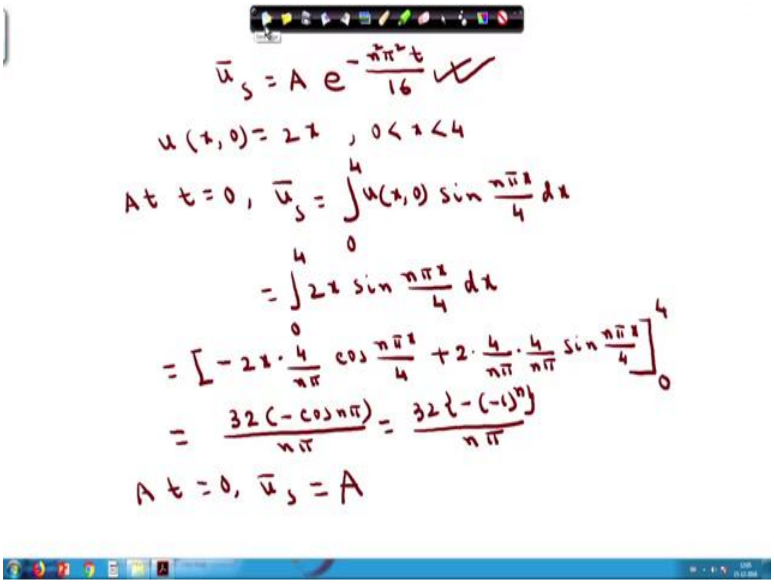

So, directly we can write down the solution of the ODE as

$$\overline{u_s} = A\,e^{-\frac{n^2\pi^2}{16}t}$$

Now, our next job is to find out the value of this constant A.

$$\text{at } t = 0,\ \ \overline{u_s} = A$$

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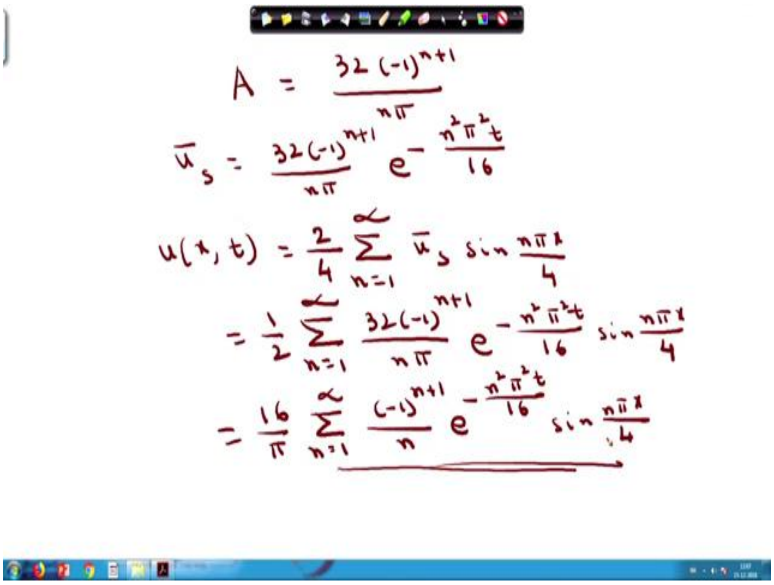

Now from the condition $u(x,0) = 2x$, we have,

$$\overline{u_s}(n,0) = \mathcal{F}_s[2x] = \frac{32}{n\pi}(-1)^{n+1}$$

$$\therefore A = \frac{32}{n\pi}(-1)^{n+1}$$

$$\therefore \overline{u_s} = \frac{32}{n\pi}(-1)^{n+1}\, e^{-\frac{n^2\pi^2}{16}t}$$

Now using the inverse finite Fourier sine transform, we obtain,

$$u(x,t) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32(-1)^{n+1}}{n\pi}\; e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\; e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

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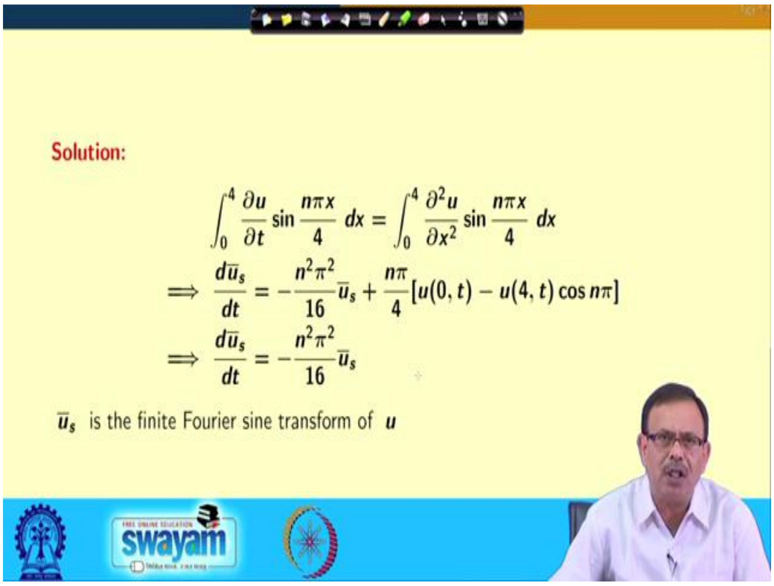

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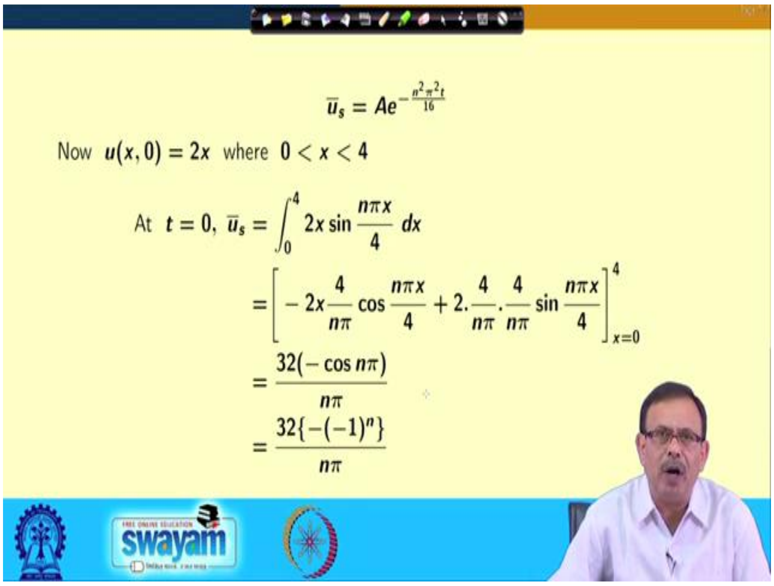

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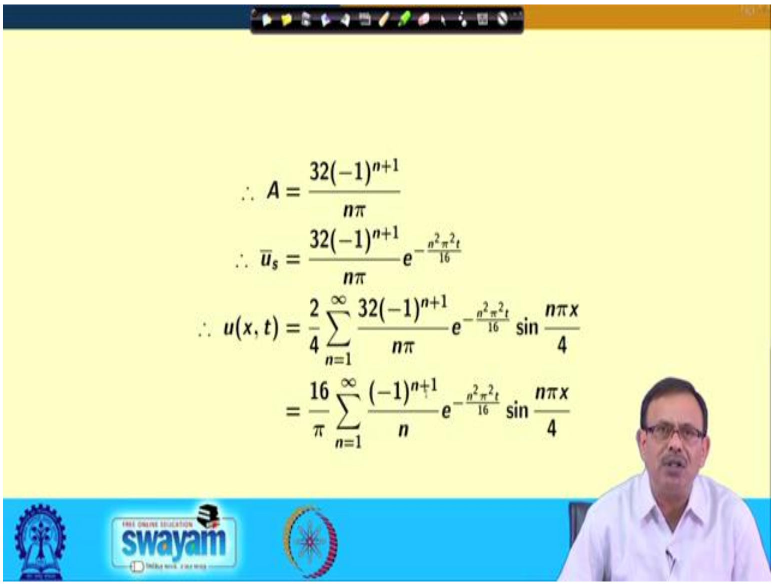

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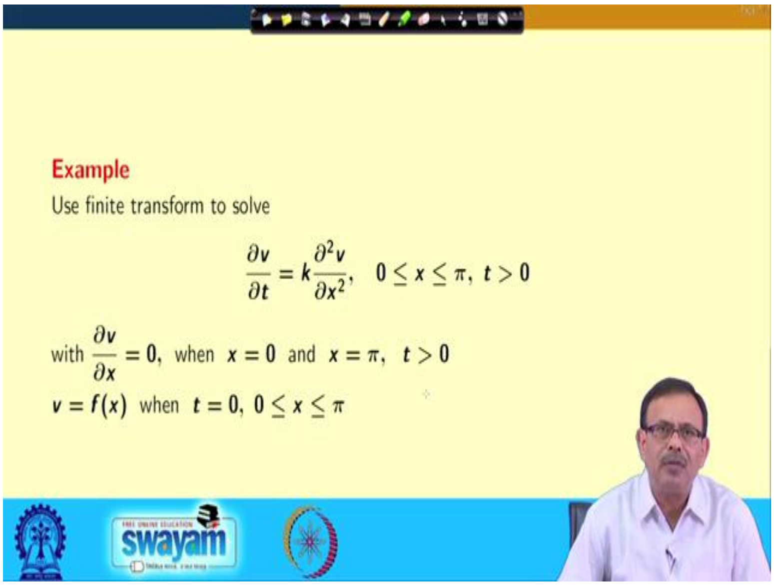

Let us take another example of some other form. We will solve this problem in the next lecture.

Thank you.