Transform Calculus and it is Applications in Differential Equations
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Lecture - 51
Introduction to Finite Fourier Transform

Till now, whatever problems we have solved, the range of those independent variables is either from 0 to ∞ or from $-\infty$ to ∞. But in real life physical problems, sometimes it may happen that they have some finite values, that is it may lie from say 2 to 10 or it may lie from 0 to π or the range is from 0 to some finite value l like that way.

So, whenever we have the finite range for independent variables, we cannot use the transform techniques especially the Fourier or Laplace techniques directly. So, we have to think for something else for the real life physical problems where the independent variable varies within a finite range.

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For that, we will now study the Finite Fourier Transform.

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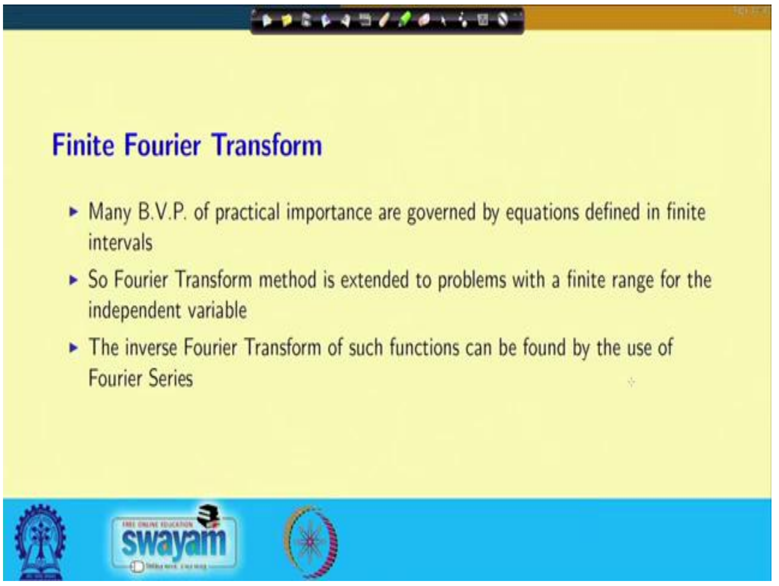

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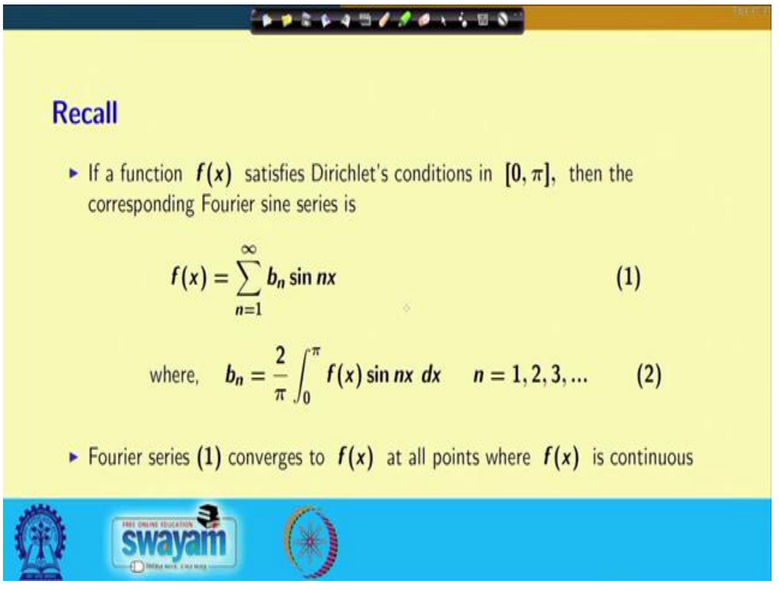

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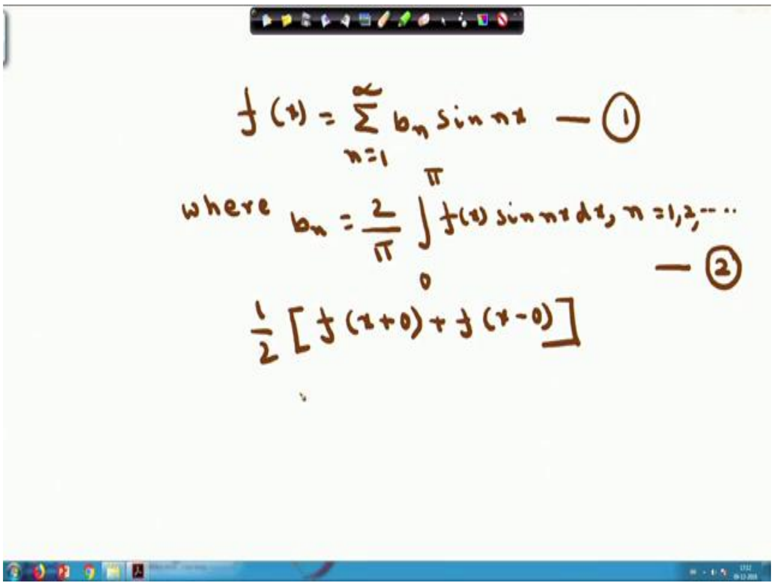

If a function $f(x)$ satisfies the Dirichlet's condition in $[0, \pi]$, then the corresponding Fourier sine series will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx \;, \qquad n = 1,2,3,\cdots \qquad (2)$$

Please note that, Fourier series (1) converges to $f(x)$ at all points where $f(x)$ is continuous and it converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at the points of discontinuities.

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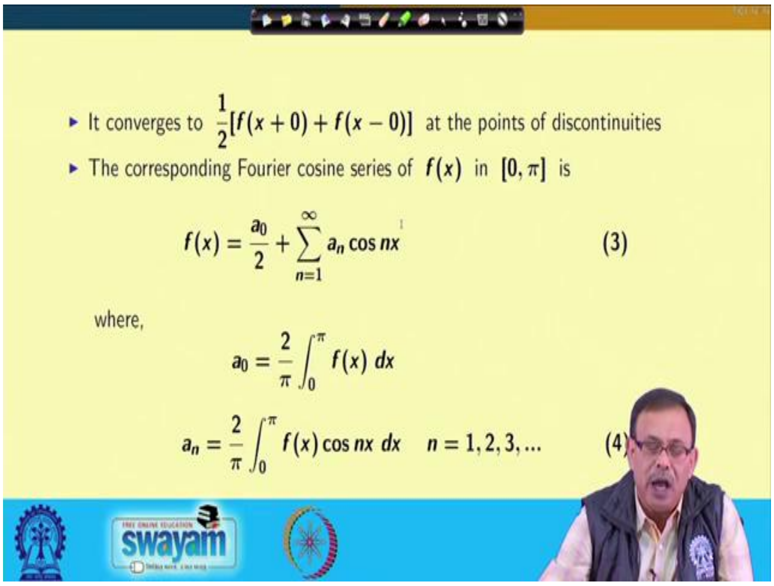

Therefore, this is the Fourier sine series of a function which is defined by (1) and the coefficient b_n is defined by (2).

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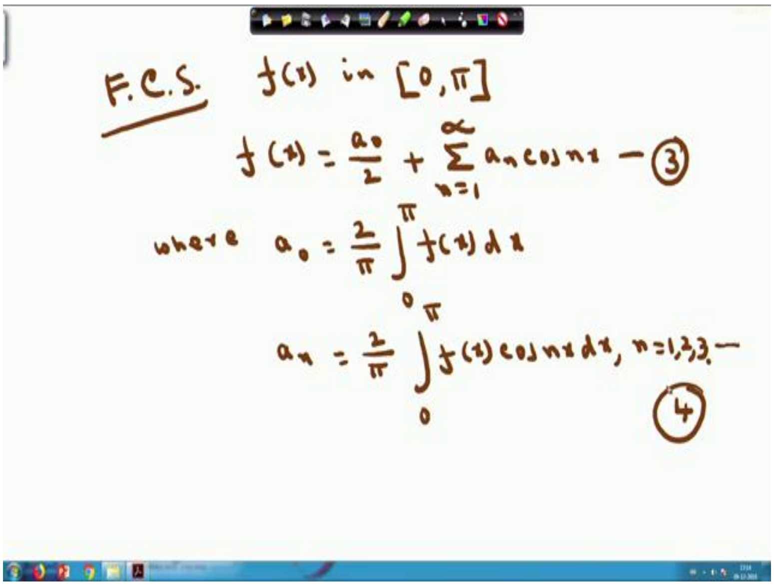

Similarly, Fourier cosine series will be

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \qquad (3)$$

where,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx\ , \qquad n = 1,2,3,\cdots \tag{4}$$

So, Fourier cosine transform is defined by equation (3) where a_0 and a_n are defined by equation (4).

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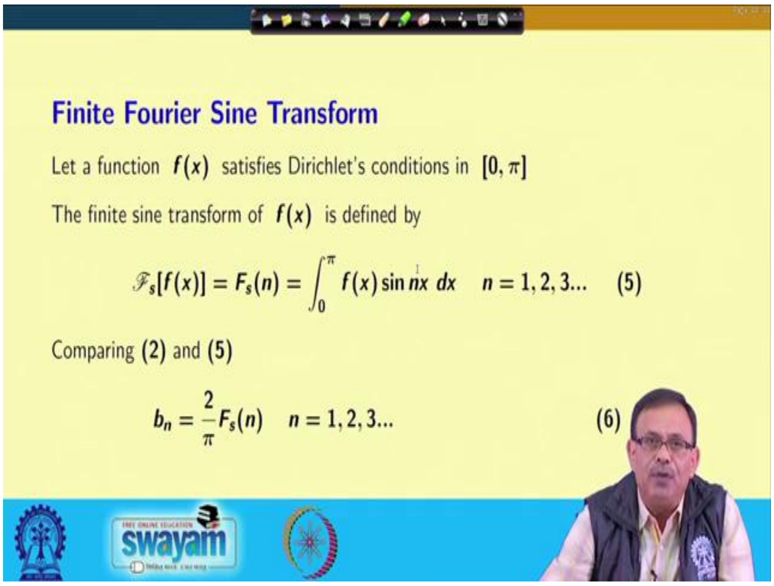

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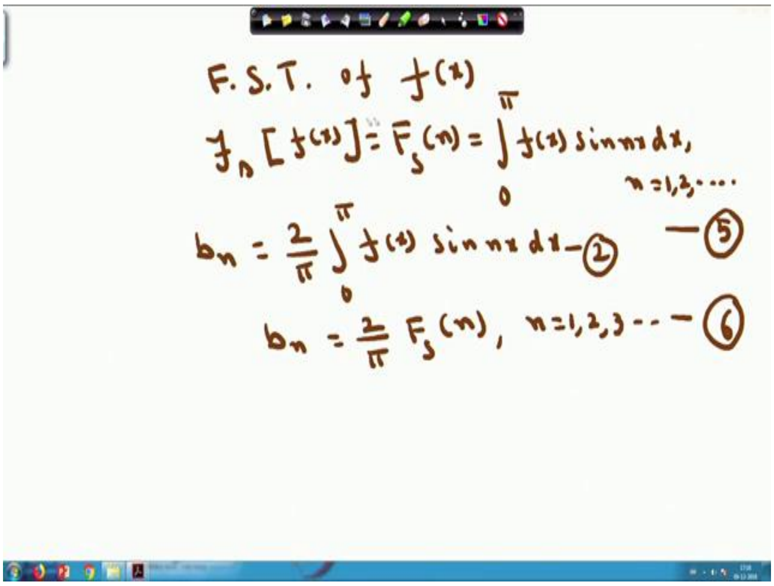

So, now let us come to the finite Fourier Sine Transform. Let a function $f(x)$ satisfies Dirichlet's condition in $[0, \pi]$, then finite Fourier sine Transform is defined as,

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^{\pi} f(x) \sin nx \ dx, \qquad n = 1,2,3 \cdots \tag{5}$$

So, if we compare equation (2) and equation (5) then we will have,

$$b_n = \frac{2}{\pi} F_s(n), \qquad n = 1,2,3 \cdots \tag{6}$$

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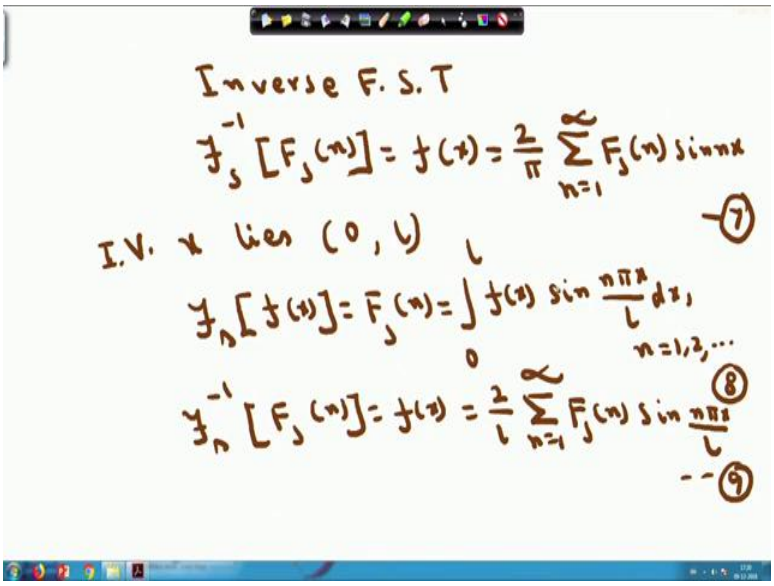

The Inverse Finite Fourier Sine Transform is defined as

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin nx \tag{7}$$

So, please note that we are defining inverse finite Fourier sine transform in terms of Fourier series. If the independent variable x lies in the interval say $[0, l]$, that is any general interval, then finite Fourier sine Transform is defined as,

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^l f(x) \sin\frac{n\pi x}{l}\, dx\,, \qquad n = 1,2,3 \cdots \tag{8}$$

and the inverse finite Fourier sine transform is defined as,

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n) \sin\frac{n\pi x}{l} \tag{9}$$

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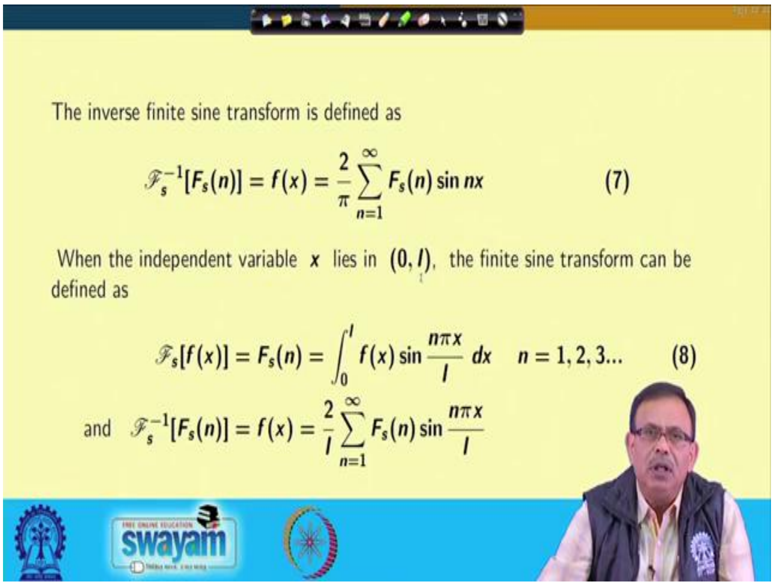

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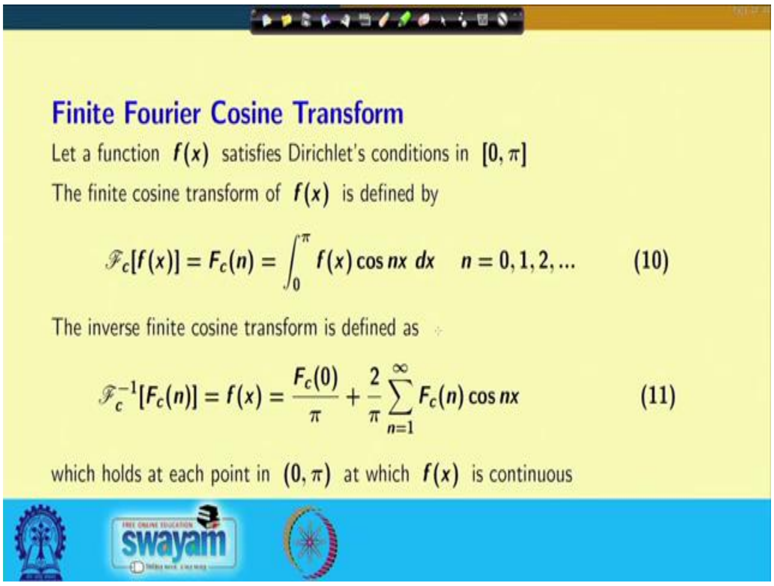

Similarly, we can define the finite Fourier cosine transform as

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^{\pi} f(x) \cos nx \; dx\,, \qquad n = 0,1,2,3 \cdots \qquad (10)$$

and the inverse finite Fourier cosine transform is defined as,

$$\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n) \cos nx \qquad (11)$$

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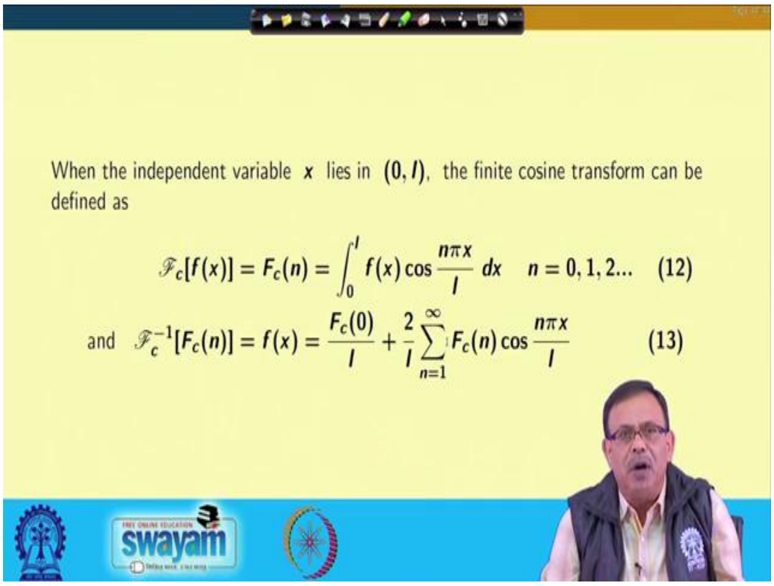

And when the independent variable x lies in $[0, l]$ then finite Fourier cosine Transform is defined as,

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx \,, \qquad n = 0,1,2,3 \cdots \tag{12}$$

and the inverse finite Fourier cosine transform is defined as,

$$\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{l} + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l} \tag{13}$$

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Now let us see the finite transform of derivatives, which will be required whenever we want to find out the solution of second order ODEs.

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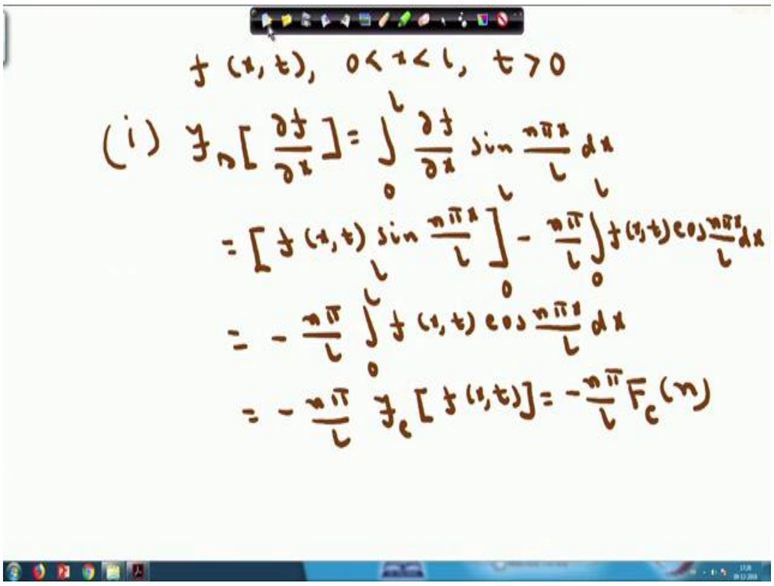

First we will find the Finite Fourier sine transform of $\frac{\partial}{\partial x} f(x,t)$ in $0 < x < l,\ t > 0$ with respect to x. From the definition we have,

$$\mathcal{F}_s\left[\frac{\partial f}{\partial x}\right] = \int_0^l \frac{\partial f}{\partial x} \sin\frac{n\pi x}{l}\, dx$$

Using integration by parts on the RHS we will obtain,

$$\mathcal{F}_s\left[\frac{\partial f}{\partial x}\right] = \left[f(x,t)\ \sin\frac{n\pi x}{l}\right]_{x=0}^{l} - \frac{n\pi}{l}\int_0^l f(x,t)\cos\frac{n\pi x}{l}\, dx$$

Now first part of RHS will be 0 and therefore we will have,

$$\mathcal{F}_s\left[\frac{\partial f}{\partial x}\right] = -\frac{n\pi}{l}\int_0^l f(x,t)\cos\frac{n\pi x}{l}\, dx$$
$$= -\frac{n\pi}{l}\mathcal{F}_c[f(x,t)]$$
$$= -\frac{n\pi}{l}F_c(n)$$

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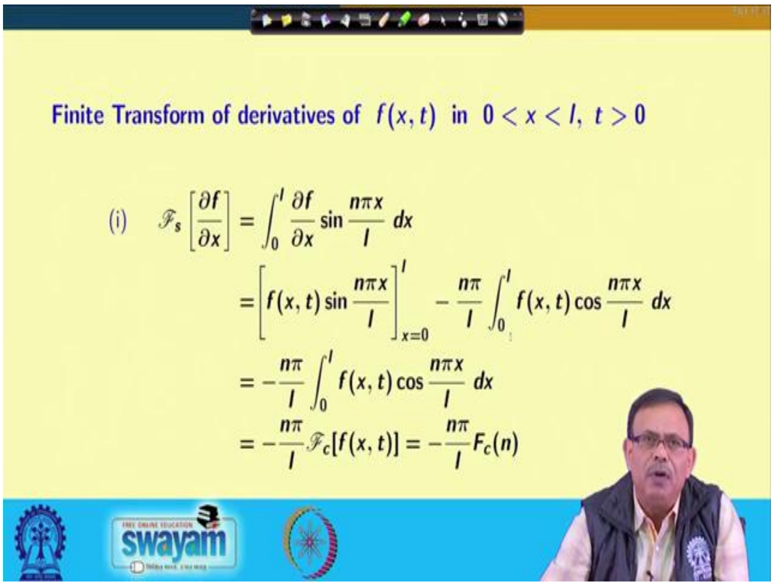

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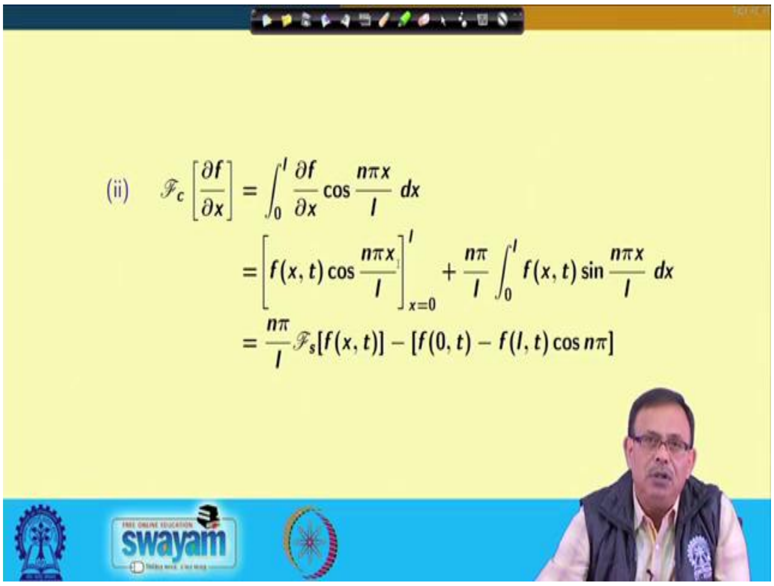

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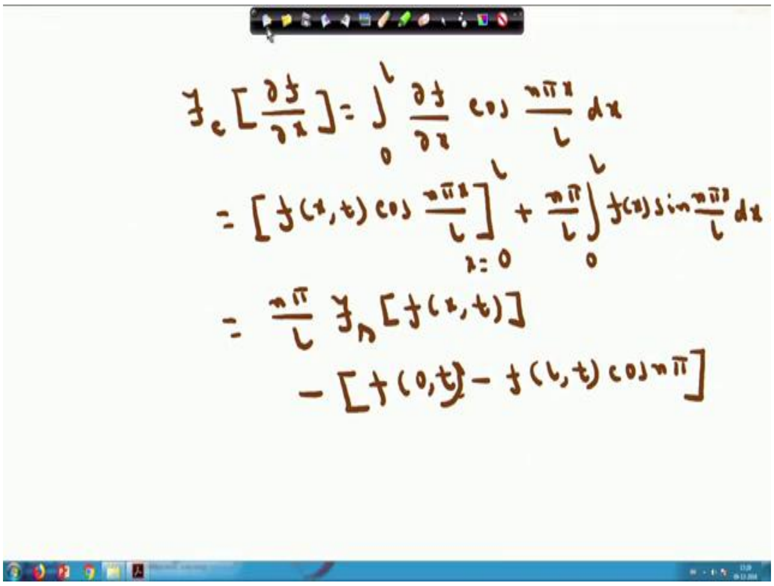

Now, we will find the Finite Fourier cosine transform of $\frac{\partial}{\partial x} f(x,t)$ in $0 < x < l,\ t > 0$ with respect to x. From the definition we have,

$$\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right] = \int_0^l \frac{\partial f}{\partial x} \cos\frac{n\pi x}{l}\ dx$$

Using integration by parts on the RHS, we will obtain,

$$\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right] = \left[f(x,t)\ \cos\frac{n\pi x}{l}\right]_{x=0}^{l} + \frac{n\pi}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\ dx$$

$$= \frac{n\pi}{l}\mathcal{F}_s[f(x,t)] - [f(0,t) - f(l,t)\cos n\pi]$$

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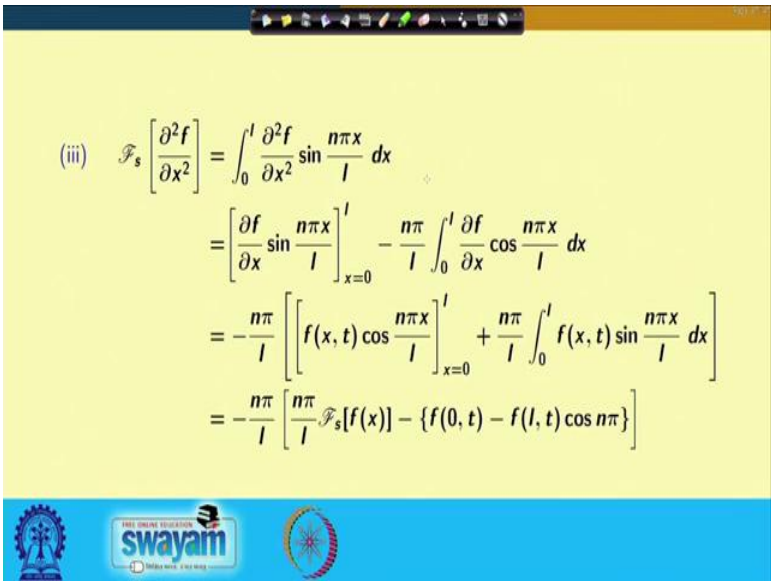

Let us see the finite Fourier sine transform of $\frac{\partial^2}{\partial x^2} f(x,t)$

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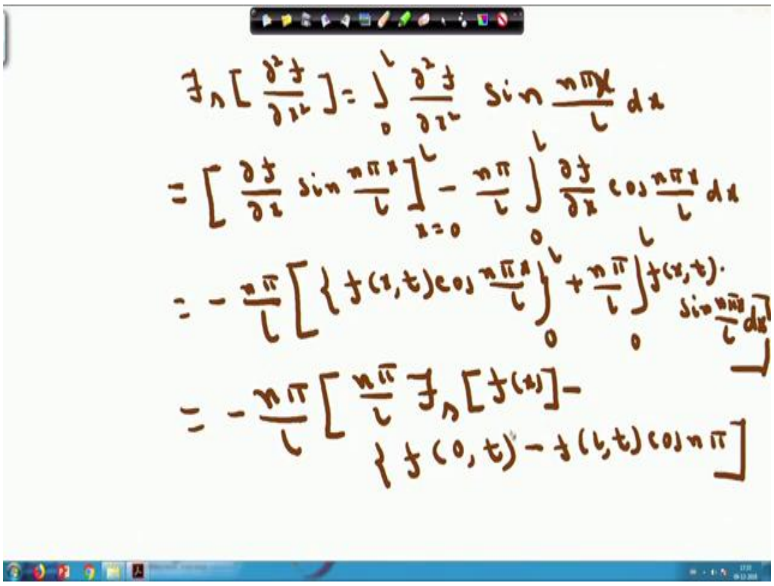

From the definition, we have,

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = \int_0^l \frac{\partial^2 f}{\partial x^2}\sin\frac{n\pi x}{l}\ dx$$

Using integration by parts on the RHS, we will obtain,

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = \left[\frac{\partial f}{\partial x}\ \sin\frac{n\pi x}{l}\right]_{x=0}^{l} - \frac{n\pi}{l}\int_0^l \frac{\partial f}{\partial x}\cos\frac{n\pi x}{l}\ dx$$

Now first part of RHS will be 0 and therefore we will have,

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = -\frac{n\pi}{l}\int_0^l \frac{\partial f}{\partial x}\cos\frac{n\pi x}{l}\ dx$$

$$= -\frac{n\pi}{l}\left(\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right]\right)$$

Now putting the value of $\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right]$, we will get,

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = -\frac{n\pi}{l}\left[\frac{n\pi}{l}\mathcal{F}_s[f(x,t)] - \{f(0,t) - f(l,t)\cos n\pi\}\right]$$

$$= -\frac{n^2\pi^2}{l^2}\mathcal{F}_s[f(x,t)] + \frac{n\pi}{l}[f(0,t) - f(l,t)\cos n\pi]$$

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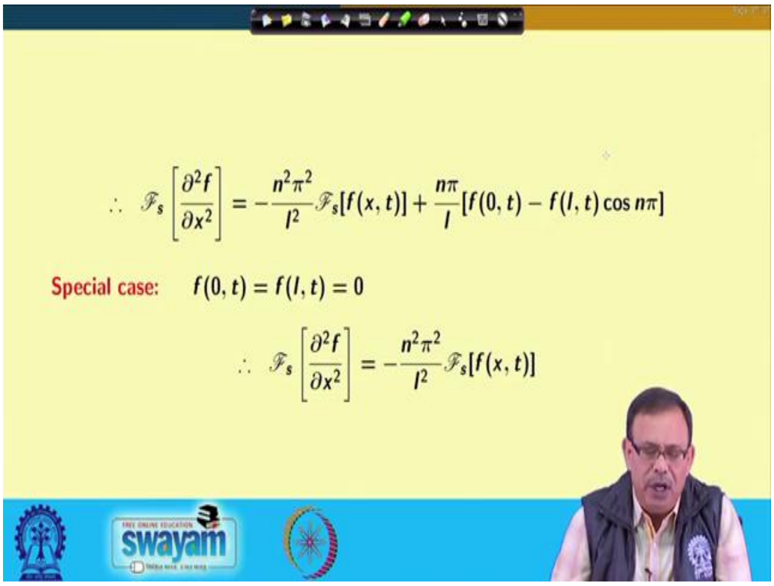

If $f(0,t) = f(l,t) = 0$, then

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}\right] = -\frac{n^2\pi^2}{l^2}\mathcal{F}_s[f(x,t)]$$

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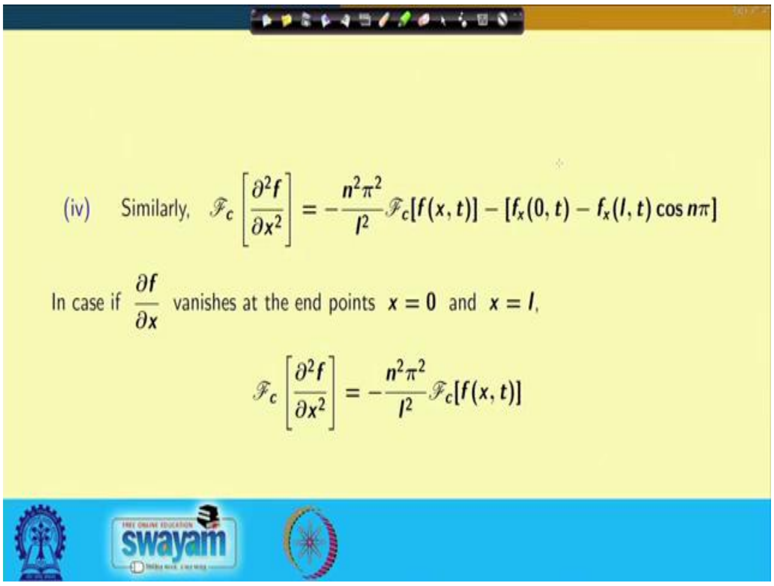

On the same way if we proceed, then we will find that finite Fourier cosine transform of $\frac{\partial^2 f}{\partial x^2}$ is given as,

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}\right] = -\frac{n^2\pi^2}{l^2}\mathcal{F}_c[f(x,t)] - [f_x(0,t) - f_x(l,t)\cos n\pi]$$

If $\frac{\partial f}{\partial x}$ vanishes at the end points $x = 0$ and $x = l$, then,

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}\right] = -\frac{n^2\pi^2}{l^2}\mathcal{F}_c[f(x,t)]$$

So, these 4 formulae are very much required and we have to remember them so that whenever we will try to solve the problems, then we can use these particular formulae. Thank you.