

Transform Calculus and its Applications in Differential Equations
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Lecture - 51
Introduction to Finite Fourier Transform

Till now, whatever problems we have solved, the range of those independent variables is either from 0 to ∞ or from $-\infty$ to ∞ . But in real life physical problems, sometimes it may happen that they have some finite values, that is it may lie from say 2 to 10 or it may lie from 0 to π or the range is from 0 to some finite value l like that way.

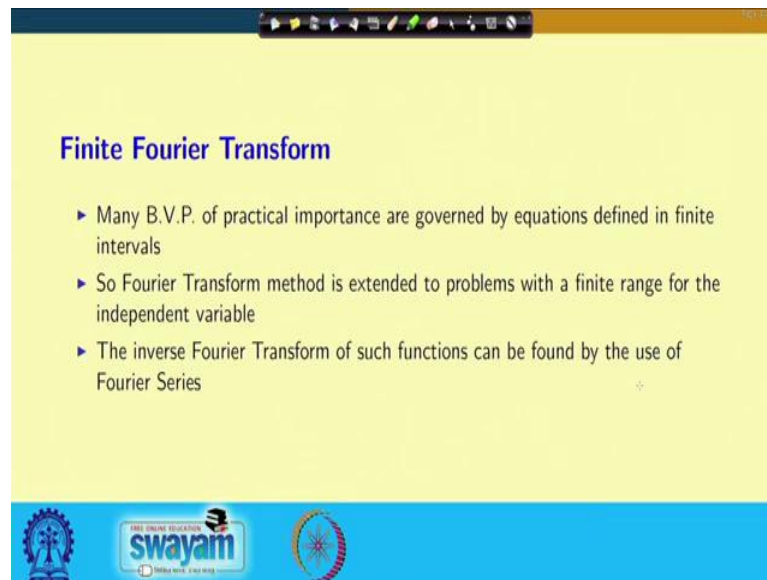
So, whenever we have the finite range for independent variables, we cannot use the transform techniques especially the Fourier or Laplace techniques directly. So, we have to think for something else for the real life physical problems where the independent variable varies within a finite range.

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For that, we will now study the Finite Fourier Transform.

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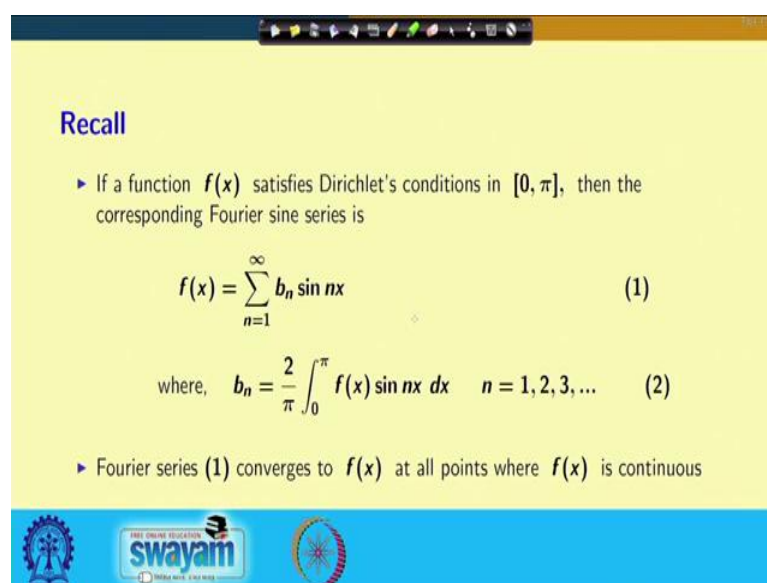


Finite Fourier Transform

- ▶ Many B.V.P. of practical importance are governed by equations defined in finite intervals
- ▶ So Fourier Transform method is extended to problems with a finite range for the independent variable
- ▶ The inverse Fourier Transform of such functions can be found by the use of Fourier Series

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Recall

- ▶ If a function $f(x)$ satisfies Dirichlet's conditions in $[0, \pi]$, then the corresponding Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

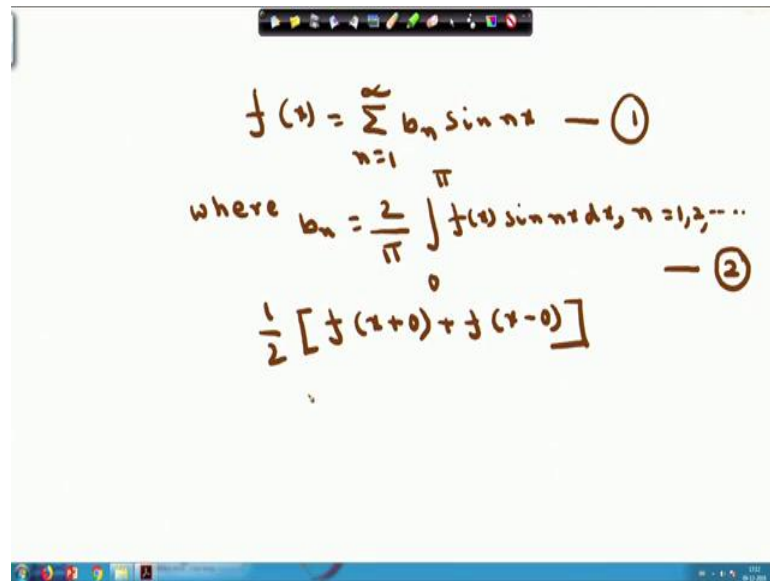
where, $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad n = 1, 2, 3, \dots \quad (2)$

- ▶ Fourier series (1) converges to $f(x)$ at all points where $f(x)$ is continuous

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Kindly refer to the slides.

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$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$
$$\text{where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n=1,2,\dots \quad \text{--- (2)}$$
$$\frac{1}{2} [f(x+0) + f(x-0)]$$

If a function $f(x)$ satisfies the Dirichlet's condition in $[0, \pi]$, then the corresponding Fourier sine series will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, 3, \dots \quad (2)$$

Please note that, Fourier series (1) converges to $f(x)$ at all points where $f(x)$ is continuous and it converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at the points of discontinuities.

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► It converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at the points of discontinuities

► The corresponding Fourier cosine series of $f(x)$ in $[0, \pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (3)$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \quad n = 1, 2, 3, \dots \quad (4)$$

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Therefore, this is the Fourier sine series of a function which is defined by (1) and the coefficient b_n is defined by (2).

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F.C.S. $f(x)$ in $[0, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (3)$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad n=1, 2, 3, \dots \quad (4)$$

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Similarly, Fourier cosine series will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (3)$$

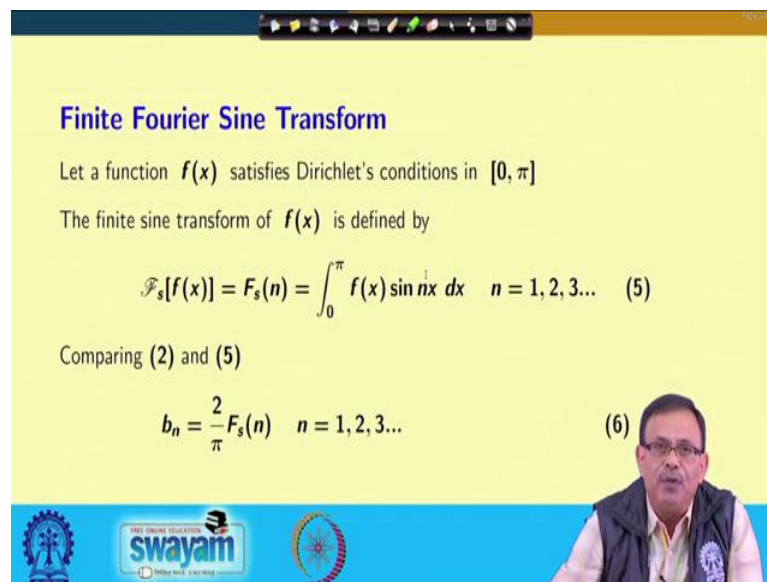
where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad n = 1, 2, 3, \dots \quad (4)$$

So, Fourier cosine transform is defined by equation (3) where a_0 and a_n are defined by equation (4).

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The slide is titled "Finite Fourier Sine Transform" and contains the following text and equations:

Let a function $f(x)$ satisfies Dirichlet's conditions in $[0, \pi]$

The finite sine transform of $f(x)$ is defined by

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^{\pi} f(x) \sin nx dx \quad n = 1, 2, 3, \dots \quad (5)$$

Comparing (2) and (5)

$$b_n = \frac{2}{\pi} F_s(n) \quad n = 1, 2, 3, \dots \quad (6)$$

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F.S.T. of $f(x)$
 $f_n [f(x)] = F_s(n) = \int_0^\pi f(x) \sin nx dx, \quad n=1,2,3,\dots$
 $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx - (2) \quad \text{--- (5)}$
 $b_n = \frac{2}{\pi} F_s(n), \quad n=1,2,3,\dots \quad \text{--- (6)}$

So, now let us come to the finite Fourier Sine Transform. Let a function $f(x)$ satisfies Dirichlet's condition in $[0, \pi]$, then finite Fourier sine Transform is defined as,

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^\pi f(x) \sin nx \, dx, \quad n = 1, 2, 3, \dots \quad (5)$$

So, if we compare equation (2) and equation (5) then we will have,

$$b_n = \frac{2}{\pi} F_s(n), \quad n = 1, 2, 3, \dots \quad (6)$$

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Inverse F. S. T

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin nx \quad (7)$$

I.V. x lies $(0, l)$

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \quad n=1,2,\dots \quad (8)$$

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l} \quad (9)$$

The Inverse Finite Fourier Sine Transform is defined as

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin nx \quad (7)$$

So, please note that we are defining inverse finite Fourier sine transform in terms of Fourier series. If the independent variable x lies in the interval say $[0, l]$, that is any general interval, then finite Fourier sine Transform is defined as,

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1,2,3 \dots \quad (8)$$

and the inverse finite Fourier sine transform is defined as,

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l} \quad (9)$$

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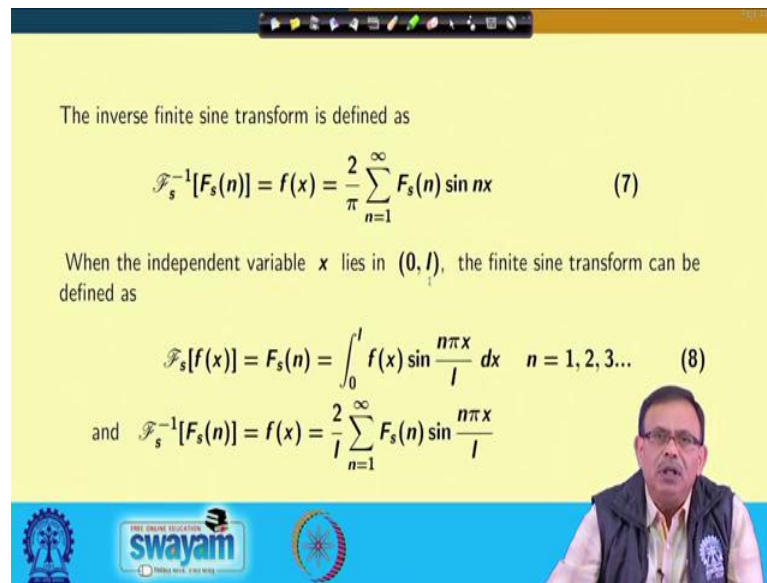
The inverse finite sine transform is defined as

$$\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin nx \quad (7)$$

When the independent variable x lies in $(0, l)$, the finite sine transform can be defined as

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad n = 1, 2, 3, \dots \quad (8)$$

and $\mathcal{F}_s^{-1}[F_s(n)] = f(x) = \frac{2}{l} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l}$



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Finite Fourier Cosine Transform

Let a function $f(x)$ satisfies Dirichlet's conditions in $[0, \pi]$

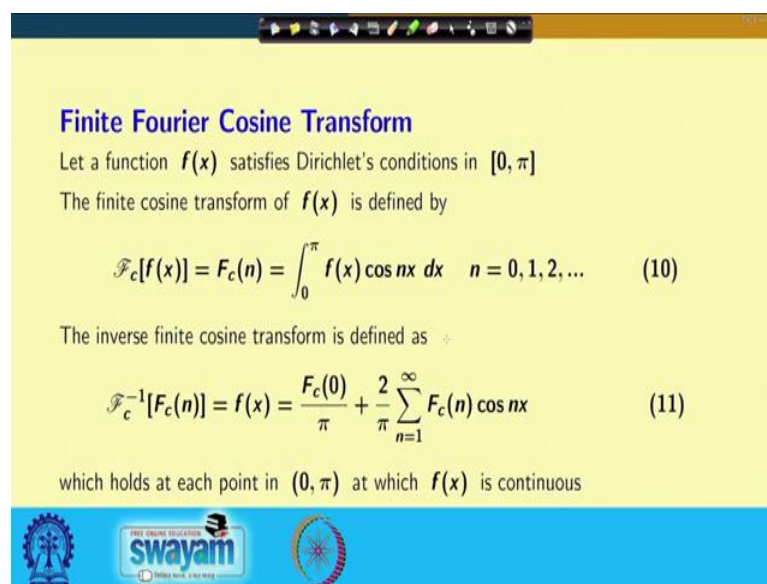
The finite cosine transform of $f(x)$ is defined by

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^{\pi} f(x) \cos nx dx \quad n = 0, 1, 2, \dots \quad (10)$$

The inverse finite cosine transform is defined as

$$\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} F_c(n) \cos nx \quad (11)$$

which holds at each point in $(0, \pi)$ at which $f(x)$ is continuous



Similarly, we can define the finite Fourier cosine transform as

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, 3, \dots \quad (10)$$

and the inverse finite Fourier cosine transform is defined as,

$$\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} F_c(n) \cos nx \quad (11)$$

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When the independent variable x lies in $(0, l)$, the finite cosine transform can be defined as

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad n = 0, 1, 2, \dots \quad (12)$$

and $\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{l} + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l} \quad (13)$

And when the independent variable x lies in $[0, l]$ then finite Fourier cosine Transform is defined as,

$$\mathcal{F}_c[f(x)] = F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, 3 \dots \quad (12)$$

and the inverse finite Fourier cosine transform is defined as,

$$\mathcal{F}_c^{-1}[F_c(n)] = f(x) = \frac{F_c(0)}{l} + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l} \quad (13)$$

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Now let us see the finite transform of derivatives, which will be required whenever we want to find out the solution of second order ODEs.

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The image shows a handwritten derivation on a whiteboard. It starts with the function $f(x, t)$ for $0 < x < l$ and $t > 0$. The derivation shows the finite transform of the derivative $\frac{\partial f}{\partial x}$ with respect to x . The steps are as follows:

$$\begin{aligned} (i) \quad \mathcal{F}_n \left[\frac{\partial f}{\partial x} \right] &= \int_0^l \frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} dx \\ &= \left[f(x, t) \sin \frac{n\pi x}{l} \right]_0^l - \frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx \\ &= - \frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx \\ &= - \frac{n\pi}{l} \mathcal{F}_c [f(x, t)] = - \frac{n\pi}{l} F_c(n) \end{aligned}$$

First we will find the Finite Fourier sine transform of $\frac{\partial}{\partial x} f(x, t)$ in $0 < x < l, t > 0$ with respect to x . From the definition we have,

$$\mathcal{F}_s \left[\frac{\partial f}{\partial x} \right] = \int_0^l \frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} dx$$

Using integration by parts on the RHS we will obtain,

$$\mathcal{F}_s \left[\frac{\partial f}{\partial x} \right] = \left[f(x, t) \sin \frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx$$

Now first part of RHS will be 0 and therefore we will have,

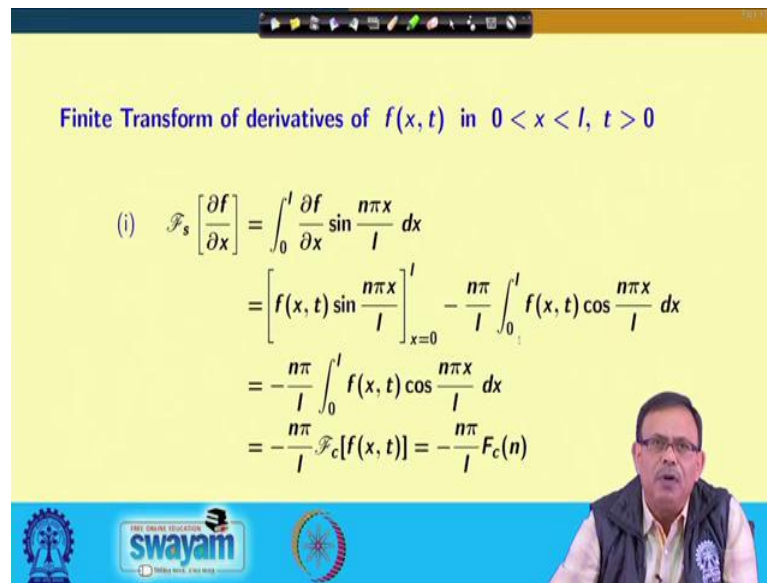
$$\begin{aligned} \mathcal{F}_s \left[\frac{\partial f}{\partial x} \right] &= -\frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx \\ &= -\frac{n\pi}{l} \mathcal{F}_c[f(x, t)] \\ &= -\frac{n\pi}{l} F_c(n) \end{aligned}$$

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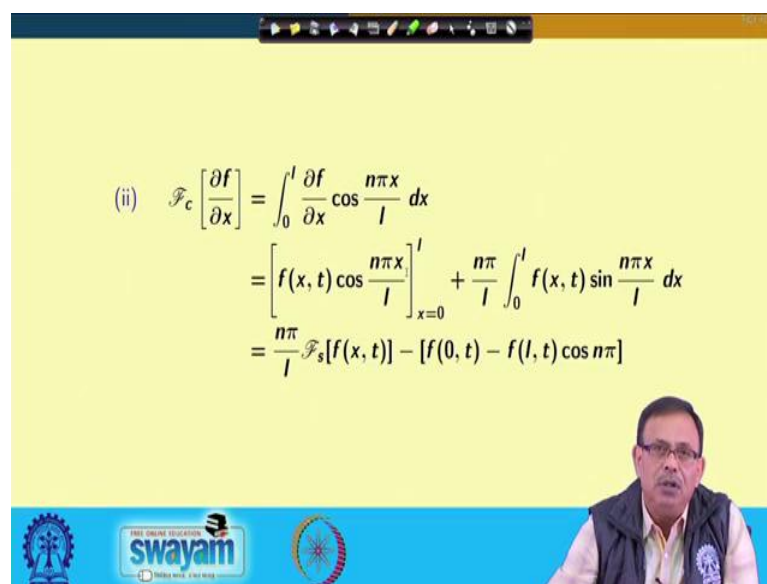


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Finite Transform of derivatives of $f(x, t)$ in $0 < x < l, t > 0$

$$\begin{aligned} \text{(i)} \quad \mathcal{F}_s \left[\frac{\partial f}{\partial x} \right] &= \int_0^l \frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} dx \\ &= \left[f(x, t) \sin \frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx \\ &= -\frac{n\pi}{l} \int_0^l f(x, t) \cos \frac{n\pi x}{l} dx \\ &= -\frac{n\pi}{l} \mathcal{F}_c[f(x, t)] = -\frac{n\pi}{l} F_c(n) \end{aligned}$$


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$$\begin{aligned} \text{(ii)} \quad \mathcal{F}_c \left[\frac{\partial f}{\partial x} \right] &= \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx \\ &= \left[f(x, t) \cos \frac{n\pi x}{l} \right]_{x=0}^l + \frac{n\pi}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \\ &= \frac{n\pi}{l} \mathcal{F}_s[f(x, t)] - [f(0, t) - f(l, t) \cos n\pi] \end{aligned}$$


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The image shows a handwritten derivation on a whiteboard. The derivation starts with the finite Fourier cosine transform of the partial derivative of a function $f(x, t)$ with respect to x . The steps are as follows:

$$\begin{aligned}\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right] &= \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx \\ &= \left[f(x, t) \cos \frac{n\pi x}{l} \right]_{x=0}^l + \frac{n\pi}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \\ &= \frac{n\pi}{l} \mathcal{F}_s[f(x, t)] \\ &\quad - [f(0, t) - f(l, t) \cos n\pi]\end{aligned}$$

Now, we will find the Finite Fourier cosine transform of $\frac{\partial}{\partial x} f(x, t)$ in $0 < x < l$, $t > 0$ with respect to x . From the definition we have,

$$\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right] = \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx$$

Using integration by parts on the RHS, we will obtain,

$$\begin{aligned}\mathcal{F}_c\left[\frac{\partial f}{\partial x}\right] &= \left[f(x, t) \cos \frac{n\pi x}{l} \right]_{x=0}^l + \frac{n\pi}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \\ &= \frac{n\pi}{l} \mathcal{F}_s[f(x, t)] - [f(0, t) - f(l, t) \cos n\pi]\end{aligned}$$

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$$\begin{aligned}
 \text{(iii) } \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] &= \int_0^l \frac{\partial^2 f}{\partial x^2} \sin \frac{n\pi x}{l} dx \\
 &= \left[\frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx \\
 &= -\frac{n\pi}{l} \left[\left[f(x, t) \cos \frac{n\pi x}{l} \right]_{x=0}^l + \frac{n\pi}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{n\pi}{l} \left[\frac{n\pi}{l} \mathcal{F}_s[f(x)] - \{f(0, t) - f(l, t) \cos n\pi\} \right]
 \end{aligned}$$

Let us see the finite Fourier sine transform of $\frac{\partial^2}{\partial x^2} f(x, t)$

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$$\begin{aligned}
 \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] &= \int_0^l \frac{\partial^2 f}{\partial x^2} \sin \frac{n\pi x}{l} dx \\
 &= \left[\frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx \\
 &= -\frac{n\pi}{l} \left[\left[f(x, t) \cos \frac{n\pi x}{l} \right]_{x=0}^l + \frac{n\pi}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{n\pi}{l} \left[\frac{n\pi}{l} \mathcal{F}_s[f(x)] - \{f(0, t) - f(l, t) \cos n\pi\} \right]
 \end{aligned}$$

From the definition, we have,

$$\mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] = \int_0^l \frac{\partial^2 f}{\partial x^2} \sin \frac{n\pi x}{l} dx$$

Using integration by parts on the RHS, we will obtain,

$$\mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] = \left[\frac{\partial f}{\partial x} \sin \frac{n\pi x}{l} \right]_{x=0}^l - \frac{n\pi}{l} \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx$$

Now first part of RHS will be 0 and therefore we will have,

$$\begin{aligned} \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] &= -\frac{n\pi}{l} \int_0^l \frac{\partial f}{\partial x} \cos \frac{n\pi x}{l} dx \\ &= -\frac{n\pi}{l} \left(\mathcal{F}_c \left[\frac{\partial f}{\partial x} \right] \right) \end{aligned}$$

Now putting the value of $\mathcal{F}_c \left[\frac{\partial f}{\partial x} \right]$, we will get,

$$\begin{aligned} \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] &= -\frac{n\pi}{l} \left[\frac{n\pi}{l} \mathcal{F}_s[f(x, t)] - \{f(0, t) - f(l, t) \cos n\pi\} \right] \\ &= -\frac{n^2\pi^2}{l^2} \mathcal{F}_s[f(x, t)] + \frac{n\pi}{l} [f(0, t) - f(l, t) \cos n\pi] \end{aligned}$$

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$$\therefore \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2\pi^2}{l^2} \mathcal{F}_s[f(x, t)] + \frac{n\pi}{l} [f(0, t) - f(l, t) \cos n\pi]$$

Special case: $f(0, t) = f(l, t) = 0$

$$\therefore \mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2\pi^2}{l^2} \mathcal{F}_s[f(x, t)]$$

If $f(0, t) = f(l, t) = 0$, then

$$\mathcal{F}_s \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2 \pi^2}{l^2} \mathcal{F}_s [f(x, t)]$$

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(iv) Similarly, $\mathcal{F}_c \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2 \pi^2}{l^2} \mathcal{F}_c [f(x, t)] - [f_x(0, t) - f_x(l, t) \cos n\pi]$

In case if $\frac{\partial f}{\partial x}$ vanishes at the end points $x = 0$ and $x = l$,

$$\mathcal{F}_c \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2 \pi^2}{l^2} \mathcal{F}_c [f(x, t)]$$

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On the same way if we proceed, then we will find that finite Fourier cosine transform of $\frac{\partial^2 f}{\partial x^2}$ is given as,

$$\mathcal{F}_c \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2 \pi^2}{l^2} \mathcal{F}_c [f(x, t)] - [f_x(0, t) - f_x(l, t) \cos n\pi]$$

If $\frac{\partial f}{\partial x}$ vanishes at the end points $x = 0$ and $x = l$, then,

$$\mathcal{F}_c \left[\frac{\partial^2 f}{\partial x^2} \right] = -\frac{n^2 \pi^2}{l^2} \mathcal{F}_c [f(x, t)]$$

So, these 4 formulae are very much required and we have to remember them so that whenever we will try to solve the problems, then we can use these particular formulae.

Thank you.