Transform Calculus and Its Applications in Differential Equations
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Lecture - 50
Solving problems on Partial Differential Equations using Transform Techniques

In the last few lectures, we have tried to find out the solution of Partial Differential Equations using the Laplace Transform and Fourier transform. In Fourier transform, we have seen that whenever we are using either Fourier transform or Fourier cosine transform or Fourier sine transform, the given PDE is converted into a first order ODE or a second order ODE and in each case, the solution can be found out very easily.

So for a given dependent variable u, we are obtaining $\bar{u}$ by applying the corresponding transform and from $\bar{u}$, using the inversion function, we can find out $u(x,t)$. In between, we have to find out the values of the constants obtained during integration because whenever we are solving the ODE, if it is the second order ODE, in that case, there will be 2 arbitrary constants whereas, for first order, there will be one constant. So using certain conditions, we have to find out the values of those constants and after that, we have to find out the value of $u(x,t)$.

So, we have solved various kinds of problems using Fourier transform, Fourier cosine transform, Fourier sine transform and we have tried to use the general functions in the conditions like $f(x)$ or $g(x)$. So that for any value of $f(x)$ or $g(x)$, we can find out the solution without resolving the entire problem again and again. So, let us take one or two more problems on this and let us see how we can find out the solution of the PDE and after that, we will go to the finite Fourier transform.

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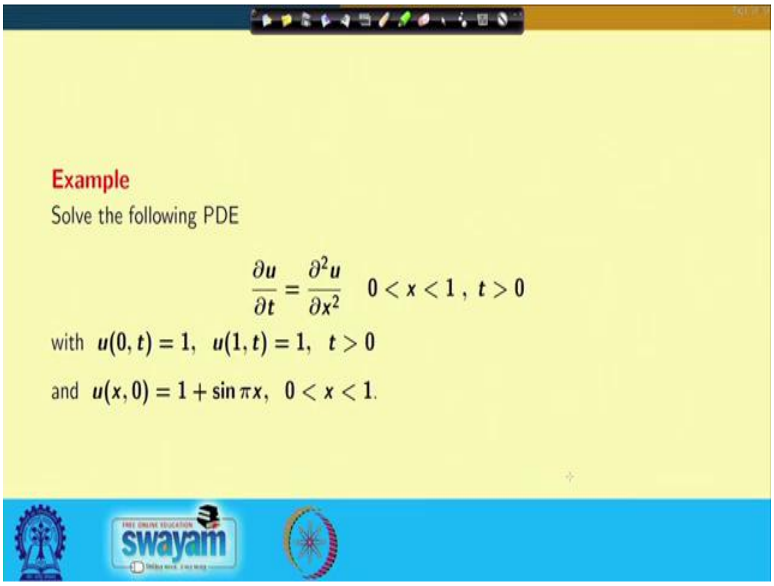

So, first let us see the example. We want to solve the following PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1,\ \ t > 0$$

$$\text{with, } \ u(0,t) = 1,\ \ u(1,t) = 1 \ \text{ for } t > 0 \text{ and } u(x,0) = 1 + \sin \pi x \ \text{ for } 0 < x < 1$$

So, here, we have 2 independent variables x and t, x lies between 0 to 1 and t is greater than 0. Since x lies between 0 to 1, therefore the question of using any transform, whether it is Laplace or Fourier or Fourier cosine or sine with respect to x, does not arise at all. Because we know that for the Laplace transform, the independent variable should vary from 0 to ∞ as well as for Fourier sine and cosine transforms, whereas for Fourier transform, the independent variable should vary from $-\infty$ to ∞.

Again, here $t > 0$, that means the range of t is from 0 to ∞. But in this particular case, we can neither use Fourier cosine nor Fourier sine transform because, u and u_t should approach 0 as x approaches or t approaches ∞, that condition is required for this case. So, we cannot use in this case, Fourier sine transform or Fourier cosine transform. Therefore, to solve this problem, we are compelled to use the Laplace transform only.

So, by this way, whenever a problem will be given to us, at first we have to determine that which would be the appropriate transform technique, by which we can solve the problem. And, again, one particular problem may be solved by using more than one transform technique, we have shown it earlier. We have solved the same problem by various techniques.

So, for this particular problem, as $t > 0$, that is range of t is from 0 to ∞ and values of u are given at x equals 0 and 1, and at t equals 0, therefore we are compelled to use Laplace transform to solve this particular problem.

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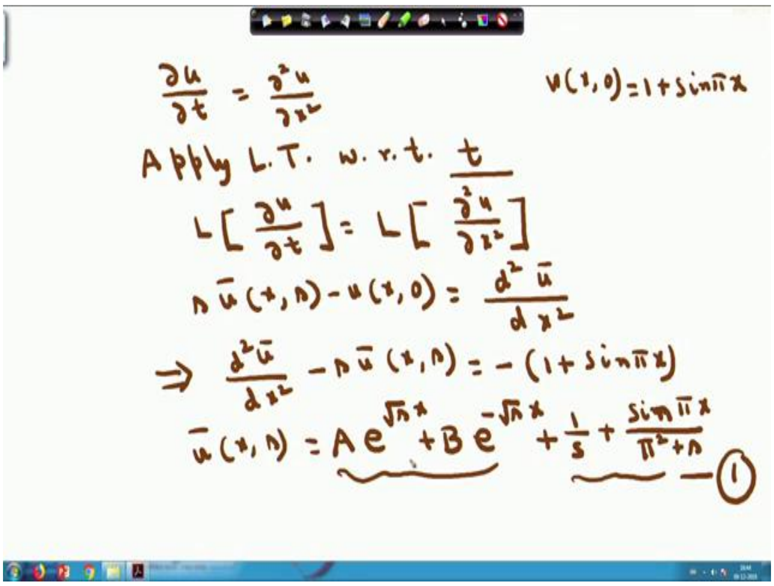

So we take Laplace Transform on both sides of the given PDE to obtain,

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s\,\bar{u}(x,s) - u(x,0) = \frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - s\bar{u} = -1 - \sin \pi x$$

Therefore, from the given PDE, we obtained an ODE.

The beauty of these techniques is that the PDE is converted into ODE or if we are trying to solve one ODE, in that case, simply the ODE is transformed into some algebraic expression which we have seen earlier, whenever we have used Laplace transform to solve one second order ODE. We are directly writing the solution of this second order ODE. There will be two parts, C.F. and P.I.

Auxiliary equation for the obtained ODE is,

$$m^2 - s = 0 \Rightarrow m = \pm\sqrt{s}$$

Therefore, C.F. of the ODE is given as,

$$\mathrm{C.F.} = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$$

And P.I. is given as

$$\mathrm{P.I.} = \frac{1}{\mathrm{s}} + \frac{\sin \pi x}{\pi^2 + s}$$

Therefore, the general solution is given as,

$$\bar{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{\mathrm{s}} + \frac{\sin \pi x}{\pi^2 + s} \qquad (1)$$

From the given initial conditions, we have, $u(0,t) = 1$ and $u(1,t) = 1$. So, we can say that,

$$\bar{u}(0,s) = \frac{1}{s} \text{ and } \bar{u}(1,s) = \frac{1}{s}$$

$$\therefore (1) \Rightarrow A + B + \frac{1}{s} = \frac{1}{s} \text{ and}$$

$$Ae^{\sqrt{s}} + Be^{-\sqrt{s}} + \frac{1}{\mathrm{s}} = \frac{1}{s}$$

$$\therefore A + B = 0,\ \ Ae^{\sqrt{s}} + Be^{-\sqrt{s}} = 0$$

Now, if we take the coefficient determinant of these two equations, we will have

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{s}} & e^{-\sqrt{s}} \end{vmatrix} = e^{-\sqrt{s}} - e^{\sqrt{s}} \neq 0$$

Thus, the determinant does not vanish. So, we can say that only trivial solution is possible for A and B. Trivial solution will be $A = B = 0$. This is because it is a homogeneous system of equations.

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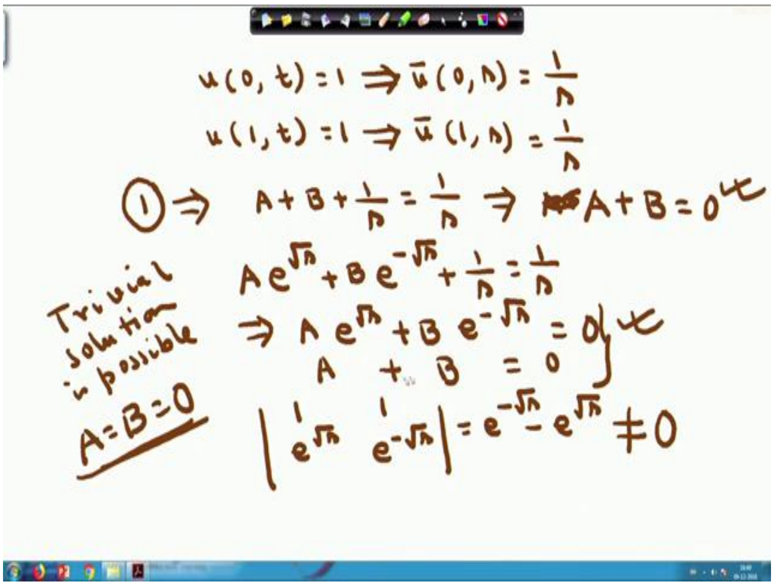

Therefore, (1) reduces to

$$\bar{u}(x,s) = \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s}$$

Now, taking inverse Laplace transform on both sides, we have,

$$\begin{aligned} u(x,t) &= L^{-1}\{\bar{u}(x,s)\} \\ &= L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{\sin \pi x}{\pi^2 + s}\right\} \\ &= 1 + \sin \pi x \, L^{-1}\left\{\frac{1}{\pi^2 + s}\right\} \\ &= 1 + e^{-\pi^2 t} \sin \pi x \end{aligned}$$

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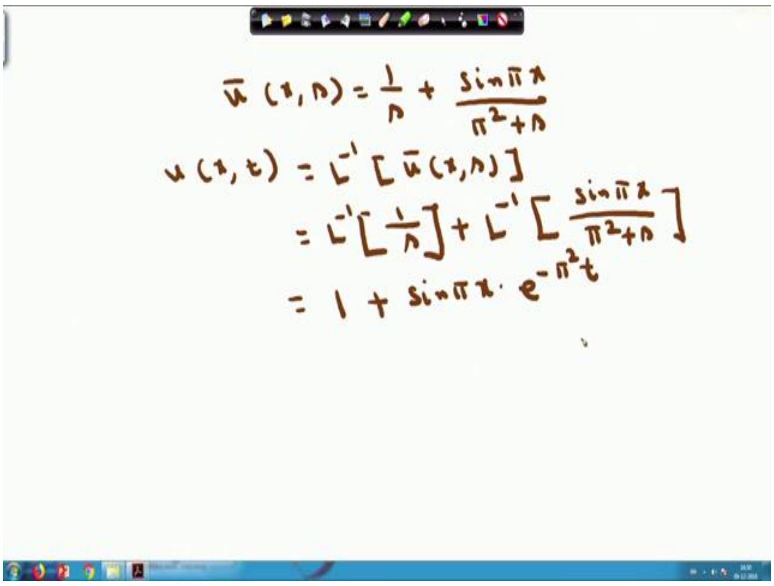

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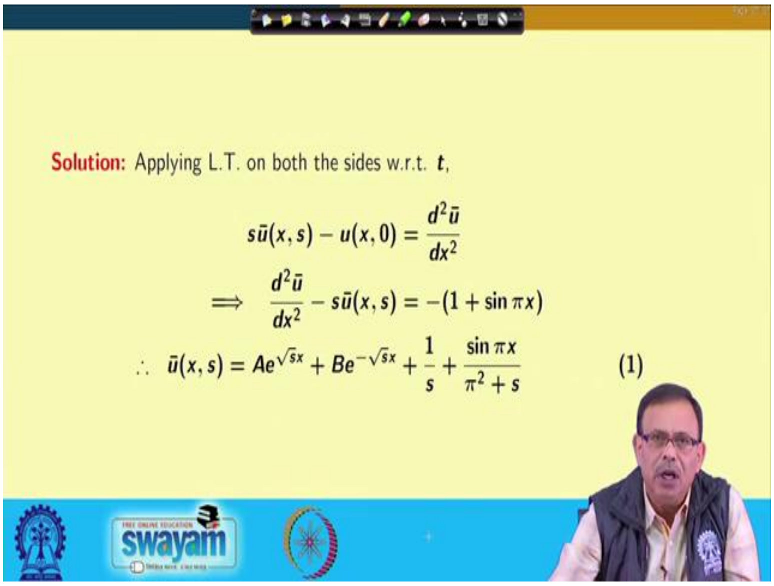

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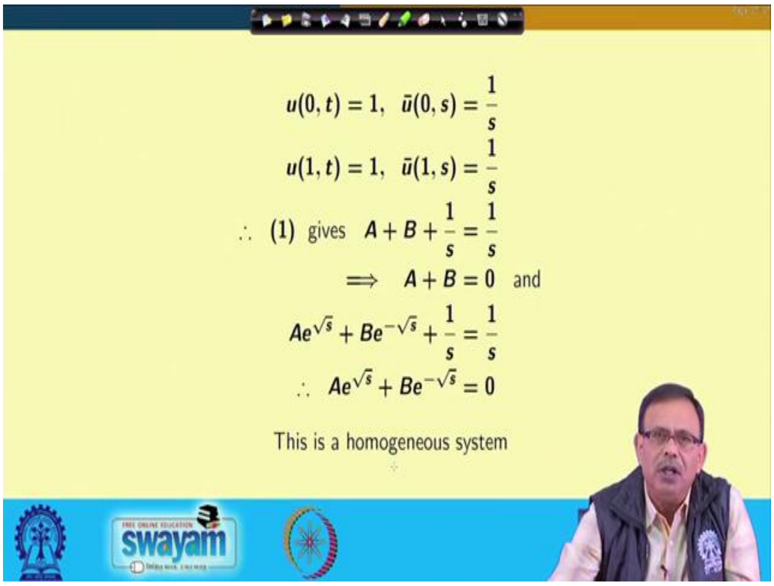

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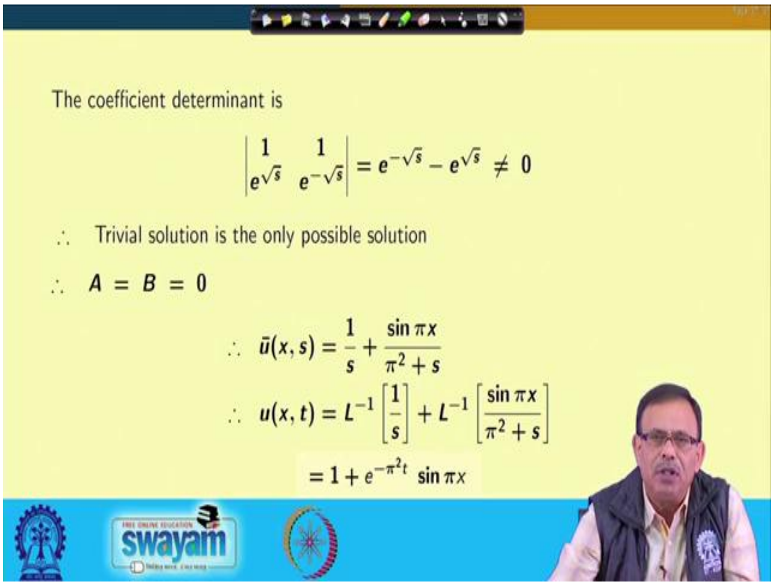

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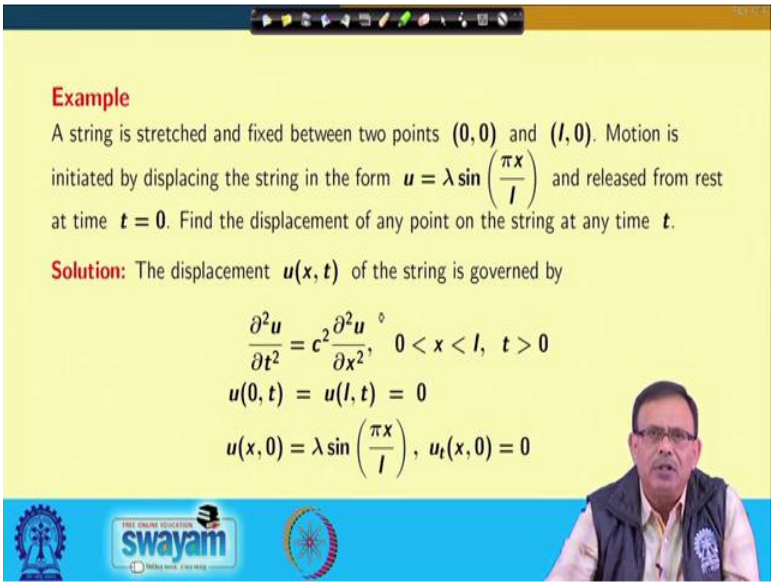

Let us take another problem. A string is stretched and fixed between 2 points, (0,0) and $(l, 0)$. Motion is initiated by displacing the string in the form, $u = \lambda \sin\left(\frac{\pi x}{l}\right)$ and released from rest at time t equals 0. We have to find the displacement of any point on the string at any point of time t. So, basically we need to find the displacement that is $u(x, t)$.

So, effectively we have to formulate the problem first, then only we can think about the solution process. So, it may not be necessary that the problem is always stated in the form of a differential equation. Sometimes, a problem may be given like this, from where we have to formulate the problem and after formulating the problem, we have to find out the solution of that particular problem.

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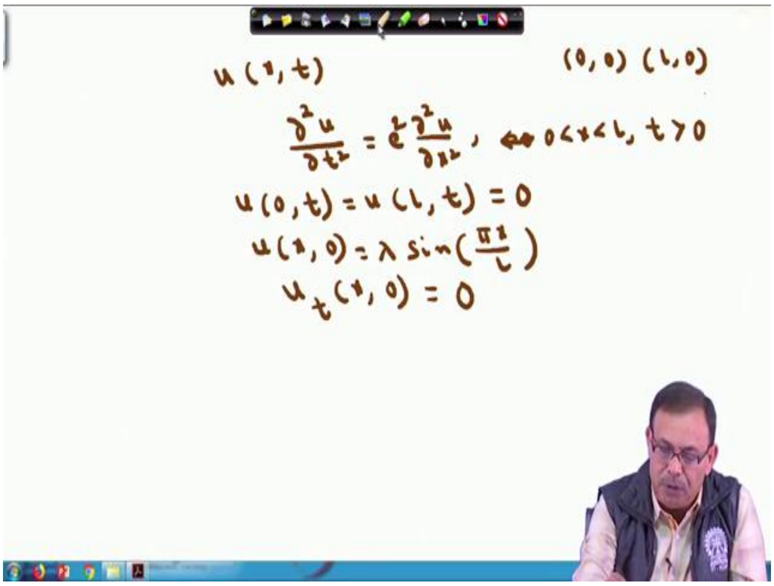

So, to formulate the problem, the displacement which we can assume as $u(x,t)$ is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < l, \;\; t > 0$$

$$\text{with,} \;\; u(0,t) = u(l,t) = 0 \;\; \text{for } t > 0$$

$$u(x,0) = \lambda \sin\left(\frac{\pi x}{l}\right) \text{ for } 0 < x < l \;\; \text{ and } \;\; u_t(x,0) = 0$$

Now we try to find out the solution of this particular problem.

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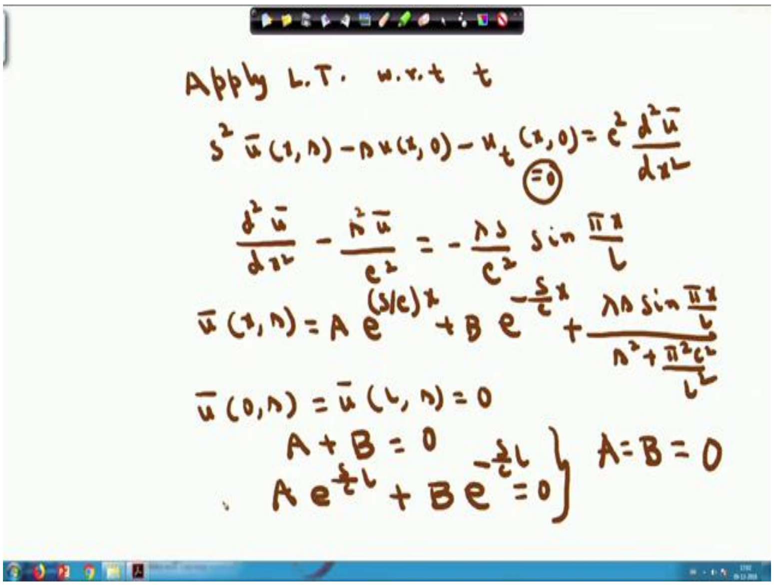

We apply Laplace transform with respect to the variable t. So that we will obtain,

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = c^2 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) = c^2\frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{s\lambda}{c^2}\sin\left(\frac{\pi x}{l}\right)$$

Auxiliary equation for the obtained ODE is,

$$m^2 - \frac{s^2}{c^2} = 0 \Rightarrow m = \pm\frac{s}{c}$$

Therefore, C.F. of the ODE is given as,

$$\text{C.F.} = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x}$$

And P.I. is given as

$$\text{P.I.} = \frac{\lambda s \sin\left(\frac{\pi x}{l}\right)}{s^2 + \frac{\pi^2 c^2}{l^2}}$$

Therefore, the general solution is given as,

$$\bar{u}(x,s) = Ae^{\frac{s}{c}x} + Be^{-\frac{s}{c}x} + \frac{\lambda s \sin\left(\frac{\pi x}{l}\right)}{s^2 + \frac{\pi^2 c^2}{l^2}} \tag{2}$$

From the given initial conditions, we have, $u(0,t) = 0$ and $u(l,t) = 0$. So, we can say that,

$$\bar{u}(0,s) = 0 \text{ and } \bar{u}(l,s) = 0$$

$$\therefore A + B = 0,\ \ Ae^{\frac{s}{c}l} + Be^{-\frac{s}{c}l} = 0$$

Now, if we take the coefficient determinant of these two equations, we will have

$$\begin{vmatrix} 1 & 1 \\ e^{\frac{s}{c}l} & e^{-\frac{s}{c}l} \end{vmatrix} = e^{-\frac{s}{c}l} - e^{\frac{s}{c}l} \neq 0$$

So, we obtain a system of homogeneous equations as earlier. Therefore, we can say that only trivial solution is possible for A and B. Trivial solution will be $A = B = 0$.

Therefore, (2) reduces to

$$\bar{u}(x,s) = \frac{\lambda s \sin\left(\frac{\pi x}{l}\right)}{s^2 + \frac{\pi^2 c^2}{l^2}}$$

Now, taking inverse Laplace transform on both sides, we have,

$$\begin{aligned} u(x,t) &= L^{-1}\{\bar{u}(x,s)\} \\ &= \lambda \sin\left(\frac{\pi x}{l}\right) L^{-1}\left\{\frac{s}{s^2 + \frac{\pi^2 c^2}{l^2}}\right\} \\ &= \lambda \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c}{l} t\right) \end{aligned}$$

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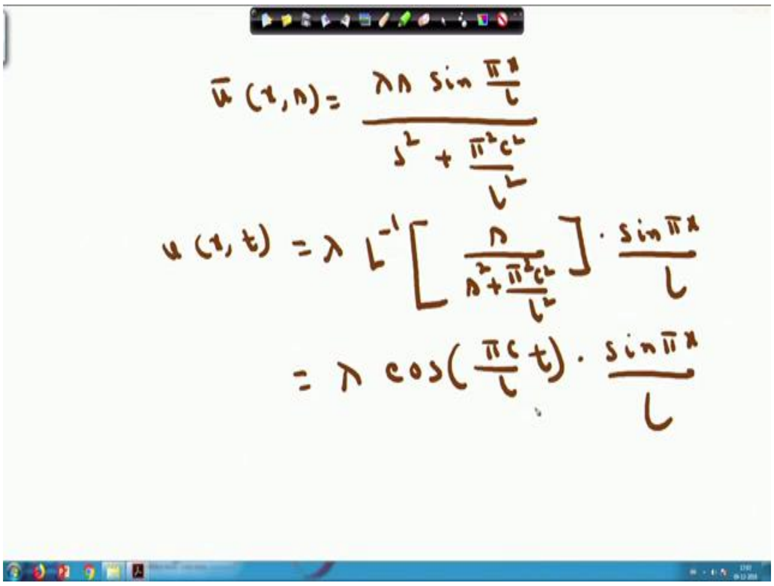

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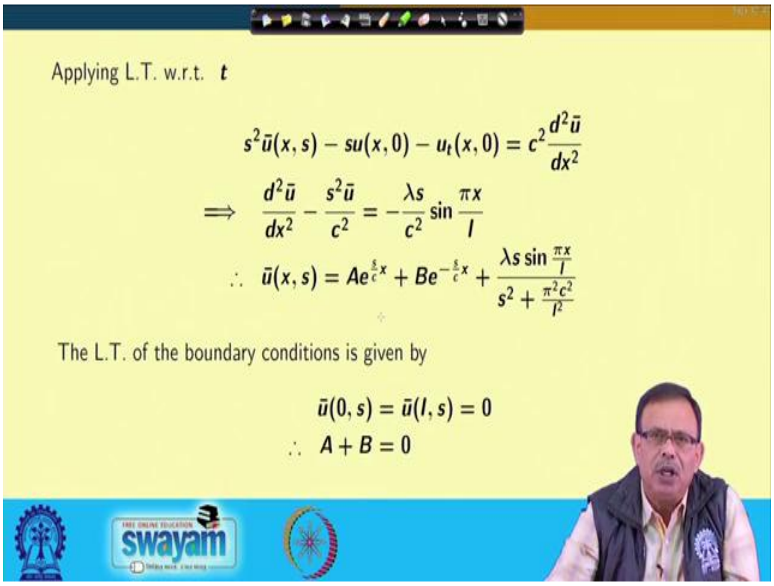

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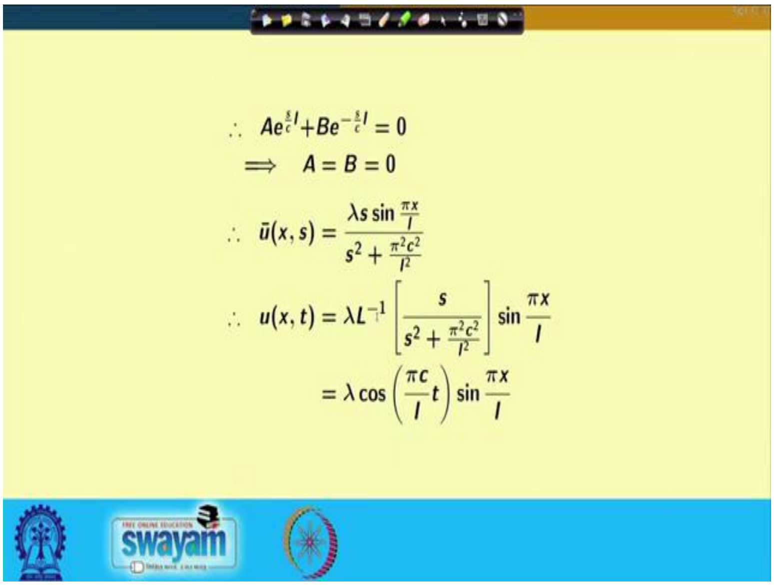

Thank you.