Transform Calculus and Its Applications in Differential Equations

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Lecture – 05
Laplace Transform of Derivative and Integration of a Function – II

In the earlier lecture, we have seen that, if we know the Laplace transform of a function, then we have derived the formulas for evaluating the Laplace transform of its derivative as well as integral.

Now, let us move on to the next theorem that is multiplication by powers of t.

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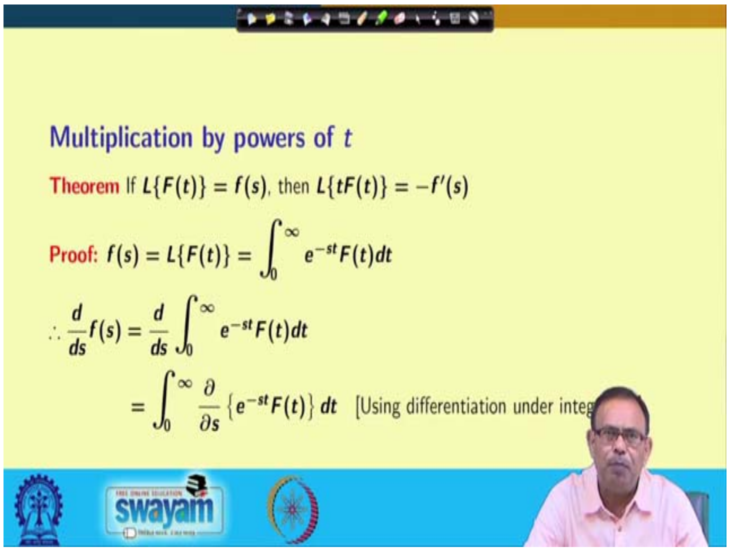

If Laplace transform of $F(t)$ is $f(s)$, then Laplace transform of $tF(t)$ equals $[-f'(s)]$, which means, if we know $L\{F(t)\} = f(s)$, then

$$L\{tF(t)\} = -f'(s).$$

Let us see the proof.

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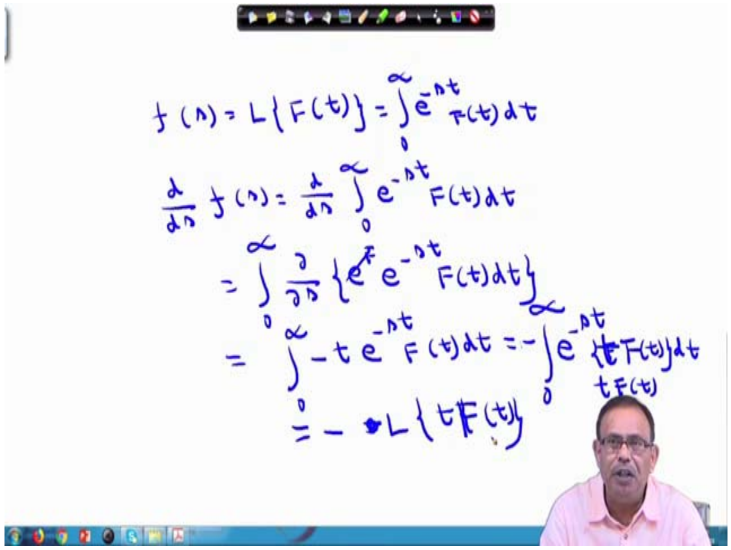

By definition,

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt$$

So, now we differentiate both sides with respect to s.

$$\Rightarrow \frac{d}{ds} f(s) = f'(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t) dt$$

Using differentiation under the sign of integration, we have,

$$f'(s) \;= \int_0^\infty \frac{\partial}{\partial s} \{e^{-st}\} F(t) dt$$

$$= -\int_0^\infty t e^{-st} F(t) dt.$$

This can be re-written as

$$f'(s) = -\int_0^\infty e^{-st} \{tF(t)\} dt$$

$$= -L\{tF(t)\} \quad \text{(by definition of Laplace Transform of } tF(t))$$

Therefore, we obtain the desired result as

$$L\{tF(t)\} = -f'(s).$$

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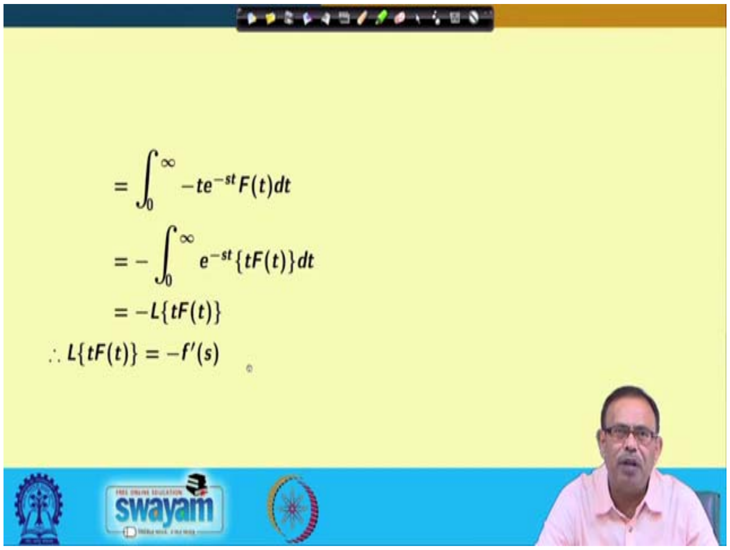

Let us see the next theorem.

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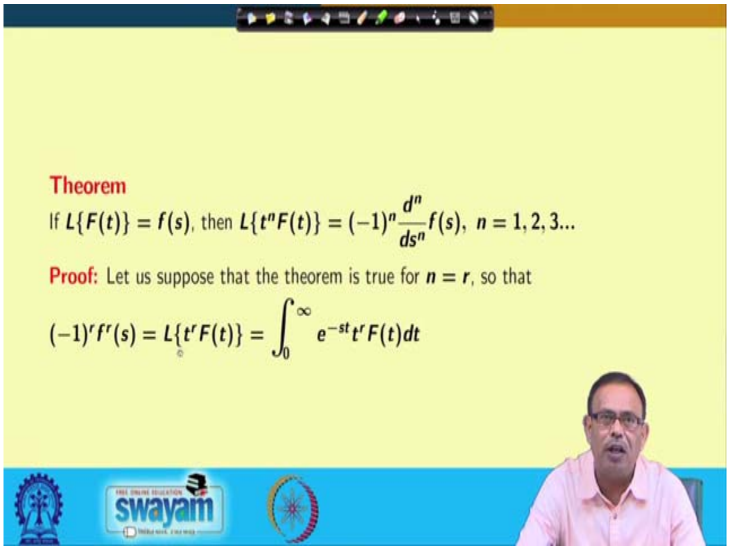

If Laplace transform of $F(t)$ equals $f(s)$, then Laplace transform of $t^n F(t)$ is equal to $(-1)^n \frac{d^n}{ds^n} f(s)$, where n can take values $1, 2, 3, \ldots$ and so on.

$$\therefore L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s).$$

Let us see, how we can complete the proof.

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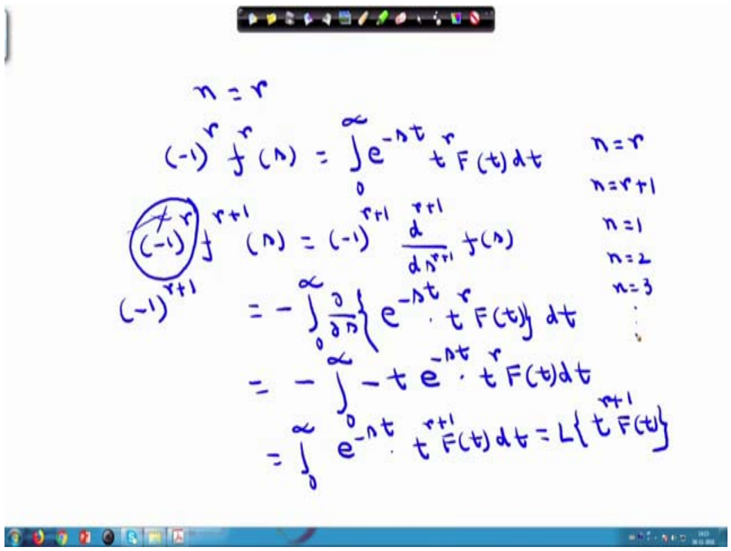

We are going to prove this with the help of mathematical induction. So, we assume that the theorem is true for some positive integer $n = r$ (say). We aim at showing that it is true for $n = r + 1$.

Once we assume that the theorem is true for $n = r$, we can write

$$L\{t^r F(t)\} = (-1)^r \frac{d^r}{ds^r} f(s). \tag{1}$$

Now we evaluate $L\{t^{r+1} F(t)\}$ i. e., for $n = r + 1$.

$$L\{t^{r+1} F(t)\} = L\{t.\, t^r F(t)\}$$

$$= -\frac{d}{ds} L\{t^r F(t)\} \quad \text{(using the result of the previous theorem)}$$

$$= -\frac{d}{ds}\left[(-1)^r \frac{d^r}{ds^r} f(s)\right] \qquad \text{[using (1)]}$$

$$= (-1)^{r+1} \frac{d^{r+1}}{ds^{r+1}} f(s)$$

which shows that the theorem holds for $n = r + 1$. Therefore by the principle of mathematical induction, we can say that the theorem is true for any positive integer n.

$$\therefore L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s).$$

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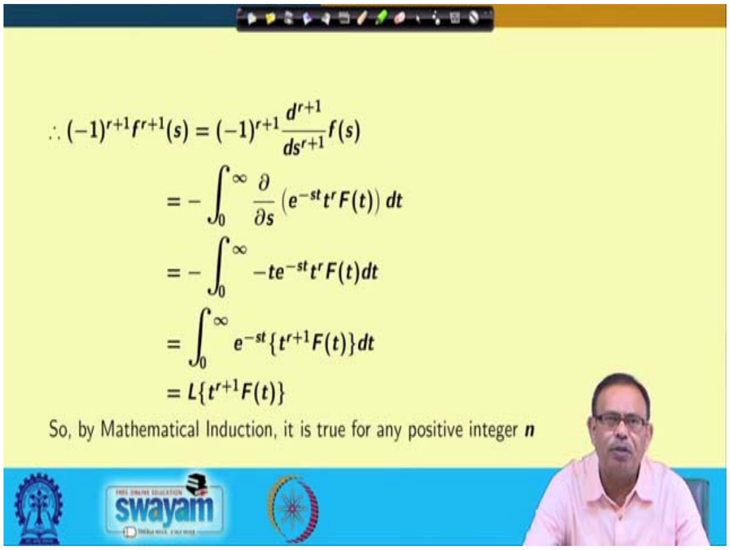

Next, let us come to the division by t.

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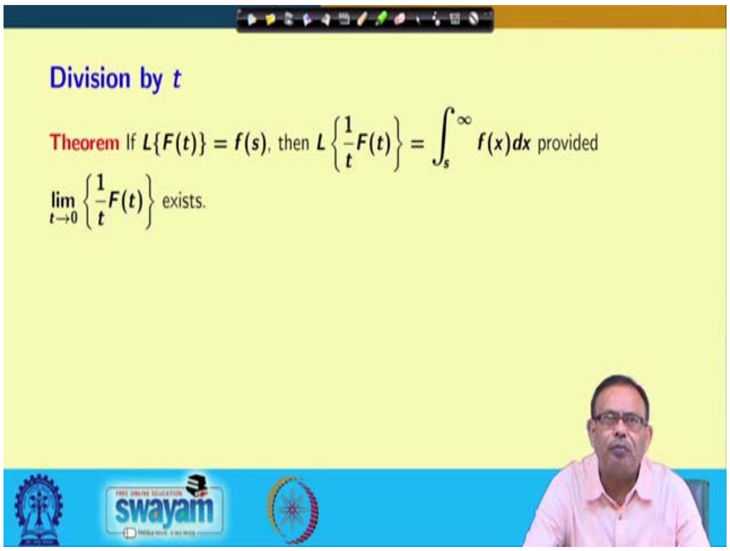

The theorem states that if Laplace transform of $F(t)$ equals $f(s)$, then Laplace transform $\frac{1}{t}F(t)$ is equal to $\int_s^\infty f(x)dx$ provided $\lim_{t\to 0}\left\{\frac{1}{t}F(t)\right\}$ exists.

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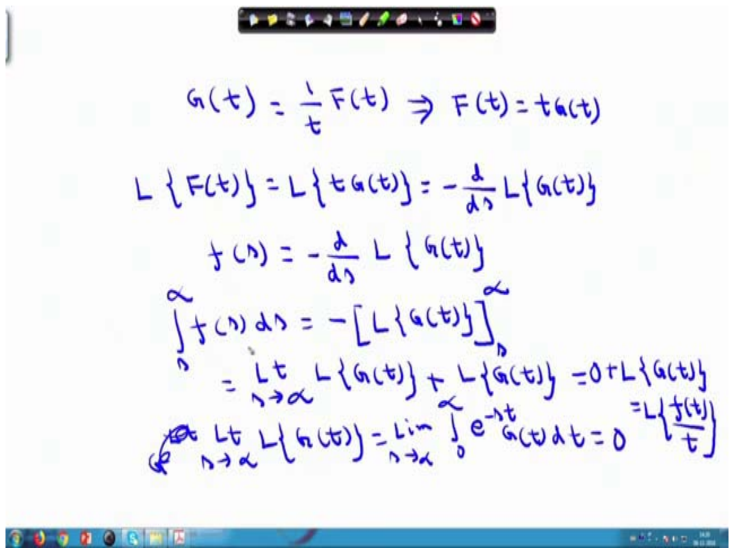

We assume

$$G(t) = \frac{1}{t}F(t)$$

$$\Rightarrow F(t) = tG(t).$$

Now we take Laplace transform on both sides.

$$\Rightarrow L\{F(t)\} = L\{tG(t)\}$$

$$= -\frac{d}{ds}L\{G(t)\}.$$

Since, $L\{F(t)\} = f(s)$, therefore we have,

$$\Rightarrow f(s) = -\frac{d}{ds}L\{G(t)\}.$$

Integrating both sides within the limits $[s, \infty)$, we have,

$$\int_s^\infty f(x)\,dx = -[L\{G(t)\}]_s^\infty$$

$$\Rightarrow \int_s^\infty f(x)\,dx = -\lim_{s\to\infty} L\{G(t)\} + L\{G(t)\}$$

Since, $\lim_{s\to\infty} L\{G(t)\} = \lim_{s\to\infty} \int_0^\infty e^{-st} G(t)\,dt = 0$, therefore,

$$\int_s^\infty f(x)\,dx = -0 + L\{G(t)\}$$

$$= L\{G(t)\}$$

$$= L\left\{\frac{1}{t}F(t)\right\}.$$

This completes the proof.

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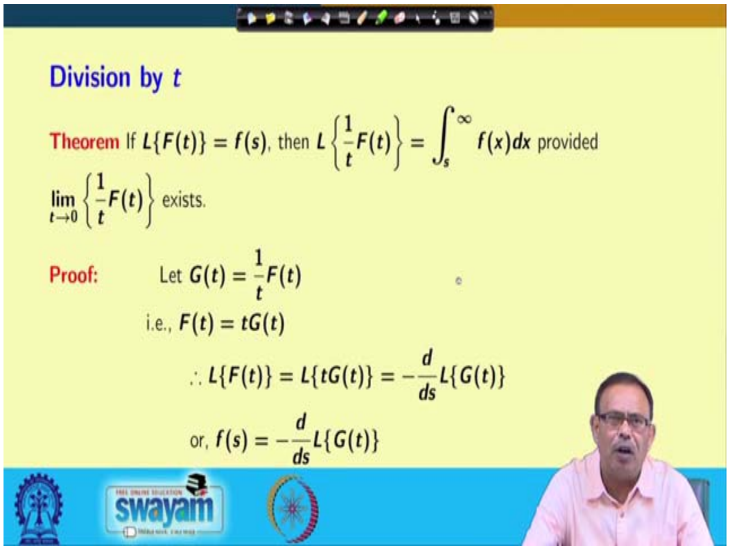

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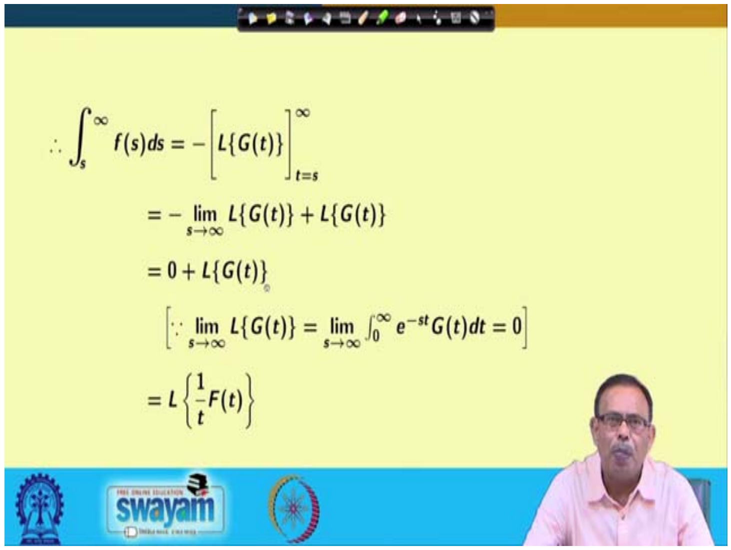

Now, let us solve certain examples on the properties and theorems that we have just discussed.

The first one is to evaluate the Laplace transform of $tcos\ at$.

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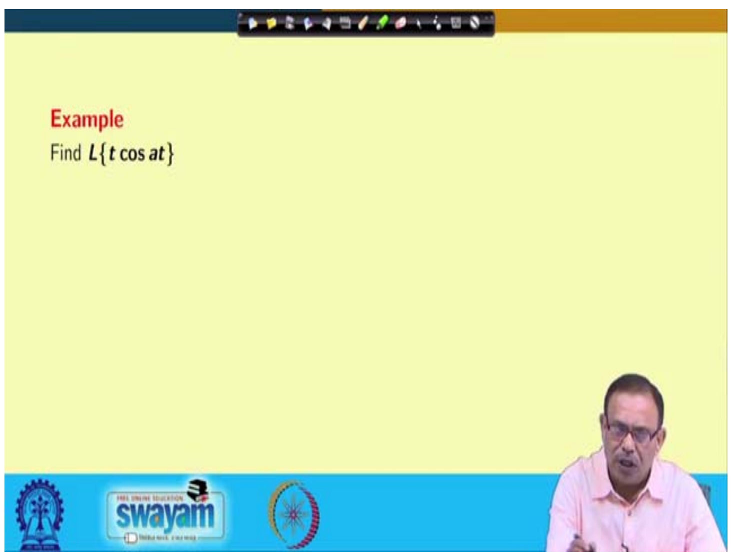

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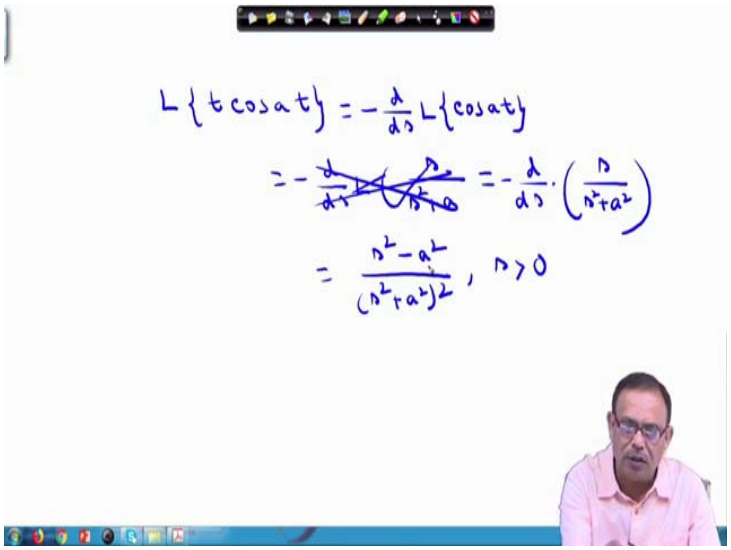

Here, we have to find out Laplace transform of $t\cos at$. So, clearly, we can use the theorem for multiplication by powers of t. Then,

$$L\{t\cos at\} = -\frac{d}{ds} L\{\cos at\}$$

We know already the Laplace transform of $\cos at$, so we get

$$L\{t\cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right)$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2},\ s > 0.$$

So, we see the usefulness of the theorems and properties and how by their application, we can obtain the desired results much easily.

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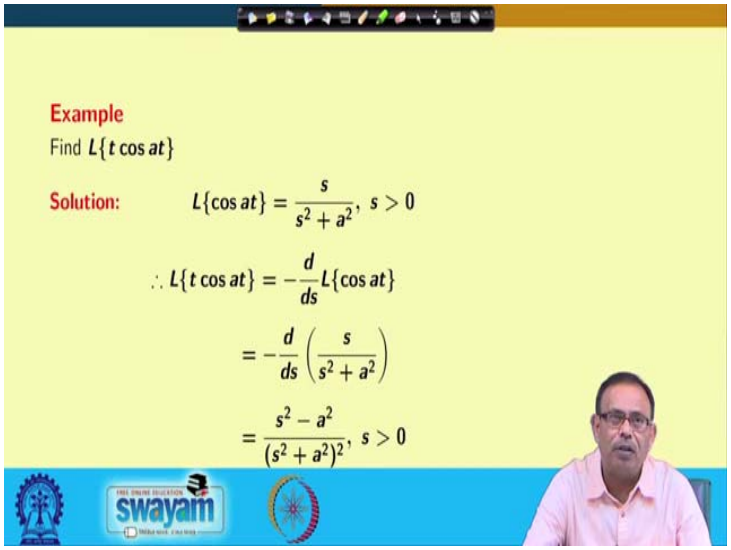

Next we come to the evaluation of $L\{t^2 \sin at\}$.

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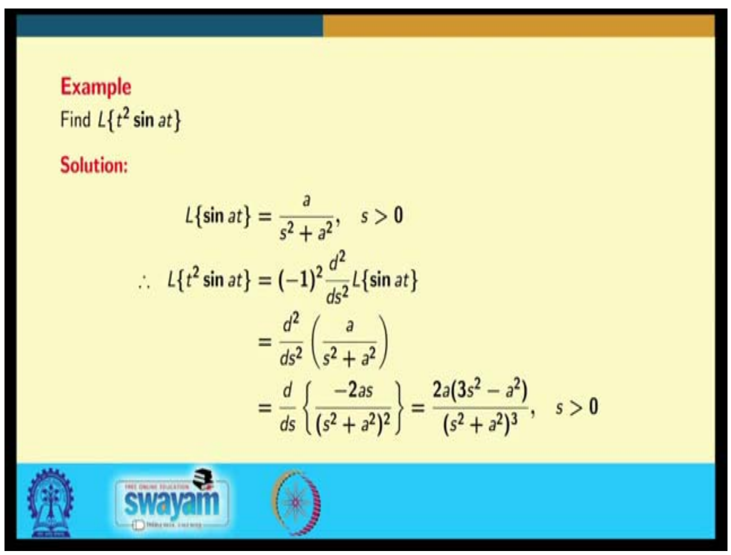

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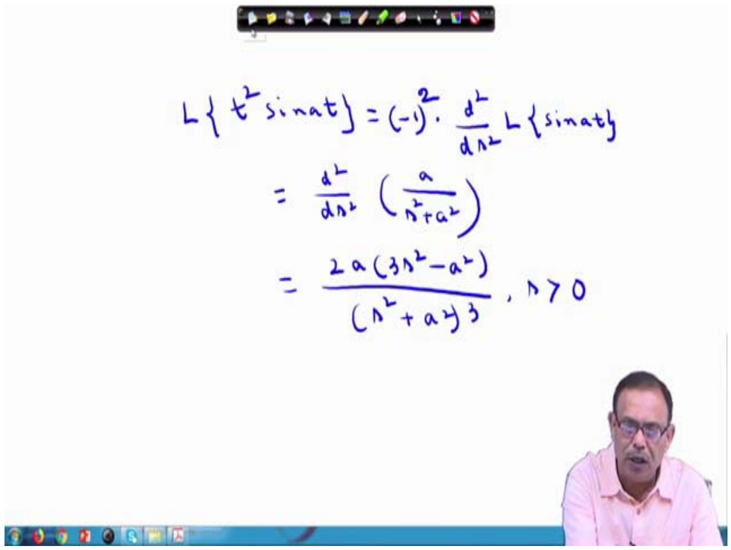

In this case also, we can obtain the result by the application of the theorem for multiplication by powers of t. As we know already,

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$\therefore L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}[L\{\sin at\}]$$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2 + a^2}\right]$$

We can easily differentiate the above twice w.r.t. s to obtain the following result:

$$L\{t^2 \sin at\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}, \qquad s > 0.$$

Again, we see that although the function is complicated, yet using the theorem, we can evaluate the transforms without any difficulty.

The next example is a little complicated one as we can see. We need to evaluate $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

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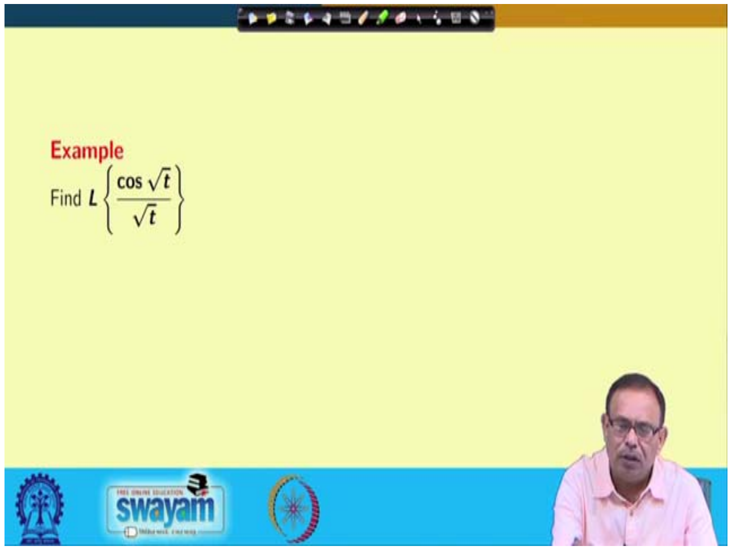

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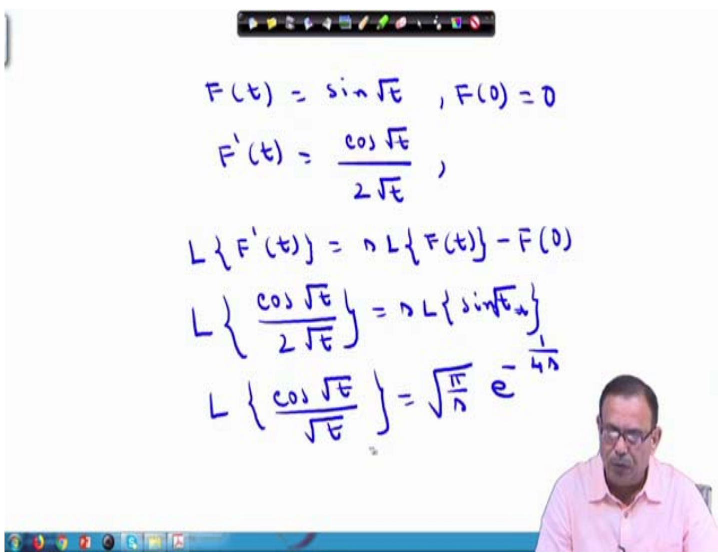

For this, we first assume

$$F(t) = \sin\sqrt{t}$$

$$\Rightarrow F(0) = 0.$$

Clearly, if we differentiate $F(t)$ w.r.t. t, then we obtain,

$$F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}.$$

Now, we take Laplace Transform on both sides

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = L\{F'(t)\}$$

Using Laplace transform of derivative of $F(t)$, we have,

$$\Rightarrow \frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = sL\{F(t)\} - F(0)$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2sL\{F(t)\} - 0$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2sL\{\sin\sqrt{t}\}$$

In lecture 3, we have already discussed the Laplace transform of $\sin\sqrt{t}$. So we can directly put the obtained value over here:

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2s\,\frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-\frac{1}{4s}}$$

$$= \sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}$$

So, although the function was complicated, through proper assumption, we managed to evaluate its Laplace transform with the help of certain properties and known results.

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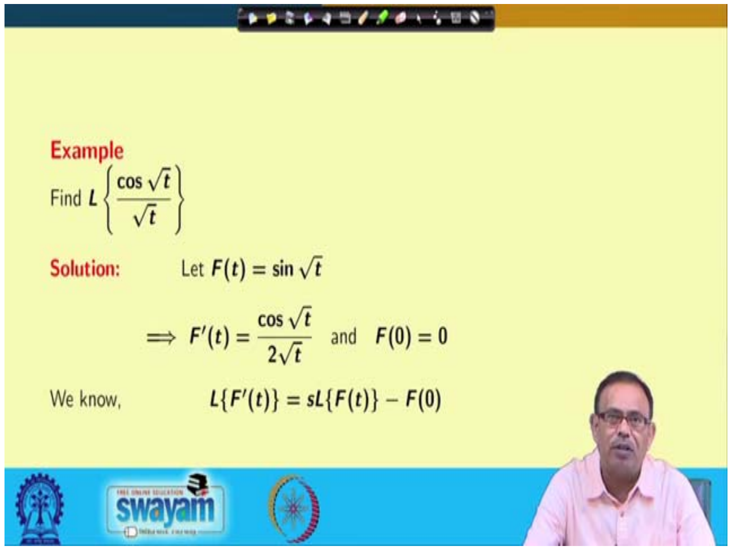

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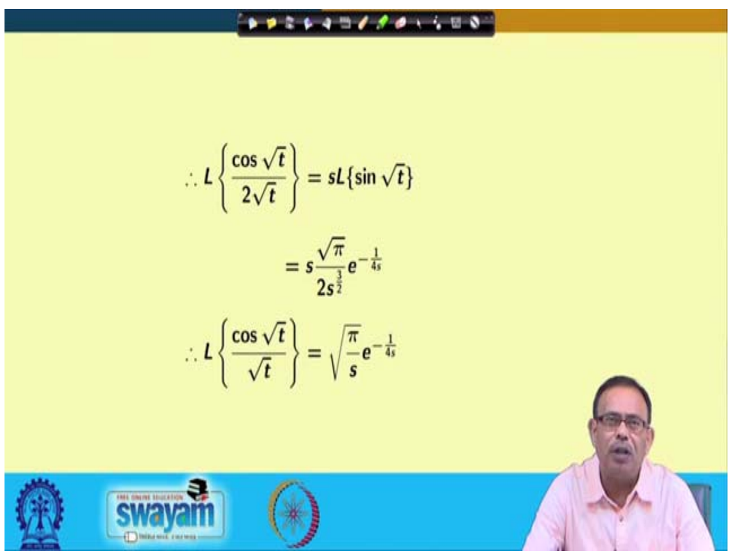

In the next example, we have to first prove $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$ and using the result, we need to evaluate $L\left\{\frac{\sin at}{t}\right\}$. Finally, it is to be checked whether $L\left\{\frac{\cos at}{t}\right\}$ exists or not.

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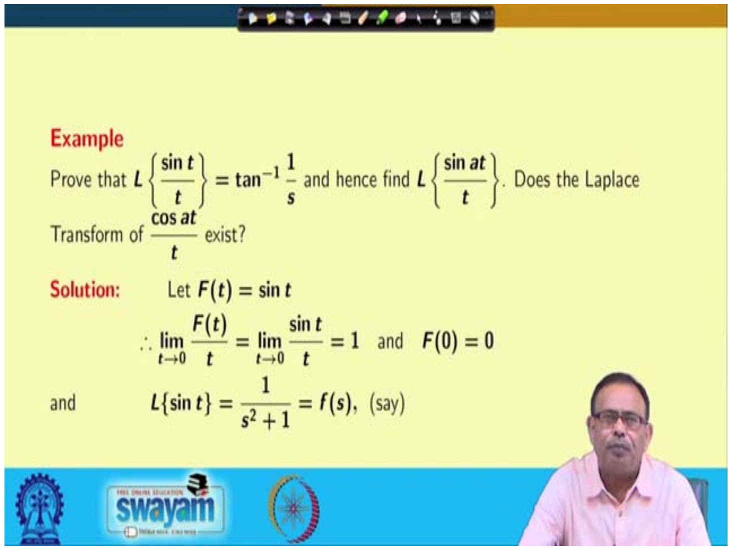

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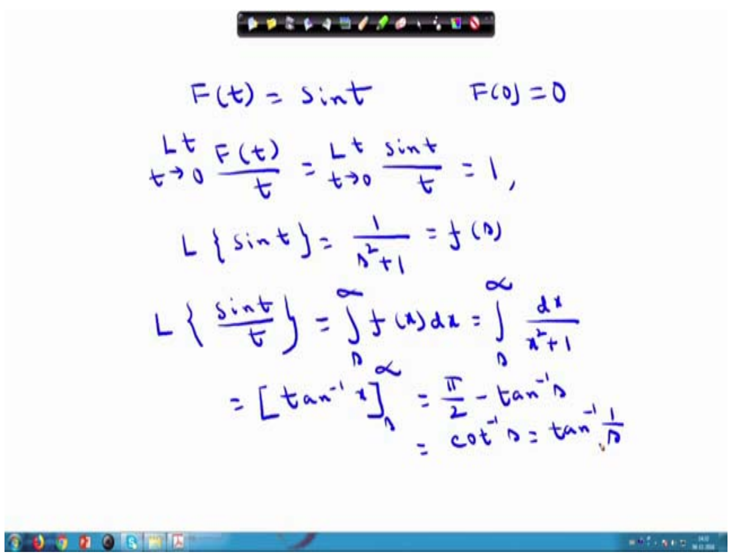

So, to prove this one, again we are starting with assuming

$$F(t) = \sin t$$

so that $F(0) = 0$. Now, $\lim_{t \to 0} \frac{F(t)}{t} = \lim_{t \to 0} \frac{\sin t}{t} = 1$. And we already know,

$$L\{\sin t\} = \frac{1}{s^2 + 1} = f(s) \quad \text{(say)}.$$

Therefore, using the result of the theorem on division by t as discussed earlirer, we can write,

$$\begin{aligned} L\left\{\frac{\sin t}{t}\right\} &= \int_s^{\infty} f(x)\, dx \\ &= \int_s^{\infty} \frac{1}{x^2 + 1} dx \\ &= [\tan^{-1} x]_s^{\infty} \\ &= \tan^{-1} \infty - \tan^{-1} s \\ &= \frac{\pi}{2} - \tan^{-1} s \\ &= \tan^{-1} \frac{1}{s}. \qquad (2) \end{aligned}$$

So, this completes the first part of the given question. Next the Laplace transform of $\frac{\sin at}{t}$ is to be evaluated.

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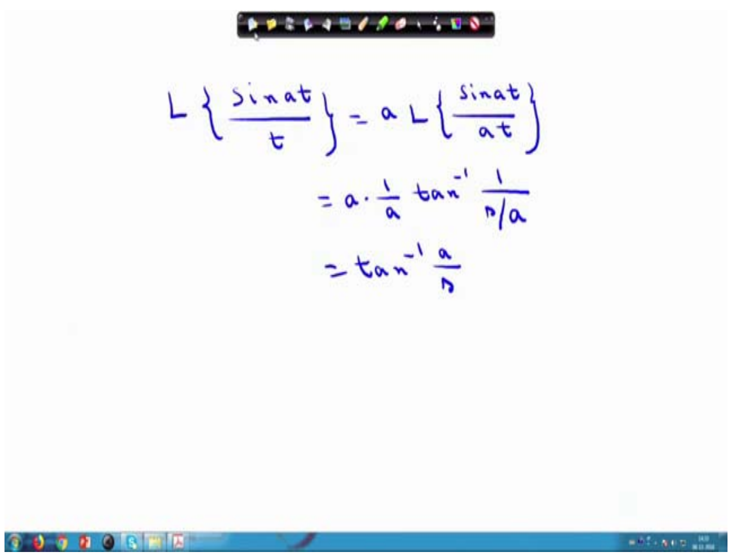

As we have already discussed change of scale property, we can use it to solve the given problem very easily. First we assume, $L\left\{\frac{\sin t}{t}\right\} = f(s)$ and from (2), we have,

$$L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}.$$

Therefore, by change of scale property, we have,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}f\left(\frac{s}{a}\right)$$

$$\Rightarrow \frac{1}{a}L\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\frac{1}{\frac{s}{a}}$$

$$\Rightarrow L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\frac{a}{s}.$$

This completes the second part of the given question.

Next, we are going to check whether $L\left\{\frac{\cos at}{t}\right\}$ exists or not.

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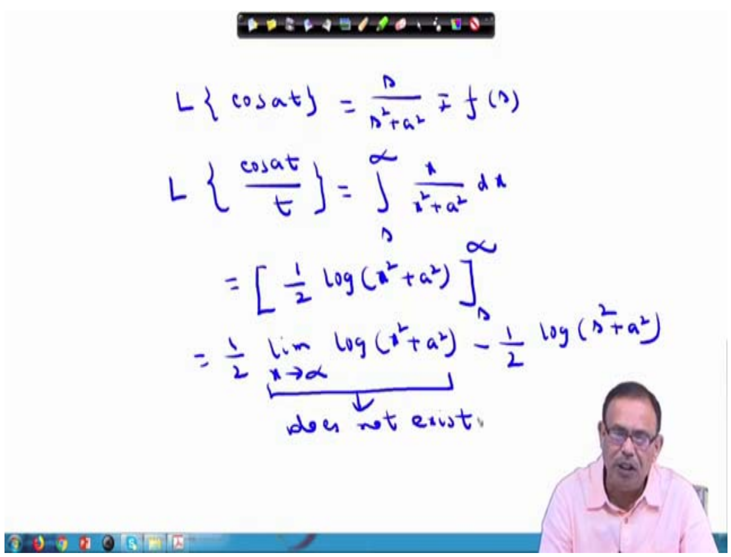

We know,

$$L\{\cos at\} = \frac{s}{s^2 + a^2} = f(s) \quad \text{(say)}$$

Then by the theorem on division by t, we can write,

$$L\left\{\frac{\cos at}{t}\right\} = \int_s^{\infty} f(x)\, dx$$

$$= \int_s^{\infty} \frac{x}{x^2 + a^2} dx$$

$$= \left[\frac{1}{2}\log(x^2 + a^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\lim_{x\to\infty} \log(x^2 + a^2) - \frac{1}{2}\log(s^2 + a^2)$$

Clearly, $\lim_{x\to\infty} \log(x^2 + a^2)$ does not exist. Therefore $L\left\{\frac{\cos at}{t}\right\}$ also does not exist.

This completes the solution to the given problem.

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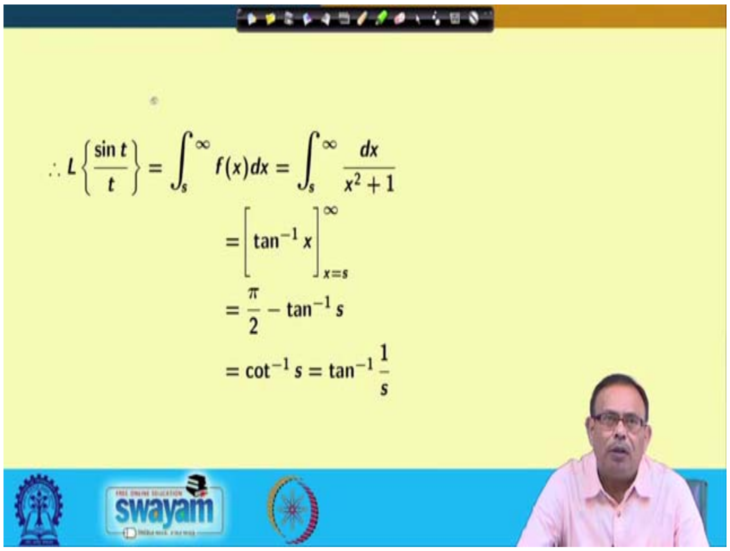

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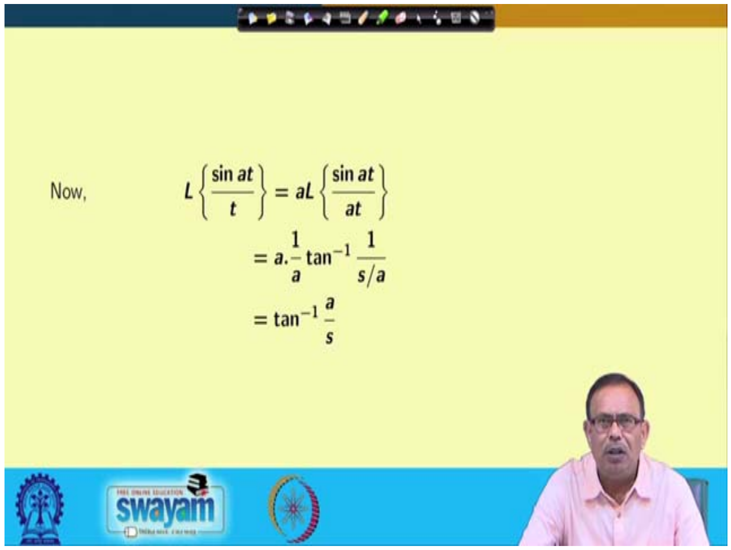

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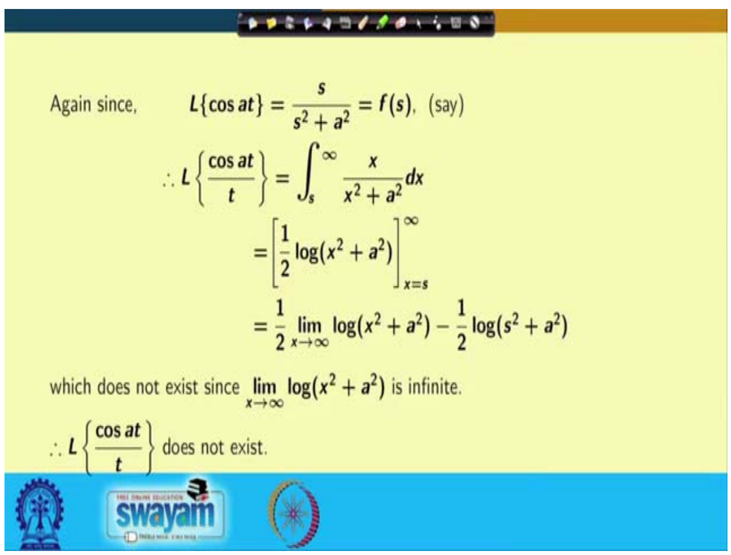

In the next lecture, we will go through some more examples. Thank you.