Transform Calculus and its Applications in Differential Equations
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Lecture - 49
Solution of Partial Differential Equations using Fourier Transform – II

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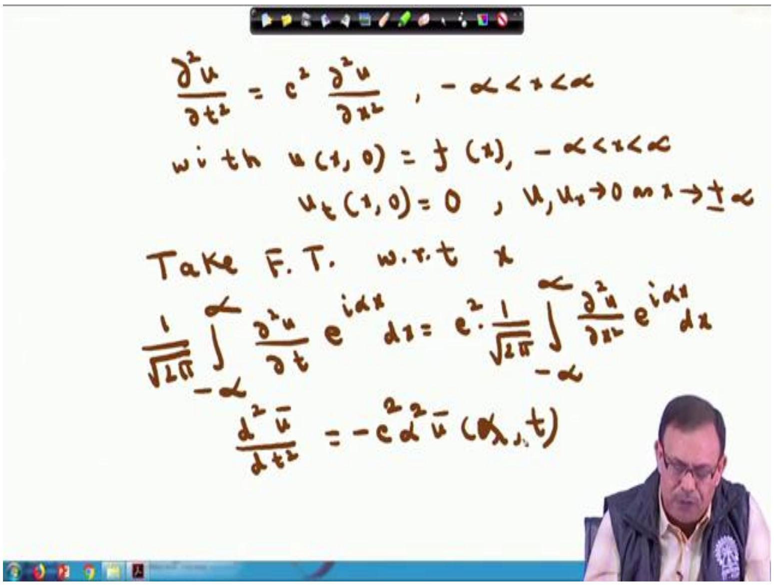

Now, let us take another problem and see how to find out the solution.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty,\ \ t > 0$$

$$\text{with } u(0,t) = 0 \text{ for } t > 0 \text{ and } u(x,0) = f(x),\ \ u_t(x,0) = 0 \text{ for } -\infty < x < \infty$$

$$u,\ \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm\infty$$

So, we can take Fourier transform with respect to x only, because about t, nothing has been told over here. So, we are taking Fourier transform with respect to x. We have,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2 \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d^2 \bar{u}}{dt^2} = c^2(-i\alpha)^2 \bar{u}(\alpha,t) \quad \text{where, } \mathcal{F}[u(x,t)] = \bar{u}(\alpha,t)$$

$$\Rightarrow \frac{d^2 \bar{u}}{dt^2} = -c^2\alpha^2\bar{u}$$

Therefore, the given PDE is reduced to a second order ODE which can be solved very easily. So, in this case, the given PDE is transformed into second order ODE, and not first order ODE. In the earlier problems, the given PDE was transformed into first order ODE, but here it is being transformed into second order ODE. Therefore, it always depends on the kind of problem given, then accordingly, it will be transformed into first order ODE or second order ODE.

Auxiliary equation for the obtained ODE is,

$$m^2 = -c^2\alpha^2 \Rightarrow m = \pm ic\alpha$$

Therefore, the general solution is given as,

$$\bar{u}(\alpha, t) = A\, cos\, c\alpha t + B \sin c\alpha t \tag{1}$$

where A and B are the constants of integration. And, we have,

$$\frac{d\bar{u}}{dt} = -Ac\alpha \sin c\alpha t + Bc\alpha \cos c\alpha t \tag{2}$$

We have the initial conditions $u(x, 0) = f(x),\ \ u_t(x, 0) = 0$. Now,

$$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0$$

$$\Rightarrow \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = 0 \text{ at } t = 0$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\alpha x} dx = 0 \text{ at } t = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u\, e^{i\alpha x} dx\right] = 0 \text{ at } t = 0$$

$$\Rightarrow \frac{d\bar{u}}{dt} = 0 \text{ at } t = 0$$

Therefore, we can say from the given conditions that,

$$\bar{u}(\alpha, 0) = F(\alpha),\ \ \bar{u}_t(\alpha, 0) = 0 \ \text{ where } \ F(\alpha) = \mathcal{F}[f(x)]$$

$\therefore$ (1) implies $A = F(\alpha)$ and

(2) implies $B = 0$.

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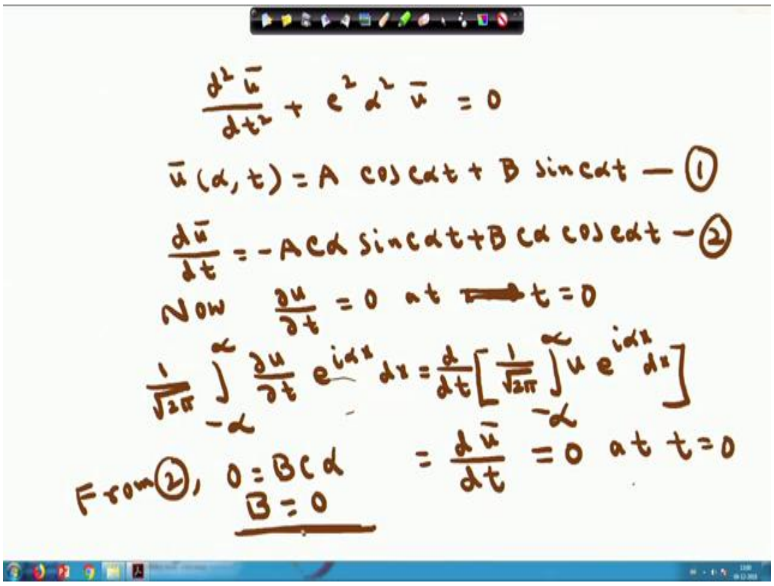

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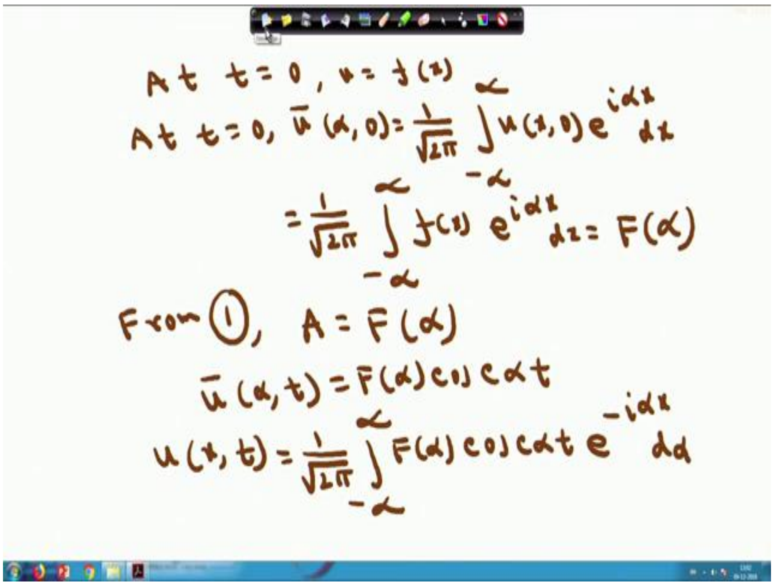

Once we have obtained the values of A and B, we can write down

$$\bar{u}(\alpha, t) = F(\alpha)\, cos\, c\alpha t$$

Now, taking the inverse Fourier transform, we have,

$$u(x, t) = \mathcal{F}^{-1}[F(\alpha) \cos c\alpha t]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) \cos c\alpha t\, e^{-i\alpha x} d\alpha$$

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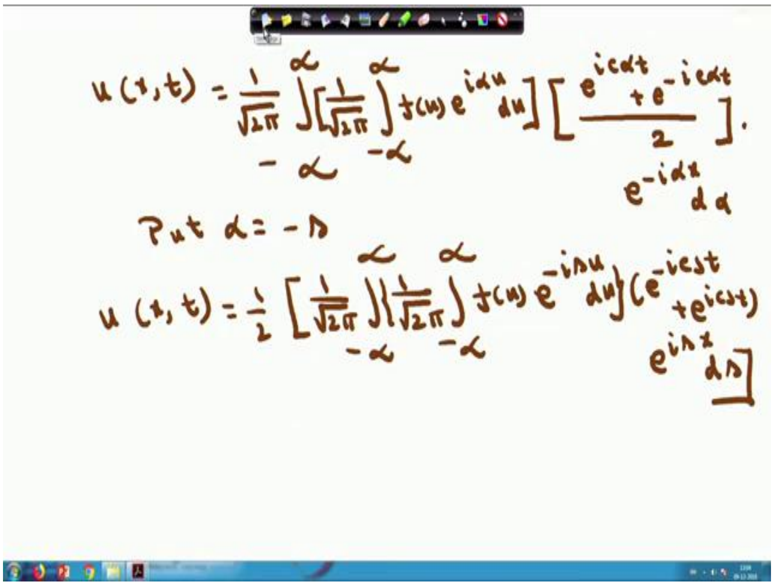

$$\therefore u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\alpha u} du \right] \left[\frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2} \right] e^{-i\alpha x} d\alpha$$

Put $\alpha = -s$ so that $d\alpha = -ds$.

$$\begin{aligned}
\therefore u(x,t) &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-isu} du \right\} \left(e^{-icst} + e^{icst} \right) e^{isx} ds \right] \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-isu} du \right\} \left\{ e^{is(x-ct)} + e^{is(x+ct)} \right\} ds \right] \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(x-ct)} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-isu} du \right\} ds \right. \\
&\qquad \left. + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(x+ct)} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-isu} du \right\} ds \right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(u) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is(x-ct-u)}\, ds \right\} du \right. \\
&\qquad \left. + \int_{-\infty}^{\infty} f(u) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is(x+ct-u)}\, ds \right\} du \right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(u)\, \delta(x-ct-u)\, du + \int_{-\infty}^{\infty} f(u) \delta(x+ct-u)\, du \right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(u)\, \delta(u-x+ct)\, du + \int_{-\infty}^{\infty} f(u) \delta(u-x-ct)\, du \right] \\
&= \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]
\end{aligned}$$

In deriving the above, we have used the well-known property of Dirac delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\, dp$$

and that Dirac delta function is an even function i.e.,

$$\delta(-x) = \delta(x)$$

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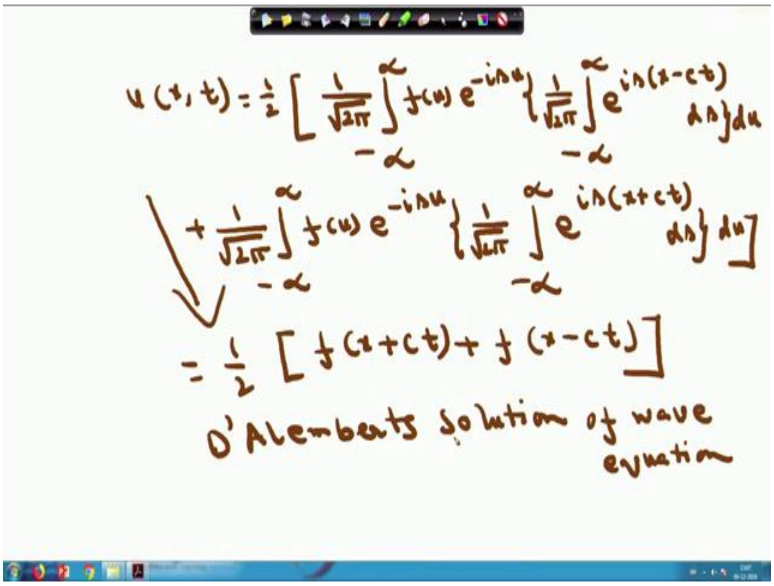

Therefore, the obtained solution

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

is known as the D'Alembert's solution of wave equation.

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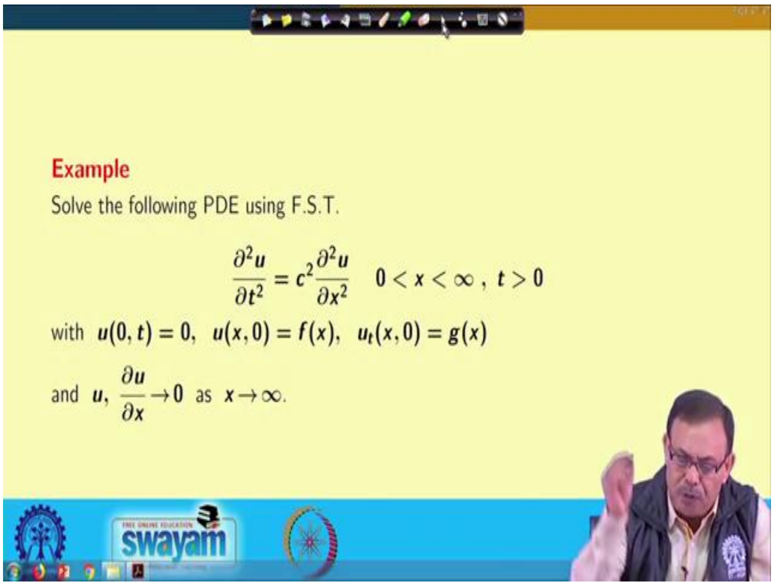

Let us move to the next problem.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty,\ \ t > 0$$

$$\text{with } u(0,t) = 0 \text{ for } t > 0 \text{ and } u(x,0) = f(x),\ \ u_t(x,0) = g(x) \text{ for } 0 < x < \infty$$

$$u,\ \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

We have to use Fourier sine transform in this case. From the given criteria, it is quite obvious that we have to use Fourier sine transform with respect to the variable x only because u and $\frac{\partial u}{\partial x}$ both are approaching 0 as x approaches ∞. The PDE is similar to the earlier problem that we solved using Fourier transform.

So, we apply Fourier sine transform with respect to x and we obtain,

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{d^2\bar{u}_s}{dt^2} = c^2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\alpha x\,dx$$

$$\Rightarrow \frac{d^2\bar{u}_s}{dt^2} = c^2\sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\sin\alpha x\right]_0^\infty - \alpha\int_0^\infty \frac{\partial u}{\partial x}\cos\alpha x\,dx\right]$$

$$\Rightarrow \frac{d^2\bar{u}_s}{dt^2} = c^2\sqrt{\frac{2}{\pi}}\left[0 - \alpha\left\{[u\cos\alpha x]_0^\infty + \alpha\int_0^\infty u\sin\alpha x\,dx\right\}\right]$$

$$\Rightarrow \frac{d^2\bar{u}_s}{dt^2} = -c^2\alpha^2\bar{u}_s(\alpha,t) \quad \text{where, } \mathcal{F}_s[u(x,t)] = \bar{u}_s(\alpha,t)$$

$$\Rightarrow \frac{d^2\bar{u}_s}{dt^2} + c^2\alpha^2\bar{u}_s = 0$$

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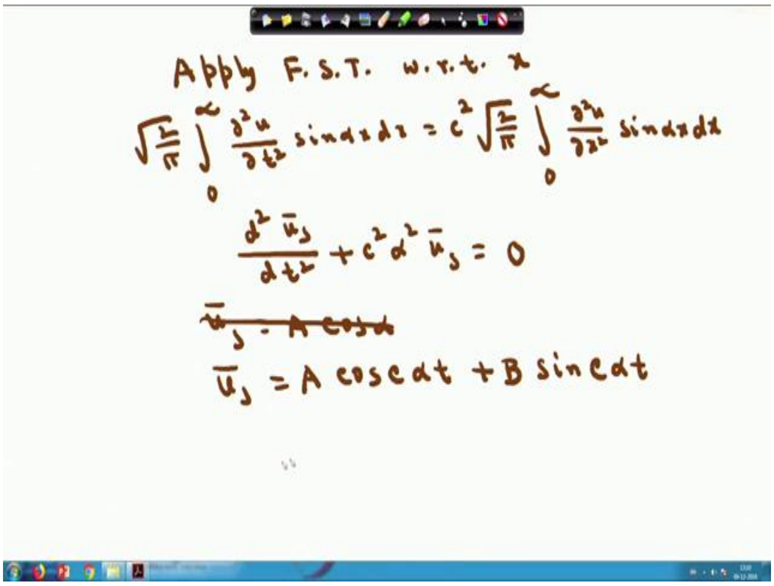

We now solve the obtained ODE as follows:

Auxiliary equation for the ODE is,

$$m^2 = -c^2\alpha^2 \Rightarrow m = \pm ic\alpha$$

Therefore, the general solution is given as,

$$\bar{u}_s(\alpha,t) = A\,cos\,c\alpha t + B\sin c\alpha t \quad (3)$$

where A and B are the constants of integration. And, we have,

$$\frac{d\bar{u}_s}{dt} = -Ac\alpha\sin c\alpha t + Bc\alpha\cos c\alpha t \quad (4)$$

We have the initial conditions $u(x,0) = f(x),\ \ u_t(x,0) = g(x)$. Now, we can say from the given conditions that,

$$\bar{u}_s(\alpha,0) = F(\alpha),\ \ \frac{d\bar{u}_s(\alpha,0)}{dt} = G(\alpha)\ \text{ where }\ F(\alpha) = \mathcal{F}[f(x)],\ \ G(\alpha) = \mathcal{F}[g(x)]$$

$$\therefore\ (3) \text{ implies } A = F(\alpha) \text{ and}$$

$$(4) \text{ implies } B = \frac{G(\alpha)}{c\alpha}.$$

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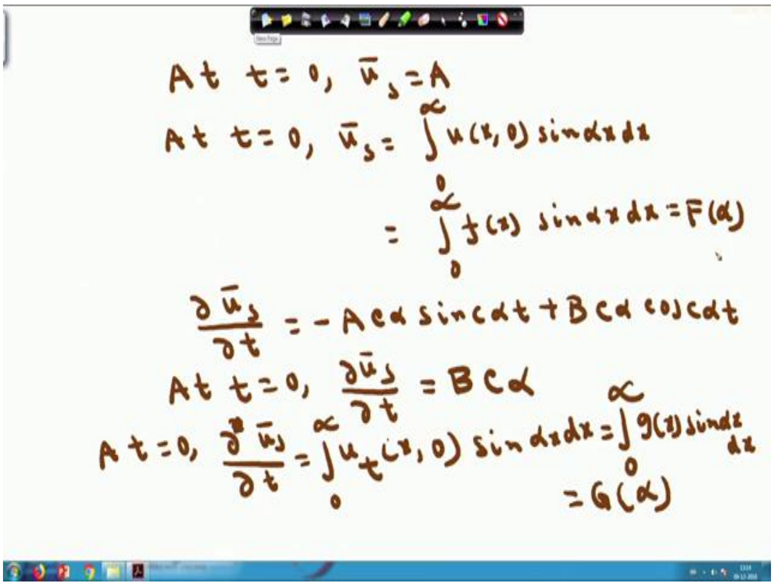

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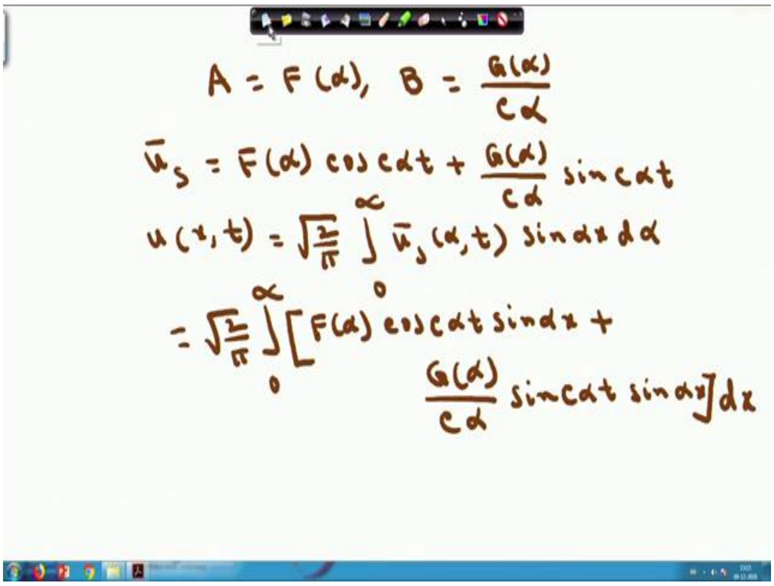

Therefore, we get,

$$\bar{u}_s(\alpha, t) = F(\alpha)\, cos\; c\alpha t + \frac{G(\alpha)}{c\alpha}\sin c\alpha t$$

Now, taking the inverse Fourier sine transform, we have,

$$\begin{aligned}
u(x,t) &= \mathcal{F}_s^{-1}[\bar{u}_s(\alpha, t)] \\
&= \mathcal{F}_s^{-1}\left[F(\alpha)\cos c\alpha t + \frac{G(\alpha)}{c\alpha}\sin c\alpha t\right] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \left[F(\alpha)\cos c\alpha t + \frac{G(\alpha)}{c\alpha}\sin c\alpha t\right]\sin \alpha x\, d\alpha \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty F(\alpha)\cos c\alpha t \sin \alpha x\, d\alpha + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{G(\alpha)}{c\alpha}\sin c\alpha t \sin \alpha x\, d\alpha \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty F(\alpha)[\sin \alpha(x+ct) + \sin \alpha(x-ct)]\, d\alpha \\
&\qquad + \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{G(\alpha)}{c\alpha}[\cos \alpha(x-ct) - \cos \alpha(x+ct)]\, d\alpha
\end{aligned} \tag{5}$$

Now, we know,

$$\begin{aligned}
f(x) &= \mathcal{F}_s^{-1}[F(\alpha)] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty F(\alpha)\sin \alpha x\, d\alpha
\end{aligned}$$

$$\therefore f(x-ct) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\alpha)\sin \alpha(x-ct)\, d\alpha$$

$$f(x+ct) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\alpha)\sin \alpha(x+ct)\, d\alpha$$

Also,

$$\begin{aligned}
g(u) &= \mathcal{F}_s^{-1}[G(\alpha)] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty G(\alpha)\sin \alpha u\, d\alpha
\end{aligned}$$

$$\therefore \int_{x-ct}^{x+ct} g(u)du = \sqrt{\frac{2}{\pi}} \int_0^\infty G(\alpha) \left[\int_{x-ct}^{x+ct} \sin \alpha u \, du \right] d\alpha$$

$$\Rightarrow \int_{x-ct}^{x+ct} g(u)du = \sqrt{\frac{2}{\pi}} \int_0^\infty G(\alpha) \left[-\frac{1}{\alpha} \cos \alpha u \right]_{x-ct}^{x+ct} d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{G(\alpha)}{\alpha} [\cos \alpha(x-ct) - \cos \alpha(x+ct)] \, d\alpha$$

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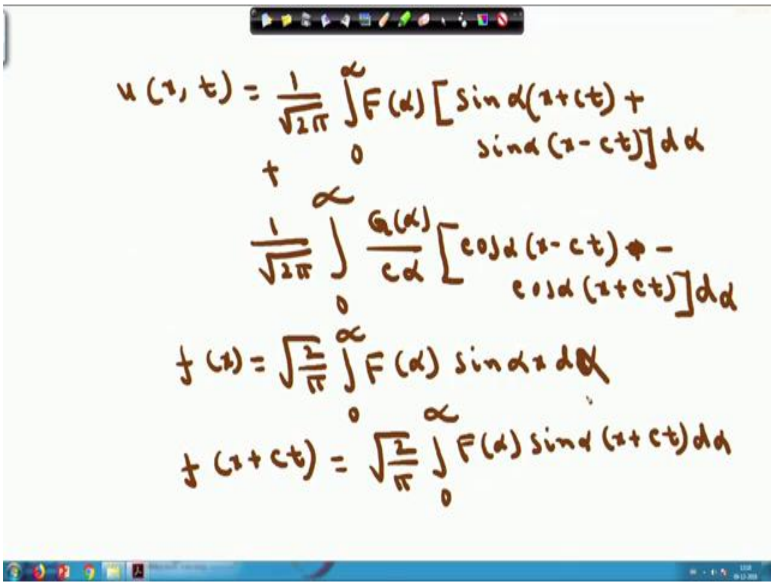

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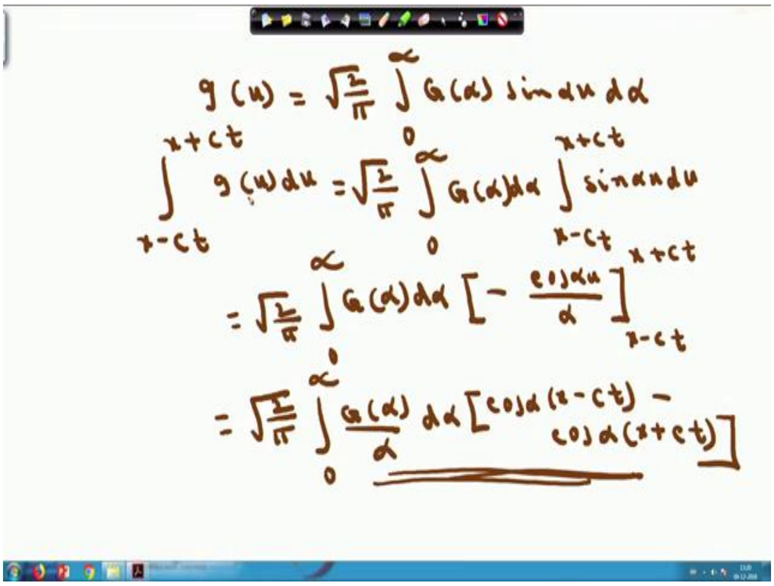

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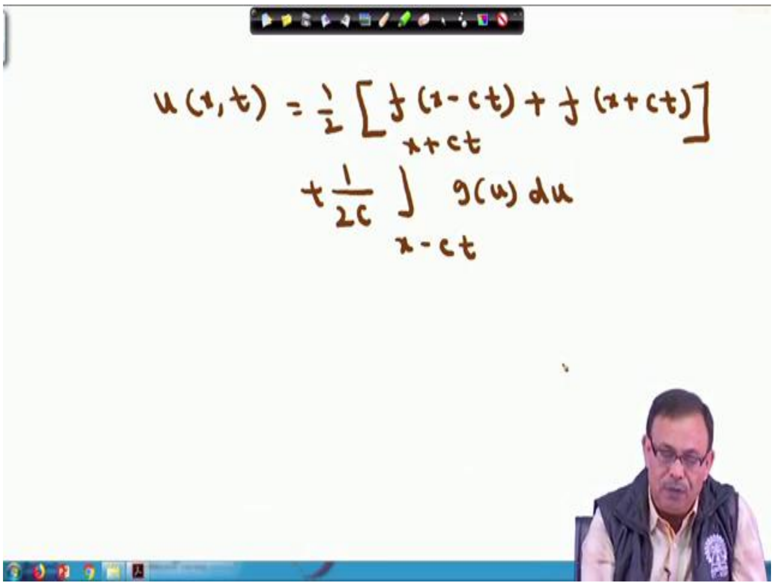

Therefore, (5) implies,

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du$$

which is the required solution.

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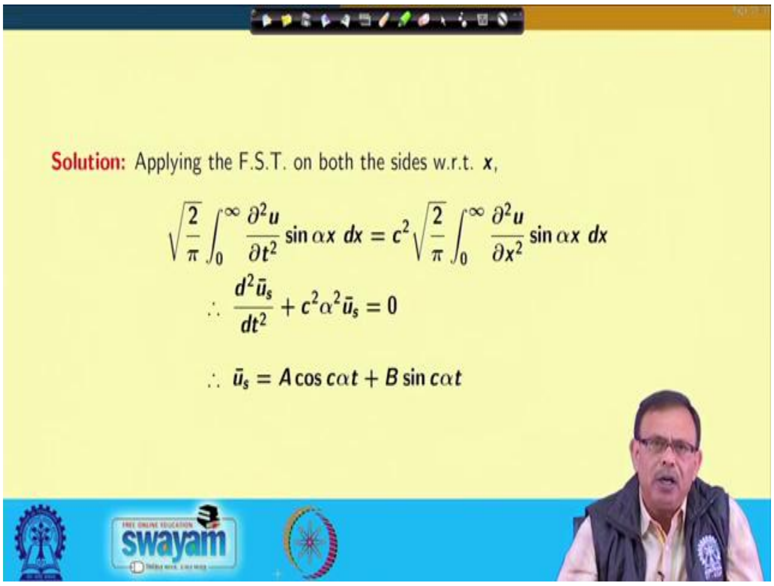

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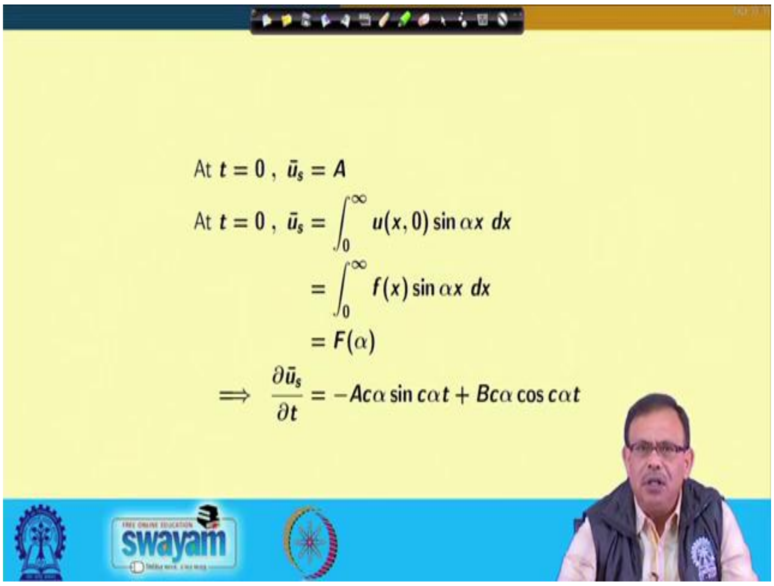

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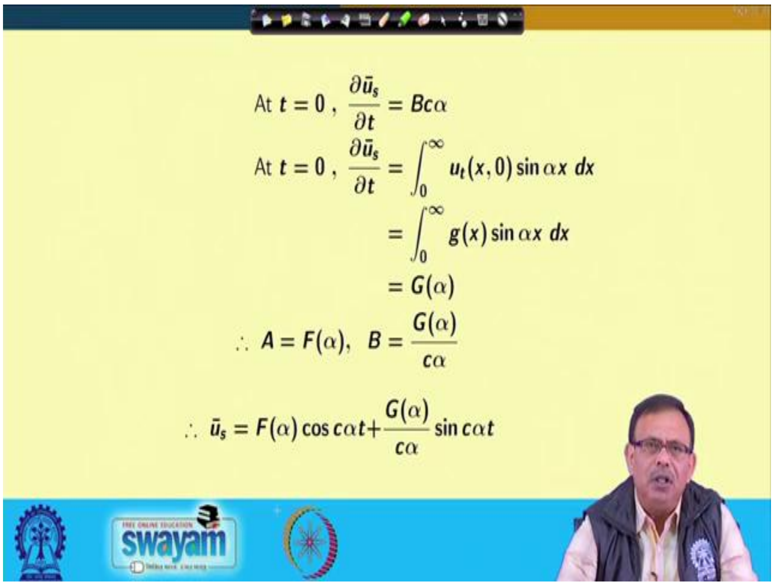

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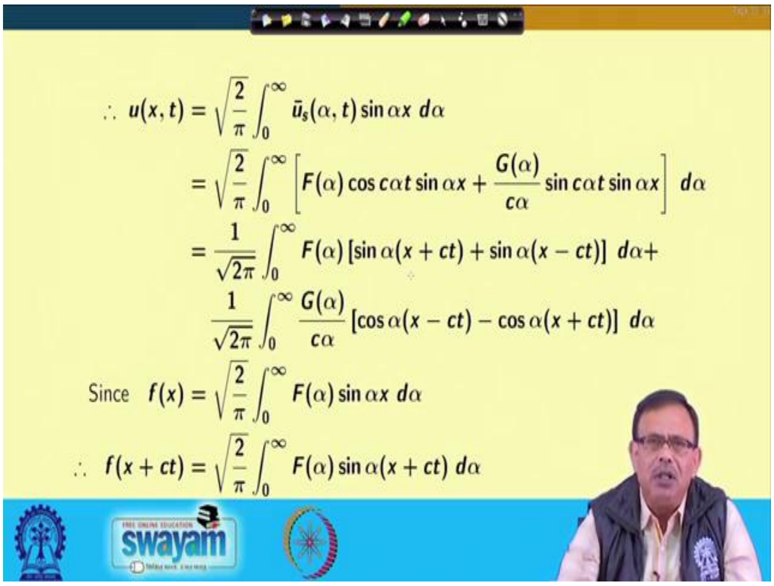

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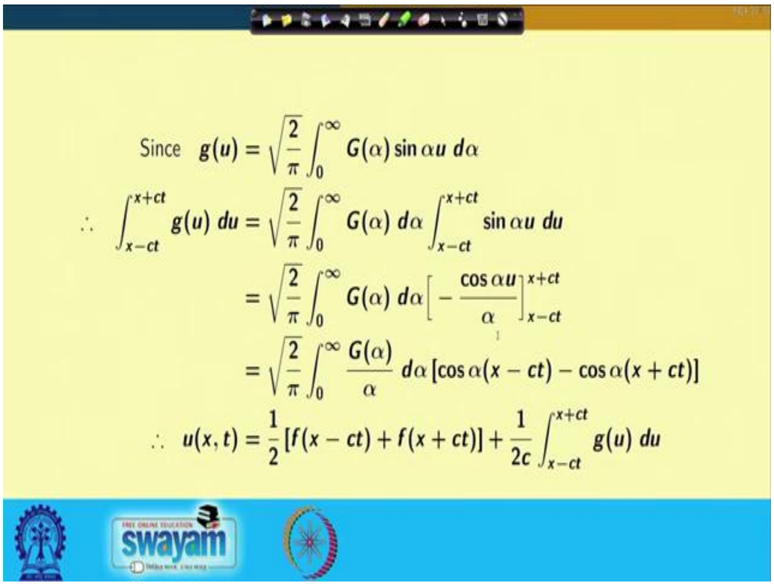

Thank you.