Transform Calculus and its Applications in Differential Equations

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Lecture – 48
Solution of Partial Differential Equations using Fourier Transform – 1

Now let us see how to find out the solution of a PDE using Fourier Transform.

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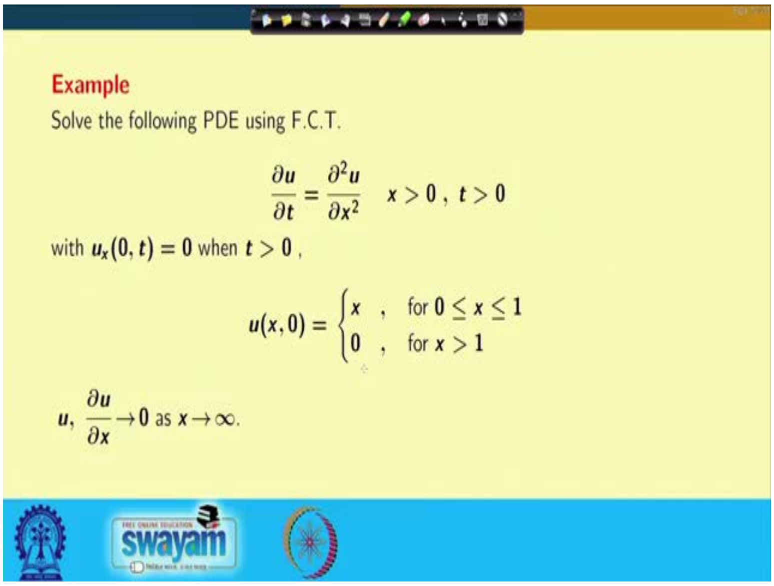

The problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \ t > 0$$

$$\text{with, } u_x(0,t) = 0 \text{ when } t > 0 \text{ and } u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$u, \frac{\partial u}{\partial x} \to 0 \ \text{ as } \ x \to \infty$$

We apply Fourier cosine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_c\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx\right] = \sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty + \alpha\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x\,dx\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = \alpha\sqrt{\frac{2}{\pi}}\left[[u\sin\alpha x]_0^\infty - \alpha\int_0^\infty u\cos\alpha x\,dx\right]$$

$$\left[\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty, \qquad u_x(0,t) = 0\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = -\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx \qquad [\because u \to 0 \text{ as } x \to \infty]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = -\alpha^2\bar{u}_c \qquad \text{where } \bar{u}_c(\alpha,t) = \mathcal{F}_c[u(x,t)]$$

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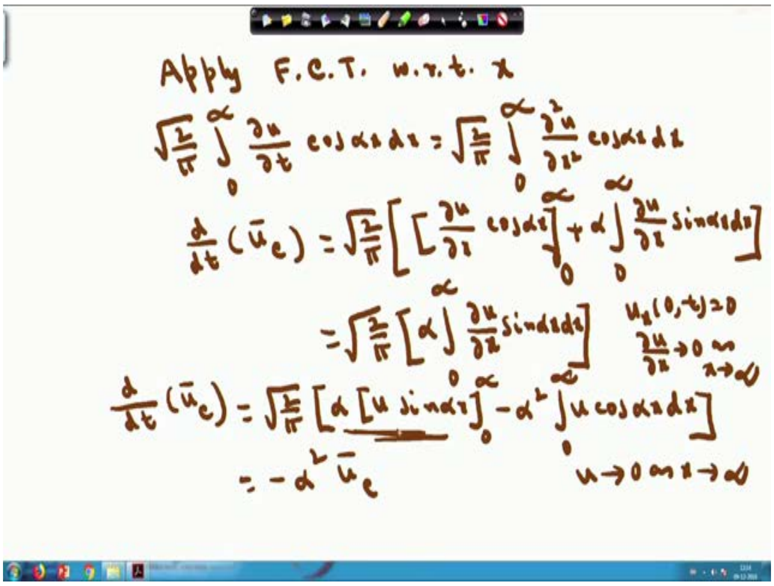

Thus, the given PDE is reduced to a first order ODE. The obtained ODE can be easily solved to get the solution as

$$\bar{u}_c(\alpha,t) = Ae^{-\alpha^2 t} \tag{1}$$

where, A is the constant of integration.

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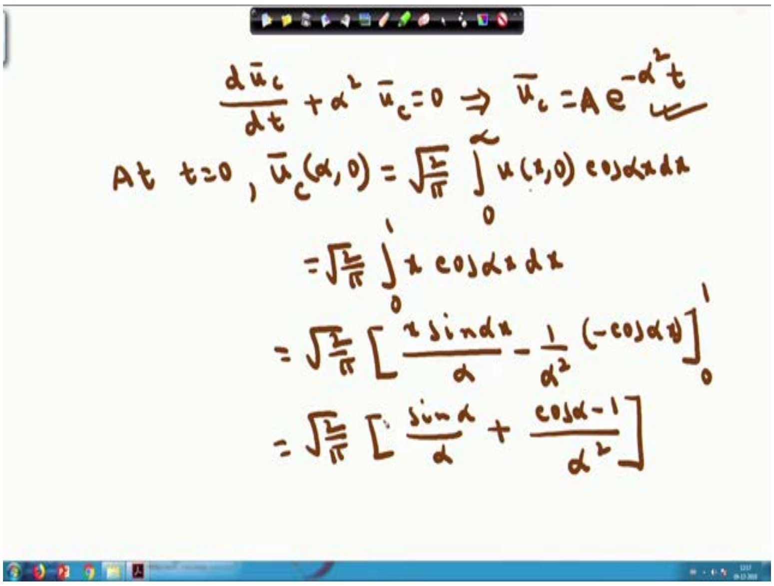

We are given the following initial condition:

$$u(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Now, at $t = 0$, we have,

$$\bar{u}_c(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\frac{x}{\alpha} \sin \alpha x \right]_0^1 - \frac{1}{\alpha} \int_0^1 \sin \alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{1}{\alpha^2} [\cos \alpha x]_0^1 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{1}{\alpha^2} (\cos \alpha - 1) \right]$$

Therefore, (1) implies

$$A = \bar{u}_c(\alpha, 0) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin\alpha}{\alpha} + \frac{1}{\alpha^2}(\cos\alpha - 1)\right]$$

Thus, from (1), we have,

$$\bar{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}\right] e^{-\alpha^2 t}$$

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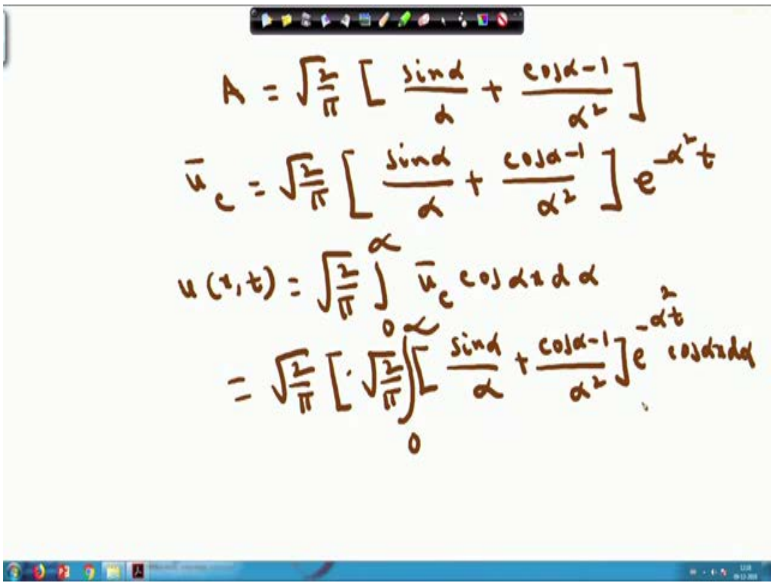

Now, taking the inverse Fourier cosine transform, we have,

$$\begin{aligned}
u(x, t) &= \mathcal{F}_c^{-1}[\bar{u}_c(\alpha, t)] \\
&= \sqrt{\frac{2}{\pi}}\, \mathcal{F}_c^{-1}\left[\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}\right] e^{-\alpha^2 t}\right] \\
&= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}\right] e^{-\alpha^2 t}\cos\alpha x\, d\alpha \\
&= \frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}\right] e^{-\alpha^2 t}\cos\alpha x\, d\alpha
\end{aligned}$$

Evaluation of the above integral will give the required solution for $u(x, t)$.

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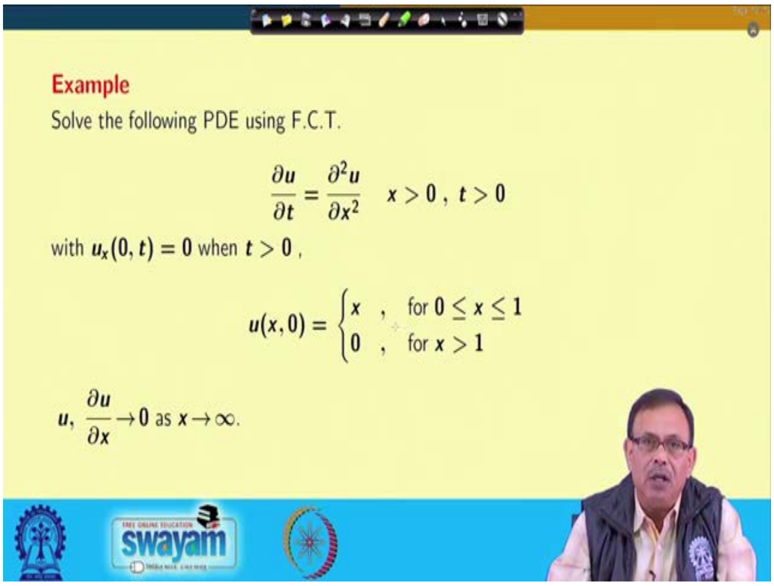

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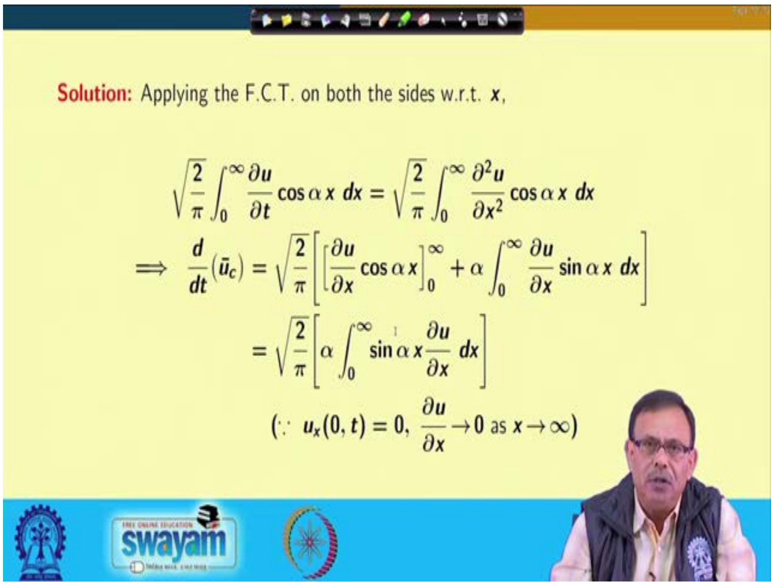

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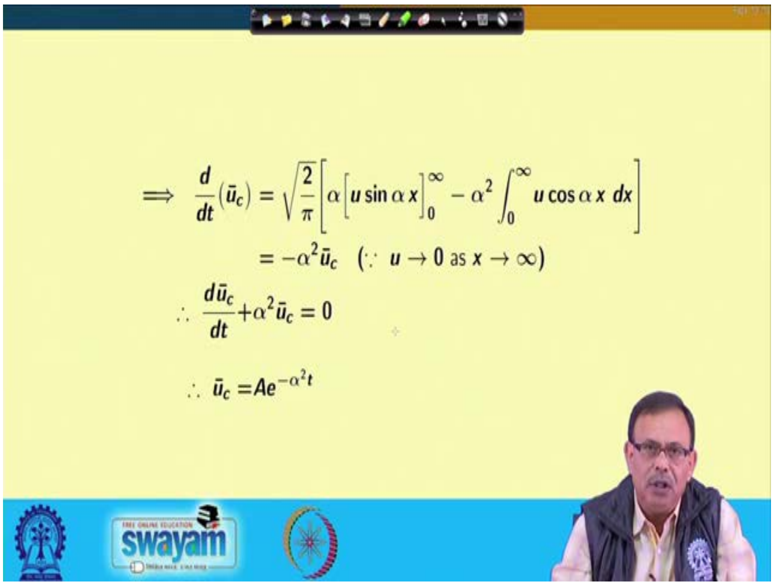

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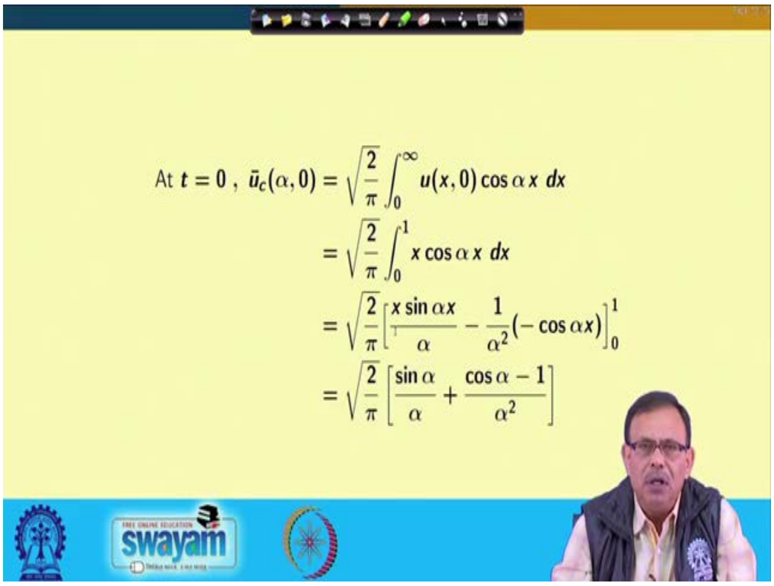

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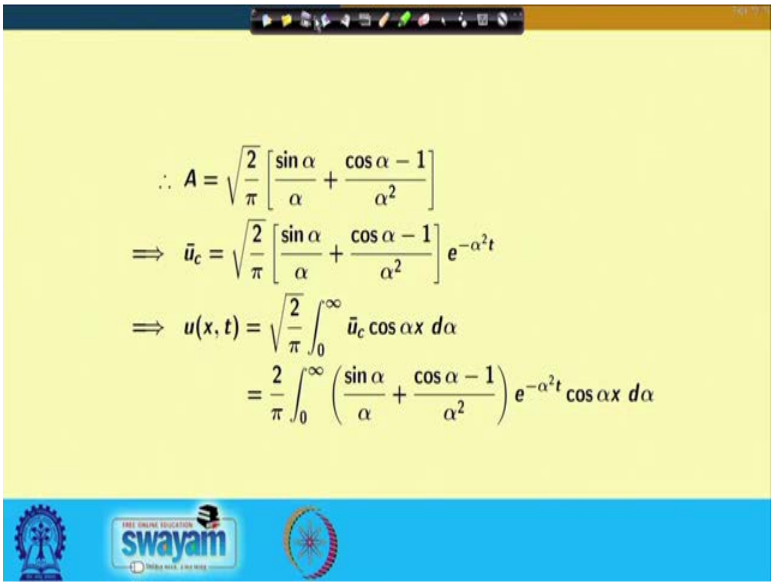

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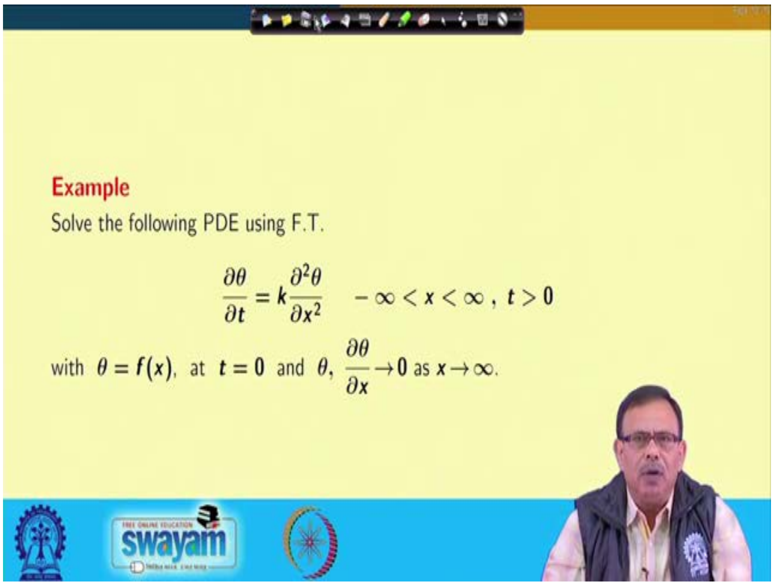

Now, let us solve another problem using Fourier transform. The problem is given as:

$$\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2} \qquad -\infty < x < \infty,\ \ t > 0$$

$$\text{with},\theta = f(x) \text{ at } t = 0 \text{ and}$$

$$\theta,\frac{\partial\theta}{\partial x} \to 0 \ \text{ as } \ x \to \pm\infty$$

We have two variables here, x and t; t is given as greater than 0 and x lies between $-\infty$ to ∞.

And we know that the range for Fourier transform is from $-\infty$ to ∞. Also it has been given that, $\theta, \frac{\partial\theta}{\partial x} \to 0$ as $x \to \pm\infty$. Therefore, from the given criteria, we can tell that, we have to use Fourier transform on this given equation with respect to the variable x, not with respect to the variable t because range of t is from 0 to ∞, not from $-\infty$ to ∞.

Here, since x varies from $-\infty$ to ∞, we can take Fourier transform.

We apply Fourier transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}\left[\frac{\partial\theta}{\partial t}\right] = k\,\mathcal{F}\left[\frac{\partial^2\theta}{\partial x^2}\right]$$

$$\Rightarrow \frac{d}{dt}\mathcal{F}[\theta] = k(-i\alpha)^2\mathcal{F}[\theta]$$

$$\Rightarrow \frac{d\bar{\theta}}{dt} = -k\alpha^2\bar{\theta} \qquad \text{where, } \bar{\theta}(\alpha,t) = \mathcal{F}[\theta(x,t)]$$

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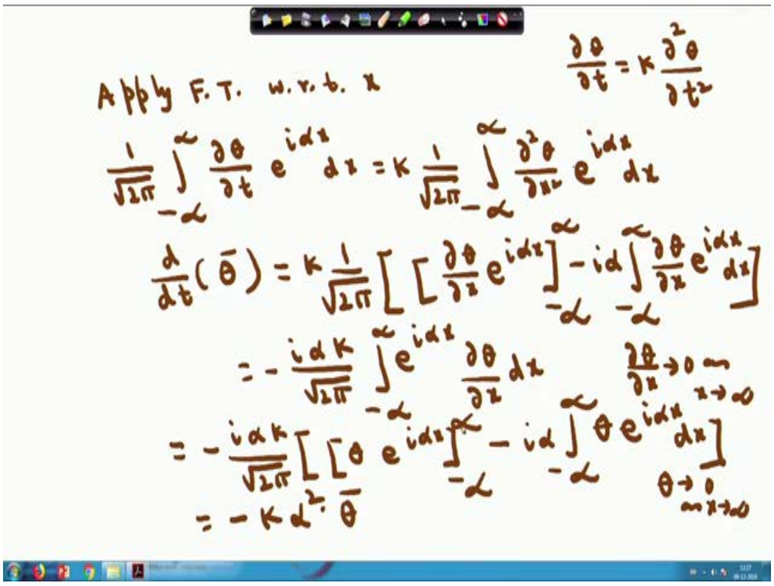

Therefore, the given PDE is reduced to a first order ODE whose solution is given as:

$$\bar{\theta}(\alpha,t) = Ae^{-k\alpha^2 t} \qquad (2)$$

where, A is the constant of integration.

We are given the initial condition as

$$\theta = f(x) \text{ at } t = 0$$

Now,

$$\begin{aligned}\bar{\theta}(\alpha, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta(x, 0)\, e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx \\ &= \bar{f}(\alpha)\end{aligned}$$

where, $\bar{f}(\alpha)$ denotes the Fourier transform of $f(x)$.

Therefore, (2) implies

$$A = \bar{\theta}(\alpha, 0) = \bar{f}(\alpha)$$

Thus, from (2), we have,

$$\bar{\theta}(\alpha, t) = \bar{f}(\alpha) e^{-k\alpha^2 t}$$

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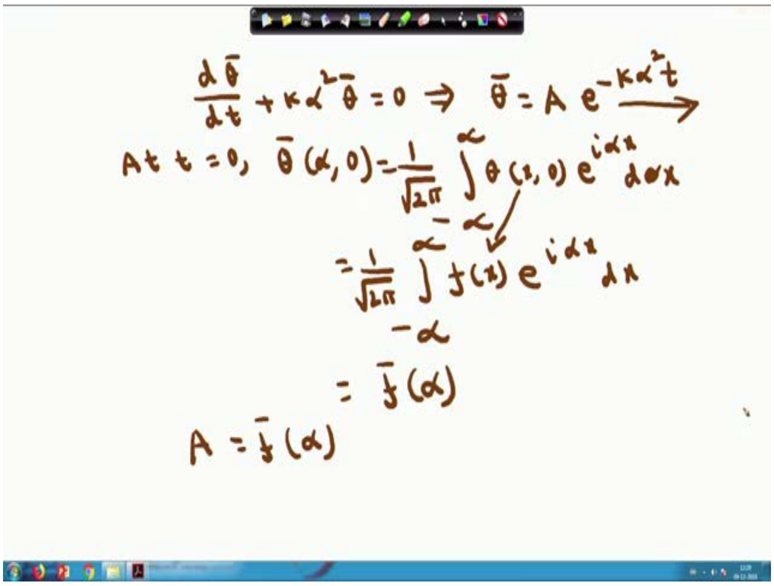

Now, taking the inverse Fourier transform, we have,

$$\theta(x,t) = \mathcal{F}^{-1}[\bar{\theta}(\alpha,t)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha$$

If $f(x)$ is known to us, then we can calculate $\bar{f}(\alpha)$ also. And hence, we can evaluate the integral to obtain the required solution for $\theta(x,t)$.

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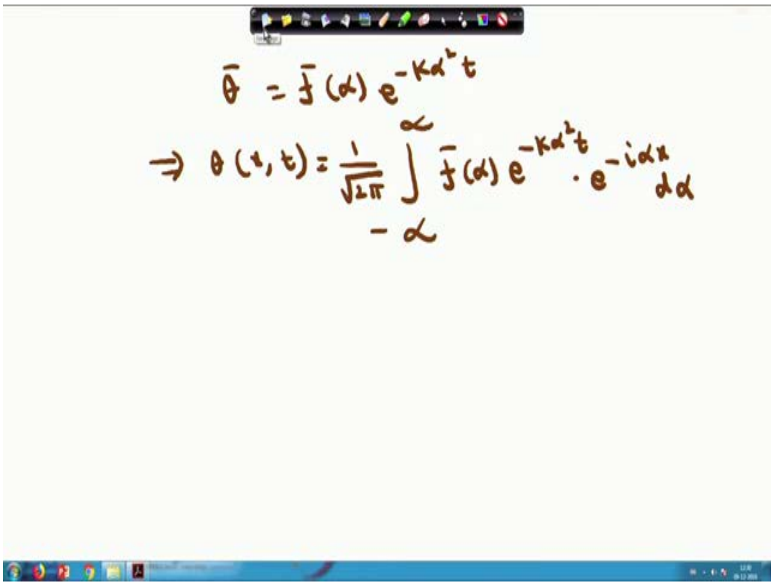

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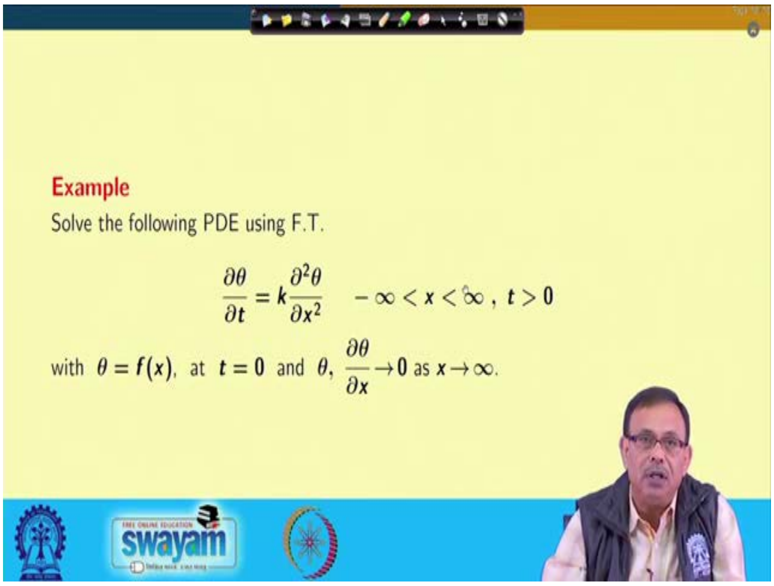

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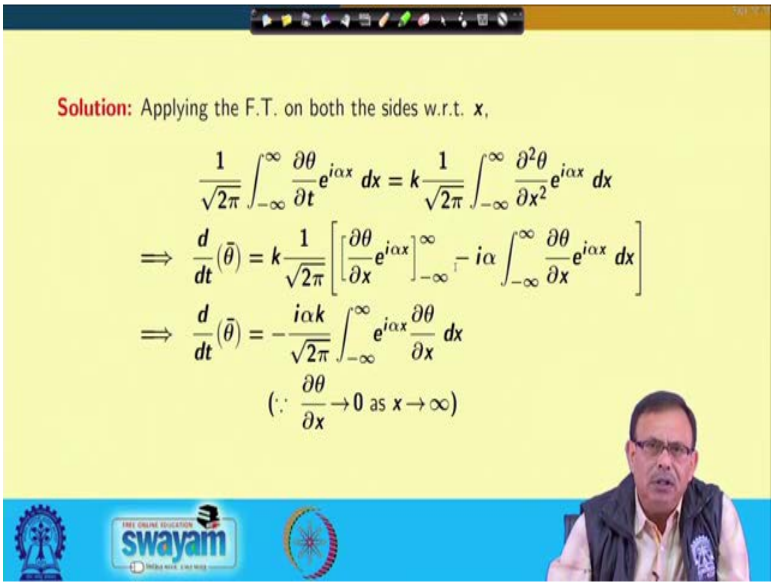

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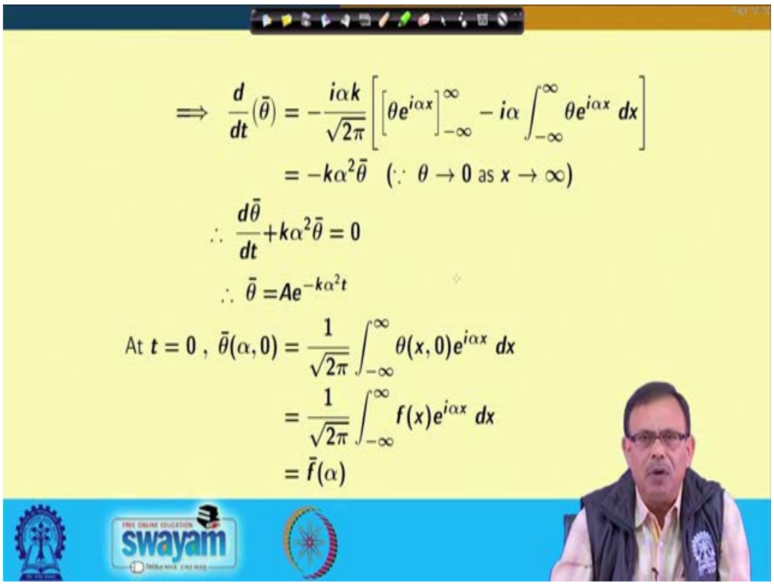

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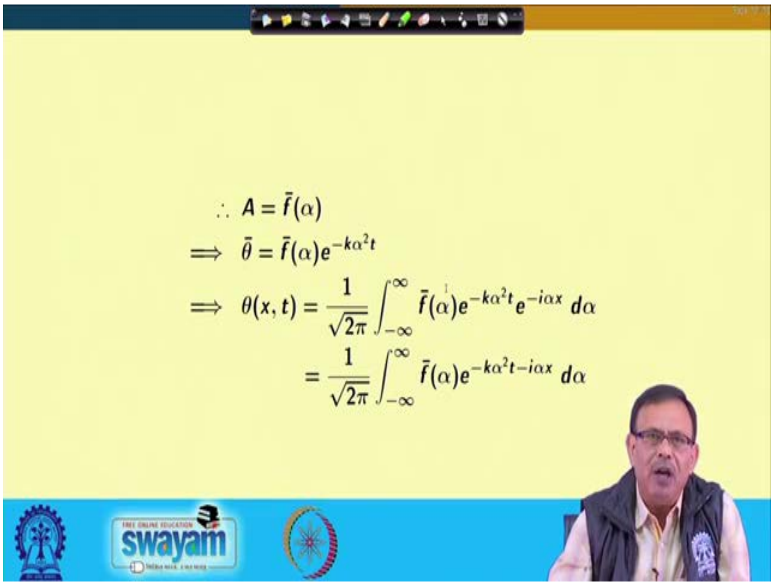

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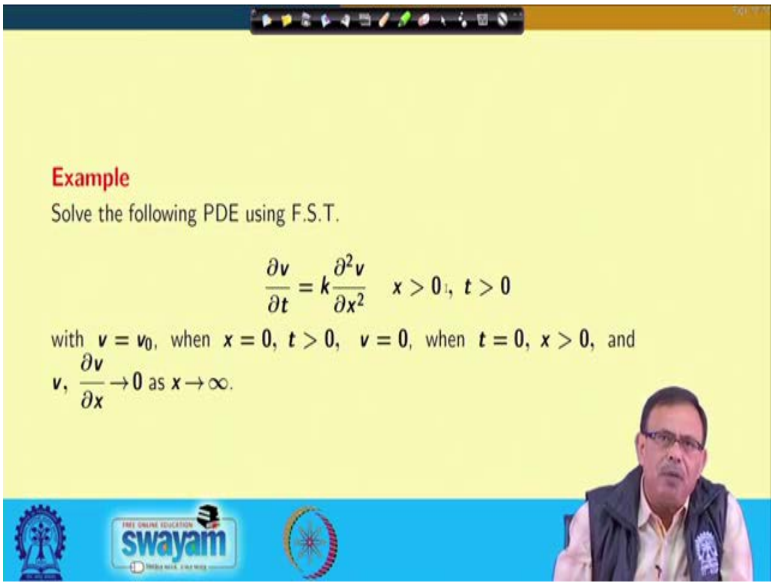

Now, let us take another problem. We have solved this particular problem earlier using Fourier cosine transform. Now we want to solve the same problem using Fourier sine transform. The problem is given as:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad x > 0,\ \ t > 0$$

$$\text{with, } v(0,t) = v_0 \text{ when } t > 0 \text{ and } v(x,0) = 0 \ \text{ when } \ x > 0$$

$$v, \frac{\partial v}{\partial x} \to 0 \ \text{ as } \ x \to \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_s\left[\frac{\partial v}{\partial t}\right] = k\mathcal{F}_s\left[\frac{\partial^2 v}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial v}{\partial t}\ \sin\alpha x\,dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 v}{\partial x^2}\ \sin\alpha x\,dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty v\sin\alpha x\,dx\right] = k\sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial v}{\partial x}\sin\alpha x\right]_0^\infty - \alpha\int_0^\infty \frac{\partial v}{\partial x}\cos\alpha x\,dx\right]$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} = -k\alpha\sqrt{\frac{2}{\pi}}\left[[v\cos\alpha x]_0^\infty + \alpha\int_0^\infty v\sin\alpha x\,dx\right] \quad \left[\because \frac{\partial v}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} = -k\alpha\sqrt{\frac{2}{\pi}}\left[-v_0 + \alpha\int_0^\infty v\sin\alpha x\,dx\right] \quad [\because v \to 0 \text{ as } x \to \infty,\ v(0,t) = v_0]$$

$$\Rightarrow \frac{d\bar{v}_s}{dt} = \sqrt{\frac{2}{\pi}}k\alpha v_0 - k\alpha^2\bar{v}_s \qquad \text{where } \bar{v}_s(\alpha,t) = \mathcal{F}_s[v(x,t)]$$

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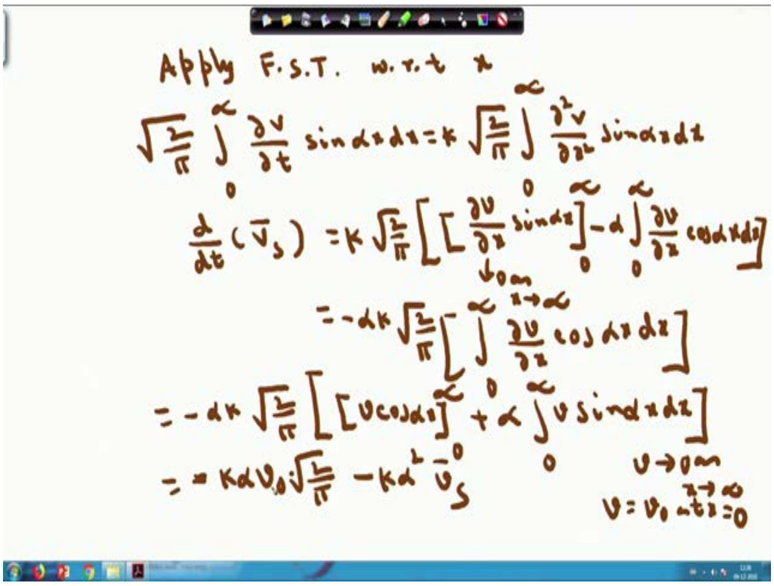

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{v}_s e^{k\alpha^2 t} = C + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k\alpha^2 t} \tag{3}$$

where, C is the constant of integration.

We have the initial condition as $v(x, 0) = 0$. Therefore,

$$\bar{v}_s(\alpha, 0) = 0$$

Thus, (3) implies

$$C = -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha}$$

Therefore, from (3), we have,

$$\bar{v}_s e^{k\alpha^2 t} = -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k\alpha^2 t}$$

$$\Rightarrow \bar{v}_s(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} \left(1 - e^{-k\alpha^2 t}\right)$$

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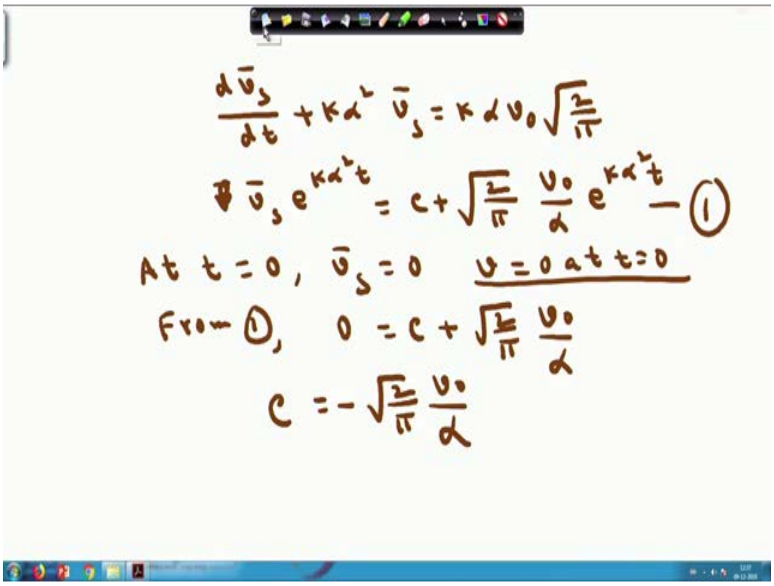

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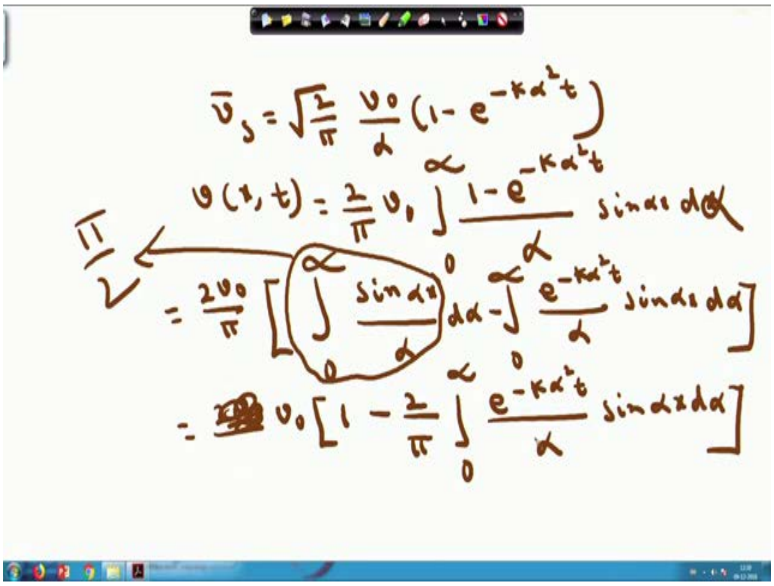

Taking the inverse Fourier sine transform, we have,

$$v(x,t) = \mathcal{F}_s^{-1}[\bar{v}_s(\alpha,t)]$$

$$\Rightarrow v(x,t) = \sqrt{\frac{2}{\pi}} v_0 \mathcal{F}_s^{-1}\left[\frac{1}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)\right]$$

$$\Rightarrow v(x,t) = \frac{2v_0}{\pi}\int_0^\infty \frac{1}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)\sin\alpha x\, d\alpha$$

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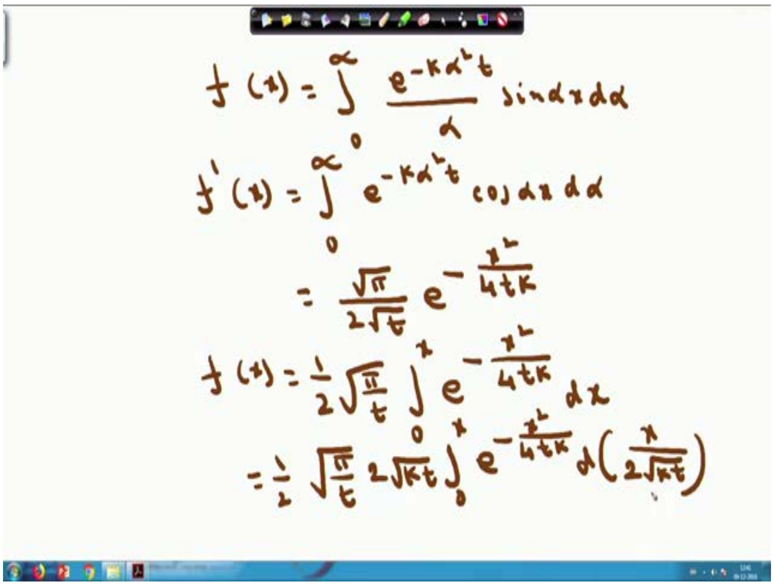

Therefore, we have,

$$v(x,t) = \frac{2v_0}{\pi}\int_0^\infty \frac{1}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)\sin \alpha x \, d\alpha$$

$$\Rightarrow v(x,t) = \frac{2v_0}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} d\alpha - \frac{2v_0}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} d\alpha$$

$$\Rightarrow v(x,t) = \frac{2v_0}{\pi}\frac{\pi}{2} - \frac{2v_0}{\pi}\int_0^\infty \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} d\alpha$$

$$\Rightarrow v(x,t) = v_0 - \frac{2v_0}{\pi} f(x)$$

where,

$$f(x) = \int_0^\infty \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} d\alpha$$

$$\Rightarrow f'(x) = \int_0^\infty e^{-k\alpha^2 t}\cos \alpha x \, d\alpha$$

$$\Rightarrow f'(x) = \frac{\sqrt{\pi}}{2\sqrt{kt}} e^{-\frac{x^2}{4tk}}$$

$$\Rightarrow f(x) = \frac{\sqrt{\pi}}{2\sqrt{kt}}\int_0^x e^{-\frac{x^2}{4tk}} dx$$

$$\Rightarrow f(x) = \sqrt{\pi}\int_0^x e^{-\frac{x^2}{4tk}} d\left(\frac{x}{2\sqrt{kt}}\right)$$

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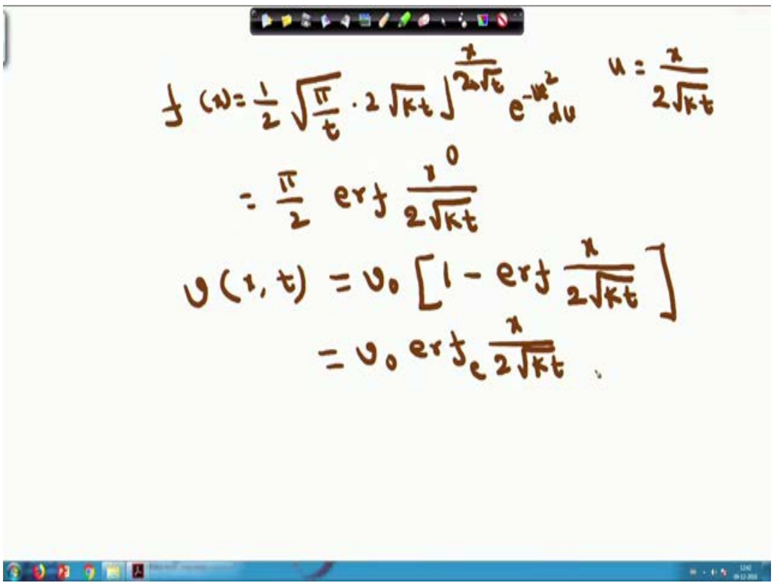

Put

$$u = \frac{x}{2\sqrt{kt}} \quad \text{so that} \quad du = \frac{dx}{2\sqrt{kt}}$$

Therefore,

$$f(x) = \sqrt{\pi}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2}\, du$$

$$\Rightarrow f(x) = \sqrt{\pi}\,\frac{\sqrt{\pi}}{2}\,\text{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$\Rightarrow f(x) = \frac{\pi}{2}\,\text{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

Therefore,

$$v(x,t) = v_0 - \frac{2v_0}{\pi}\frac{\pi}{2}\,\text{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$\Rightarrow v(x,t) = v_0 - v_0\,\text{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$\Rightarrow v(x,t) = v_0\left[1 - \text{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right] = v_0\,\text{erf}_c\left(\frac{x}{2\sqrt{kt}}\right)$$

which is the required solution.

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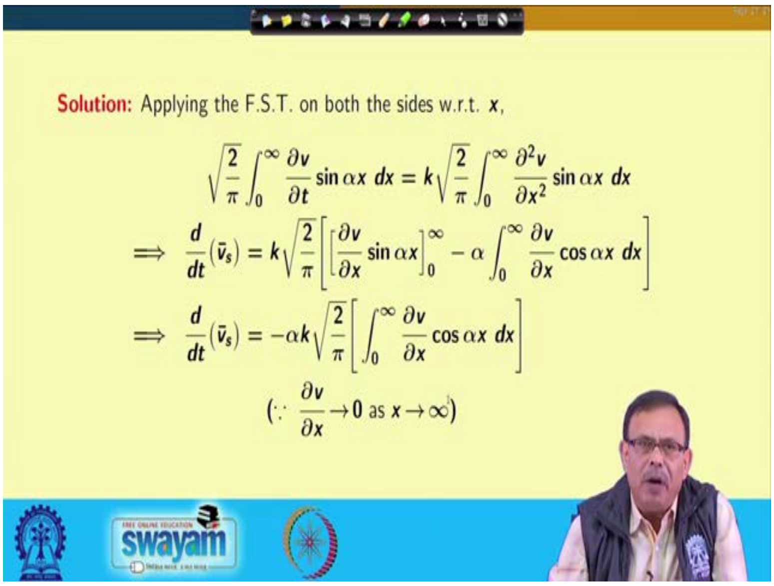

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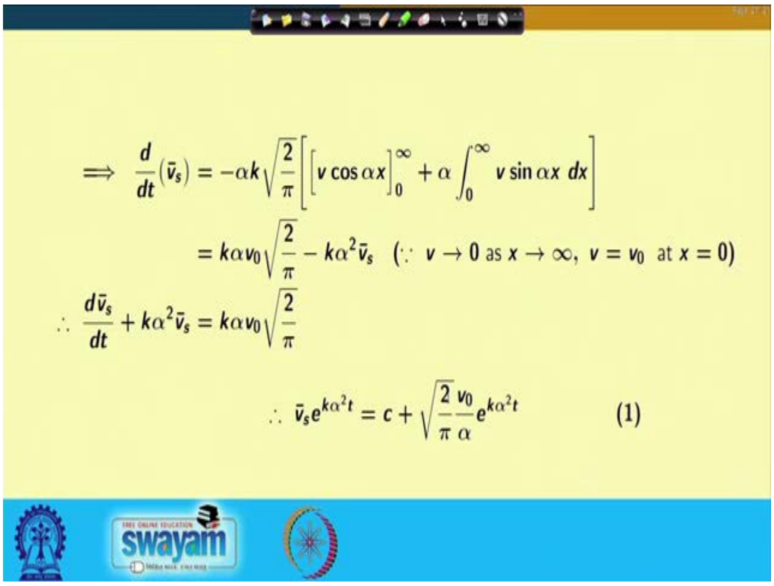

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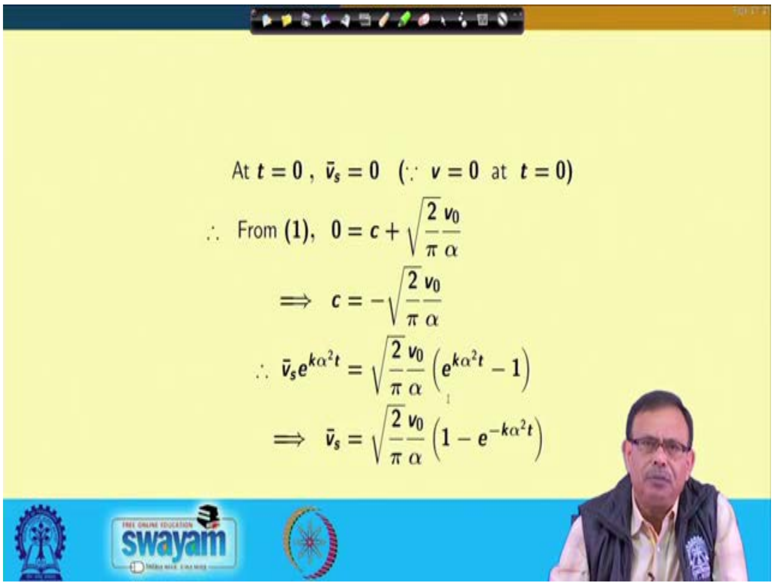

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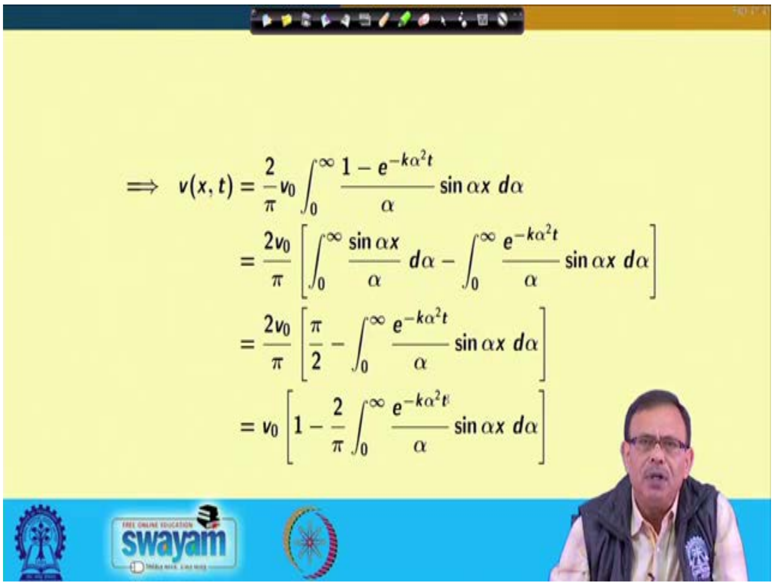

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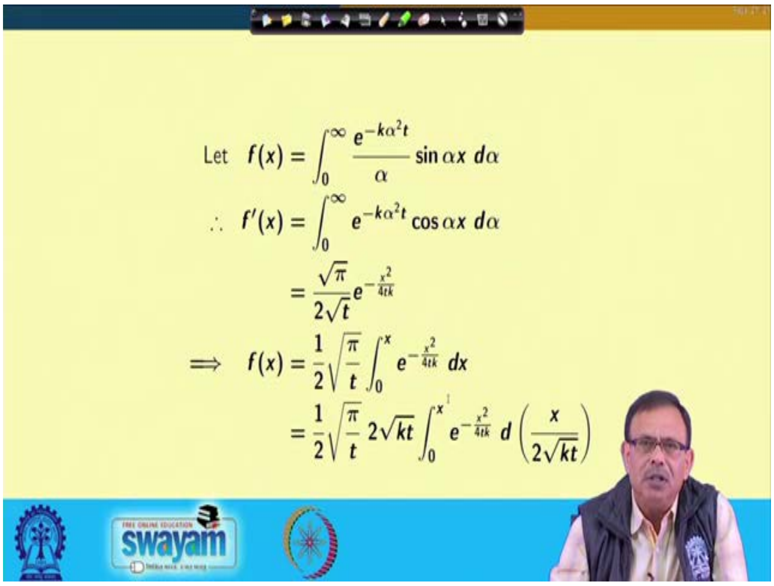

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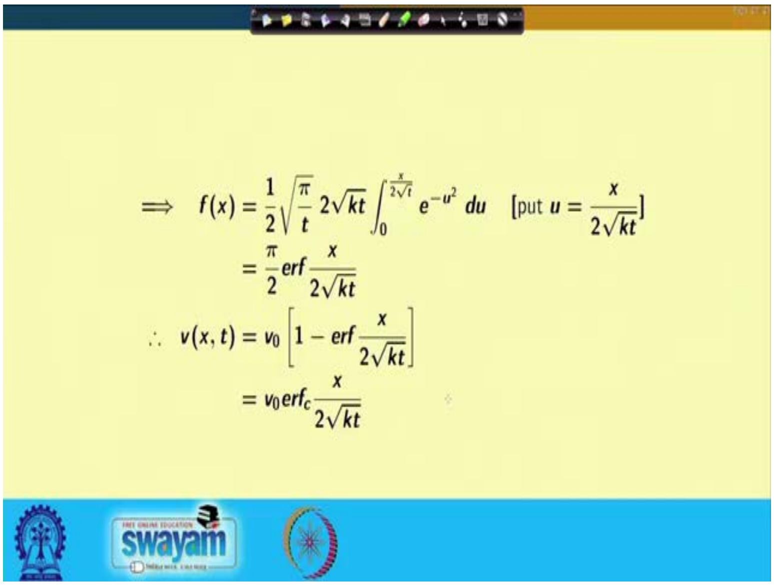

Thank you.