

Transform Calculus and its Applications in Differential Equations
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 48
Solution of Partial Differential Equations using Fourier Transform – 1

Now let us see how to find out the solution of a PDE using Fourier Transform.

(Refer Slide Time: 00:27)

The problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad x > 0, \quad t > 0$$

with, $u_x(0, t) = 0$ when $t > 0$ and $u(x, 0) = \begin{cases} x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$

$$u, \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

We apply Fourier cosine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_c \left[\frac{\partial u}{\partial t} \right] = \mathcal{F}_c \left[\frac{\partial^2 u}{\partial x^2} \right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \cos ax \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \cos ax \, dx$$

$$\begin{aligned} \Rightarrow \frac{d}{dt} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} u \cos \alpha x \, dx \right] &= \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \cos \alpha x \right]_0^{\infty} + \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \sin \alpha x \, dx \right] \\ \Rightarrow \frac{d\bar{u}_c}{dt} &= \alpha \sqrt{\frac{2}{\pi}} \left[[u \sin \alpha x]_0^{\infty} - \alpha \int_0^{\infty} u \cos \alpha x \, dx \right] \\ &\quad \left[\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty, \quad u_x(0, t) = 0 \right] \\ \Rightarrow \frac{d\bar{u}_c}{dt} &= -\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} u \cos \alpha x \, dx \quad [\because u \rightarrow 0 \text{ as } x \rightarrow \infty] \\ \Rightarrow \frac{d\bar{u}_c}{dt} &= -\alpha^2 \bar{u}_c \quad \text{where } \bar{u}_c(\alpha, t) = \mathcal{F}_c[u(x, t)] \end{aligned}$$

(Refer Slide Time: 01:33)

Apply F.C.T. w.r.t. x

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \cos \alpha x \, dx &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx \\ \frac{d}{dt} (\bar{u}_c) &= \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \cos \alpha x \right]_0^{\infty} + \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \sin \alpha x \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\alpha \int_0^{\infty} \frac{\partial u}{\partial x} \sin \alpha x \, dx \right] \quad \begin{matrix} u_x(0, t) = 0 \\ \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \end{matrix} \\ \frac{d}{dt} (\bar{u}_c) &= \sqrt{\frac{2}{\pi}} \left[\alpha [u \sin \alpha x]_0^{\infty} - \alpha^2 \int_0^{\infty} u \cos \alpha x \, dx \right] \\ &= -\alpha^2 \bar{u}_c \quad u \rightarrow 0 \text{ as } x \rightarrow \infty \end{aligned}$$

Thus, the given PDE is reduced to a first order ODE. The obtained ODE can be easily solved to get the solution as

$$\bar{u}_c(\alpha, t) = A e^{-\alpha^2 t} \quad (1)$$

where, A is the constant of integration.

(Refer Slide Time: 04:49)

$$\begin{aligned}\frac{d\bar{u}_c}{dt} + \alpha^2 \bar{u}_c &= 0 \Rightarrow \bar{u}_c = A e^{-\alpha^2 t} \\ \text{At } t=0, \bar{u}_c(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^1 u(x, 0) \cos \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin \alpha x}{\alpha} - \frac{1}{\alpha^2} (-\cos \alpha x) \right]_0^1 \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right]\end{aligned}$$

We are given the following initial condition:

$$u(x, 0) = \begin{cases} x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

Now, at $t = 0$, we have,

$$\begin{aligned}\bar{u}_c(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, 0) \cos \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \left[\left[\frac{x}{\alpha} \sin \alpha x \right]_0^1 - \frac{1}{\alpha} \int_0^1 \sin \alpha x dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{1}{\alpha^2} [\cos \alpha x]_0^1 \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{1}{\alpha^2} (\cos \alpha - 1) \right]\end{aligned}$$

Therefore, (1) implies

$$A = \bar{u}_c(\alpha, 0) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{1}{\alpha^2} (\cos \alpha - 1) \right]$$

Thus, from (1), we have,

$$\bar{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t}$$

(Refer Slide Time: 07:09)

The image shows a whiteboard with handwritten mathematical work. The equations are as follows:

$$A = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right]$$

$$\bar{u}_c = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t}$$

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{u}_c \cos \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \left[\sqrt{\frac{2}{\pi}} \right] \int_0^{\infty} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t} \cos \alpha x \, d\alpha$$

Now, taking the inverse Fourier cosine transform, we have,

$$\begin{aligned} u(x, t) &= \mathcal{F}_c^{-1}[\bar{u}_c(\alpha, t)] \\ &= \sqrt{\frac{2}{\pi}} \mathcal{F}_c^{-1} \left[\left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t} \right] \\ &= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t} \cos \alpha x \, d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t} \cos \alpha x \, d\alpha \end{aligned}$$

Evaluation of the above integral will give the required solution for $u(x, t)$.

(Refer Slide Time: 08:51)


Example
Solve the following PDE using F.C.T.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0$$

with $u_x(0, t) = 0$ when $t > 0$,

$$u(x, 0) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 0, & \text{for } x > 1 \end{cases}$$

$u, \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.




(Refer Slide Time: 08:53)

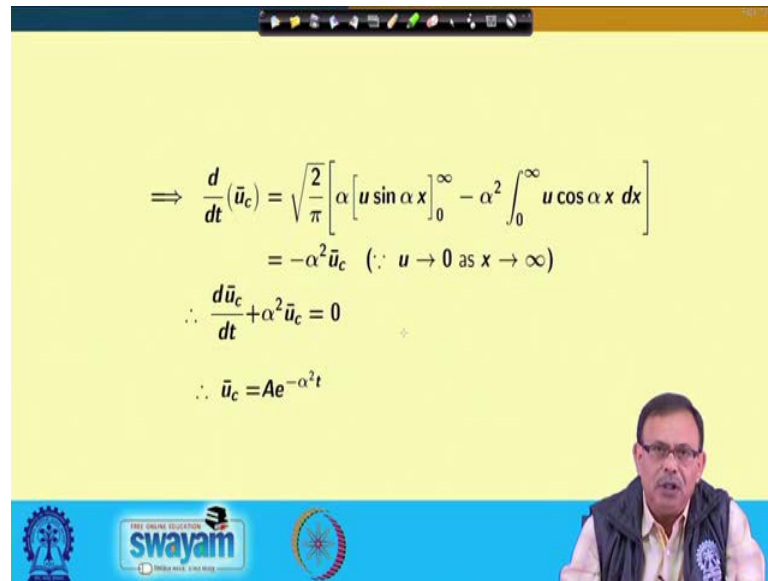
Solution: Applying the F.C.T. on both the sides w.r.t. x ,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$
$$\Rightarrow \frac{d}{dt}(\bar{u}_c) = \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \cos \alpha x \right]_0^\infty + \alpha \int_0^\infty \frac{\partial u}{\partial x} \sin \alpha x \, dx \right]$$
$$= \sqrt{\frac{2}{\pi}} \left[\alpha \int_0^\infty \sin \alpha x \frac{\partial u}{\partial x} \, dx \right]$$

($\because u_x(0, t) = 0, \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$)



(Refer Slide Time: 09:13)

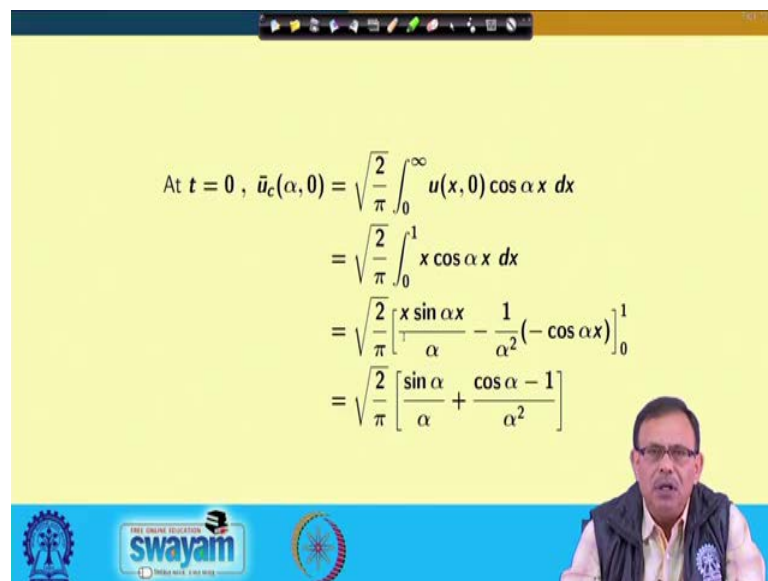


The slide displays the following mathematical derivation:

$$\begin{aligned}\Rightarrow \frac{d}{dt}(\bar{u}_c) &= \sqrt{\frac{2}{\pi}} \left[\alpha \left[u \sin \alpha x \right]_0^\infty - \alpha^2 \int_0^\infty u \cos \alpha x \, dx \right] \\ &= -\alpha^2 \bar{u}_c \quad (\because u \rightarrow 0 \text{ as } x \rightarrow \infty) \\ \therefore \frac{d\bar{u}_c}{dt} + \alpha^2 \bar{u}_c &= 0 \\ \therefore \bar{u}_c &= Ae^{-\alpha^2 t}\end{aligned}$$

The slide also features the Swamyam logo and a small video inset of the presenter in the bottom right corner.

(Refer Slide Time: 09:31)



The slide displays the following mathematical derivation for the initial condition:

$$\begin{aligned}\text{At } t = 0, \bar{u}_c(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \cos \alpha x \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos \alpha x \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin \alpha x}{\alpha} - \frac{1}{\alpha^2} (-\cos \alpha x) \right]_0^1 \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right]\end{aligned}$$

The slide also features the Swamyam logo and a small video inset of the presenter in the bottom right corner.

(Refer Slide Time: 10:01)

$$\therefore A = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right]$$
$$\Rightarrow \bar{u}_c = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right] e^{-\alpha^2 t}$$
$$\Rightarrow u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{u}_c \cos \alpha x \, d\alpha$$
$$= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha}{\alpha} + \frac{\cos \alpha - 1}{\alpha^2} \right) e^{-\alpha^2 t} \cos \alpha x \, d\alpha$$

(Refer Slide Time: 10:35)

Example

Solve the following PDE using F.T.

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \quad -\infty < x < \infty, \quad t > 0$$

with $\theta = f(x)$, at $t = 0$ and $\theta, \frac{\partial \theta}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.

Now, let us solve another problem using Fourier transform. The problem is given as:

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \quad -\infty < x < \infty, \quad t > 0$$

with, $\theta = f(x)$ at $t = 0$ and

$$\theta, \frac{\partial \theta}{\partial x} \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty$$

We have two variables here, x and t ; t is given as greater than 0 and x lies between $-\infty$ to ∞ .

And we know that the range for Fourier transform is from $-\infty$ to ∞ . Also it has been given that, $\theta, \frac{\partial \theta}{\partial x} \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore, from the given criteria, we can tell that, we have to use Fourier transform on this given equation with respect to the variable x , not with respect to the variable t because range of t is from 0 to ∞ , not from $-\infty$ to ∞ .

Here, since x varies from $-\infty$ to ∞ , we can take Fourier transform.

We apply Fourier transform with respect to x on the given equation. Therefore, we obtain,

$$\begin{aligned} \mathcal{F}\left[\frac{\partial \theta}{\partial t}\right] &= k \mathcal{F}\left[\frac{\partial^2 \theta}{\partial x^2}\right] \\ \Rightarrow \frac{d}{dt} \mathcal{F}[\theta] &= k(-i\alpha)^2 \mathcal{F}[\theta] \\ \Rightarrow \frac{d\bar{\theta}}{dt} &= -k\alpha^2 \bar{\theta} \quad \text{where, } \bar{\theta}(\alpha, t) = \mathcal{F}[\theta(x, t)] \end{aligned}$$

(Refer Slide Time: 12:53)

Apply F.T. w.r.t. x

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} e^{i\alpha x} dx = k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \theta}{\partial x^2} e^{i\alpha x} dx$$

$$\frac{d}{dt} (\bar{\theta}) = k \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \frac{\partial \theta}{\partial x} e^{i\alpha x} dx - i\alpha \int_{-\infty}^{\infty} \theta \frac{\partial e^{i\alpha x}}{\partial x} dx \right]$$

$$= -\frac{i\alpha k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial \theta}{\partial x} dx$$

$$= -\frac{i\alpha k}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \theta e^{i\alpha x} dx - i\alpha \int_{-\infty}^{\infty} \theta e^{i\alpha x} dx \right]$$

$$= -k\alpha^2 \bar{\theta}$$

Notes: $\frac{\partial \theta}{\partial x} \rightarrow 0$ as $x \rightarrow \pm\infty$, $\theta \rightarrow 0$ as $x \rightarrow \pm\infty$

Therefore, the given PDE is reduced to a first order ODE whose solution is given as:

$$\bar{\theta}(\alpha, t) = A e^{-k\alpha^2 t} \quad (2)$$

where, A is the constant of integration.

We are given the initial condition as

$$\theta = f(x) \text{ at } t = 0$$

Now,

$$\begin{aligned}\bar{\theta}(\alpha, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta(x, 0) e^{i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \\ &= \bar{f}(\alpha)\end{aligned}$$

where, $\bar{f}(\alpha)$ denotes the Fourier transform of $f(x)$.

Therefore, (2) implies

$$A = \bar{\theta}(\alpha, 0) = \bar{f}(\alpha)$$

Thus, from (2), we have,

$$\bar{\theta}(\alpha, t) = \bar{f}(\alpha) e^{-k\alpha^2 t}$$

(Refer Slide Time: 17:33)

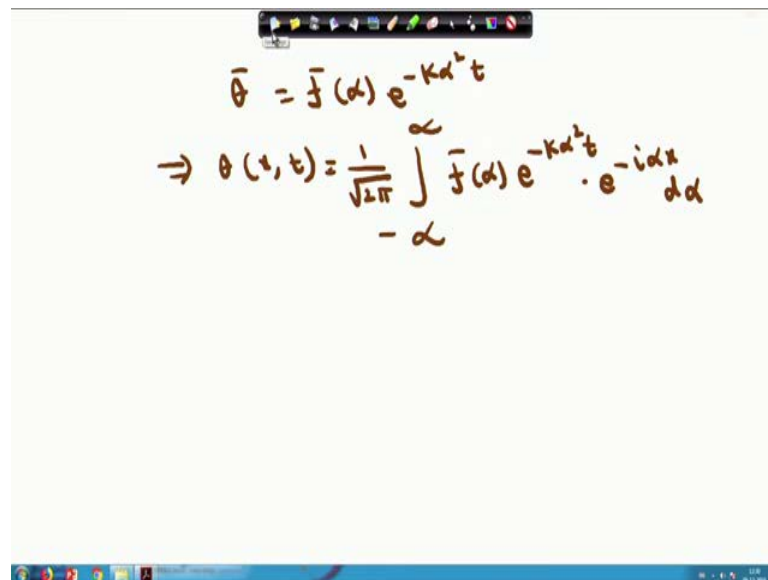
The image shows a handwritten derivation on a whiteboard. At the top, the differential equation $\frac{d\bar{\theta}}{dt} + k\alpha^2 \bar{\theta} = 0$ is written, with an arrow pointing to the solution $\bar{\theta} = A e^{-k\alpha^2 t}$. Below this, the initial condition is used: "At $t = 0$, $\bar{\theta}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta(x, 0) e^{i\alpha x} dx$ ". The integral is then simplified to $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$, which is identified as $= \bar{f}(\alpha)$. Finally, the constant A is determined to be $A = \bar{f}(\alpha)$.

Now, taking the inverse Fourier transform, we have,

$$\begin{aligned}\theta(x, t) &= \mathcal{F}^{-1}[\bar{\theta}(\alpha, t)] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha\end{aligned}$$

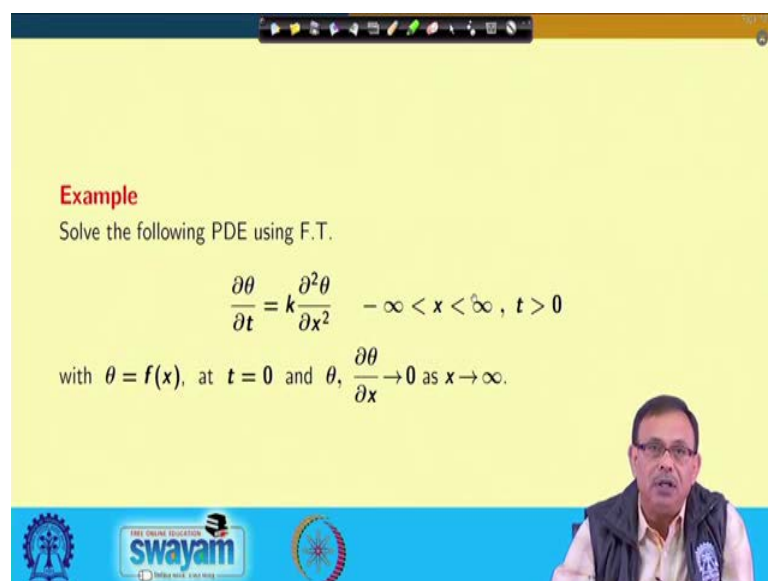
If $f(x)$ is known to us, then we can calculate $\bar{f}(\alpha)$ also. And hence, we can evaluate the integral to obtain the required solution for $\theta(x, t)$.

(Refer Slide Time: 19:25)



$$\begin{aligned}\bar{\theta} &= \bar{f}(\alpha) e^{-k\alpha^2 t} \\ \Rightarrow \theta(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-k\alpha^2 t} \cdot e^{-i\alpha x} d\alpha\end{aligned}$$

(Refer Slide Time: 20:21)



Example
Solve the following PDE using F.T.

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \quad -\infty < x < \infty, t > 0$$

with $\theta = f(x)$, at $t = 0$ and $\theta, \frac{\partial \theta}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.

(Refer Slide Time: 20:23)

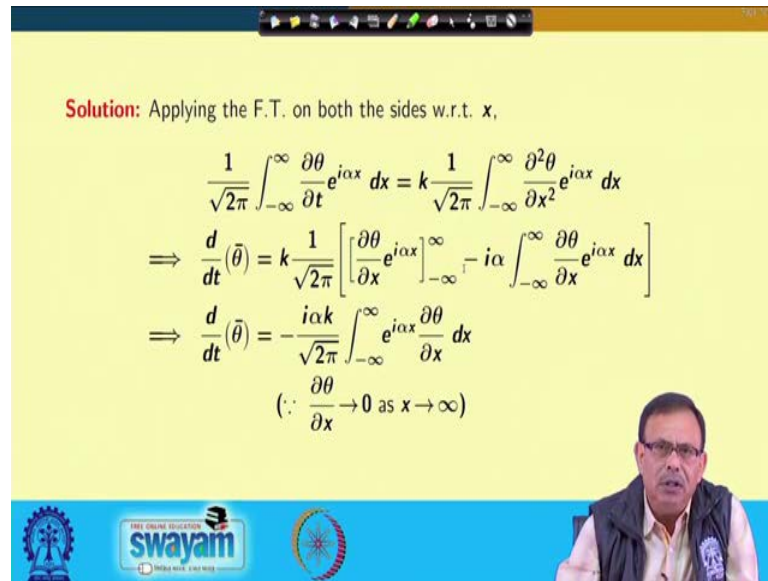
Solution: Applying the F.T. on both the sides w.r.t. x ,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} e^{i\alpha x} dx = k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 \theta}{\partial x^2} e^{i\alpha x} dx$$

$$\Rightarrow \frac{d}{dt}(\bar{\theta}) = k \frac{1}{\sqrt{2\pi}} \left[\left[\frac{\partial \theta}{\partial x} e^{i\alpha x} \right]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial x} e^{i\alpha x} dx \right]$$

$$\Rightarrow \frac{d}{dt}(\bar{\theta}) = -\frac{i\alpha k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial \theta}{\partial x} dx$$

($\because \frac{\partial \theta}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$)



(Refer Slide Time: 21:09)

$$\Rightarrow \frac{d}{dt}(\bar{\theta}) = -\frac{i\alpha k}{\sqrt{2\pi}} \left[\left[\theta e^{i\alpha x} \right]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} \theta e^{i\alpha x} dx \right]$$

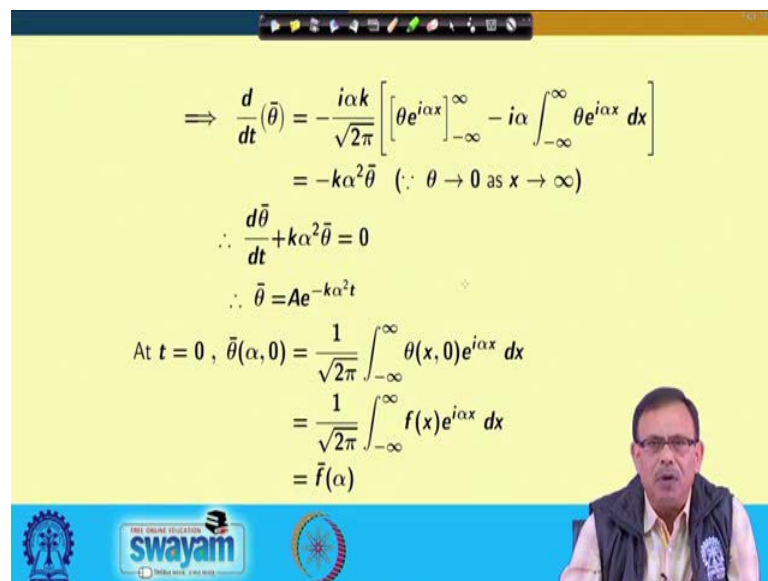
$$= -k\alpha^2 \bar{\theta} \quad (\because \theta \rightarrow 0 \text{ as } x \rightarrow \infty)$$

$$\therefore \frac{d\bar{\theta}}{dt} + k\alpha^2 \bar{\theta} = 0$$

$$\therefore \bar{\theta} = A e^{-k\alpha^2 t}$$

At $t = 0$, $\bar{\theta}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \theta(x, 0) e^{i\alpha x} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$= \bar{f}(\alpha)$$


(Refer Slide Time: 21:59)

The slide displays the following mathematical steps:

$$\begin{aligned} \therefore A &= \bar{f}(\alpha) \\ \Rightarrow \bar{\theta} &= \bar{f}(\alpha)e^{-k\alpha^2 t} \\ \Rightarrow \theta(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-k\alpha^2 t} e^{-i\alpha x} d\alpha \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-k\alpha^2 t - i\alpha x} d\alpha \end{aligned}$$

The slide also features the Swayam logo and a video feed of the presenter in the bottom right corner.

(Refer Slide Time: 22:23)

The slide presents the following example problem:

Example
Solve the following PDE using F.S.T.

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \quad x > 0, \quad t > 0$$

with $v = v_0$, when $x = 0, t > 0$, $v = 0$, when $t = 0, x > 0$, and $v, \frac{\partial v}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.

The slide also features the Swayam logo and a video feed of the presenter in the bottom right corner.

Now, let us take another problem. We have solved this particular problem earlier using Fourier cosine transform. Now we want to solve the same problem using Fourier sine transform. The problem is given as:

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} \quad x > 0, \quad t > 0$$

with, $v(0, t) = v_0$ when $t > 0$ and $v(x, 0) = 0$ when $x > 0$

$$v, \frac{\partial v}{\partial x} \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\begin{aligned}
 \mathcal{F}_s \left[\frac{\partial v}{\partial t} \right] &= k \mathcal{F}_s \left[\frac{\partial^2 v}{\partial x^2} \right] \\
 \Rightarrow \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial v}{\partial t} \sin \alpha x \, dx &= k \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 v}{\partial x^2} \sin \alpha x \, dx \\
 \Rightarrow \frac{d}{dt} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} v \sin \alpha x \, dx \right] &= k \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial v}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial v}{\partial x} \cos \alpha x \, dx \right] \\
 \Rightarrow \frac{d\bar{v}_s}{dt} &= -k\alpha \sqrt{\frac{2}{\pi}} \left[[v \cos \alpha x]_0^{\infty} + \alpha \int_0^{\infty} v \sin \alpha x \, dx \right] \quad \left[\because \frac{\partial v}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \right] \\
 \Rightarrow \frac{d\bar{v}_s}{dt} &= -k\alpha \sqrt{\frac{2}{\pi}} \left[-v_0 + \alpha \int_0^{\infty} v \sin \alpha x \, dx \right] \quad \left[\because v \rightarrow 0 \text{ as } x \rightarrow \infty, v(0, t) = v_0 \right] \\
 \Rightarrow \frac{d\bar{v}_s}{dt} &= \sqrt{\frac{2}{\pi}} k\alpha v_0 - k\alpha^2 \bar{v}_s \quad \text{where } \bar{v}_s(\alpha, t) = \mathcal{F}_s[v(x, t)]
 \end{aligned}$$

(Refer Slide Time: 23:15)

Apply F.S.T. w.r.t x

$$\begin{aligned}
 \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial v}{\partial t} \sin \alpha x \, dx &= k \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 v}{\partial x^2} \sin \alpha x \, dx \\
 \frac{d}{dt} (\bar{v}_s) &= k \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial v}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial v}{\partial x} \cos \alpha x \, dx \right] \\
 &= -k\alpha \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} \frac{\partial v}{\partial x} \cos \alpha x \, dx \right] \\
 &= -k\alpha \sqrt{\frac{2}{\pi}} \left[[v \cos \alpha x]_0^{\infty} + \alpha \int_0^{\infty} v \sin \alpha x \, dx \right] \\
 &= k\alpha v_0 \sqrt{\frac{2}{\pi}} - k\alpha^2 \bar{v}_s \quad \begin{matrix} v \rightarrow 0 \\ x \rightarrow \infty \\ v = v_0 \text{ at } x = 0 \end{matrix}
 \end{aligned}$$

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{v}_s e^{k\alpha^2 t} = C + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k\alpha^2 t} \quad (3)$$

where, C is the constant of integration.

We have the initial condition as $v(x, 0) = 0$. Therefore,

$$\bar{v}_s(\alpha, 0) = 0$$

Thus, (3) implies

$$C = -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha}$$

Therefore, from (3), we have,

$$\begin{aligned} \bar{v}_s e^{k\alpha^2 t} &= -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k\alpha^2 t} \\ \Rightarrow \bar{v}_s(\alpha, t) &= \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} (1 - e^{-k\alpha^2 t}) \end{aligned}$$

(Refer Slide Time: 26:17)

The image shows a handwritten derivation on a whiteboard. It starts with the differential equation:

$$\frac{d\bar{v}_s}{dt} + k\alpha^2 \bar{v}_s = k\alpha v_0 \sqrt{\frac{2}{\pi}}$$

Then it shows the general solution:

$$\bar{v}_s e^{k\alpha^2 t} = c + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k\alpha^2 t} \quad (1)$$

It then states the initial condition: "At $t=0$, $\bar{v}_s = 0$ $v=0$ at $t=0$ ".

From equation (1), it derives:

$$0 = c + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha}$$

Finally, it solves for c :

$$c = -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha}$$

(Refer Slide Time: 27:47)

Handwritten mathematical derivation on a whiteboard:

$$\bar{v}_s = \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} (1 - e^{-k\alpha^2 t})$$

$$v(x, t) = \frac{2}{\pi} v_0 \int_0^{\infty} \frac{1 - e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha$$

$$= \frac{2v_0}{\pi} \left[\int_0^{\infty} \frac{\sin \alpha x}{\alpha} \, d\alpha - \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \right]$$

$$= \frac{2v_0}{\pi} \left[1 - \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \right]$$

Taking the inverse Fourier sine transform, we have,

$$v(x, t) = \mathcal{F}_s^{-1}[\bar{v}_s(\alpha, t)]$$

$$\Rightarrow v(x, t) = \sqrt{\frac{2}{\pi}} v_0 \mathcal{F}_s^{-1} \left[\frac{1}{\alpha} (1 - e^{-k\alpha^2 t}) \right]$$

$$\Rightarrow v(x, t) = \frac{2v_0}{\pi} \int_0^{\infty} \frac{1}{\alpha} (1 - e^{-k\alpha^2 t}) \sin \alpha x \, d\alpha$$

(Refer Slide Time: 29:51)

Handwritten mathematical derivation on a whiteboard:

$$f(x) = \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha$$

$$f'(x) = \int_0^{\infty} e^{-k\alpha^2 t} \cos \alpha x \, d\alpha$$

$$= \frac{\sqrt{\pi}}{2\sqrt{t}} e^{-\frac{x^2}{4tk}}$$

$$f(x) = \frac{1}{2} \sqrt{\frac{\pi}{t}} \int_0^x e^{-\frac{z^2}{4tk}} \, dz$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{t}} 2\sqrt{tk} \int_0^x e^{-\frac{z^2}{4tk}} \, d\left(\frac{z}{2\sqrt{tk}}\right)$$

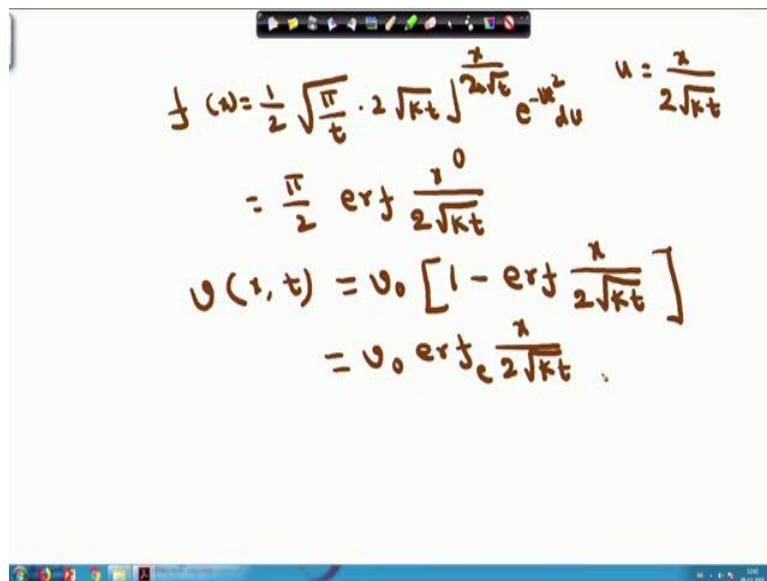
Therefore, we have,

$$\begin{aligned}
 v(x, t) &= \frac{2v_0}{\pi} \int_0^{\infty} \frac{1}{\alpha} (1 - e^{-k\alpha^2 t}) \sin \alpha x \, d\alpha \\
 \Rightarrow v(x, t) &= \frac{2v_0}{\pi} \int_0^{\infty} \frac{\sin \alpha x}{\alpha} \, d\alpha - \frac{2v_0}{\pi} \int_0^{\infty} \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} \, d\alpha \\
 \Rightarrow v(x, t) &= \frac{2v_0}{\pi} \frac{\pi}{2} - \frac{2v_0}{\pi} \int_0^{\infty} \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} \, d\alpha \\
 \Rightarrow v(x, t) &= v_0 - \frac{2v_0}{\pi} f(x)
 \end{aligned}$$

where,

$$\begin{aligned}
 f(x) &= \int_0^{\infty} \frac{\sin \alpha x}{\alpha} e^{-k\alpha^2 t} \, d\alpha \\
 \Rightarrow f'(x) &= \int_0^{\infty} e^{-k\alpha^2 t} \cos \alpha x \, d\alpha \\
 \Rightarrow f'(x) &= \frac{\sqrt{\pi}}{2\sqrt{kt}} e^{-\frac{x^2}{4tk}} \\
 \Rightarrow f(x) &= \frac{\sqrt{\pi}}{2\sqrt{kt}} \int_0^x e^{-\frac{x^2}{4tk}} \, dx \\
 \Rightarrow f(x) &= \sqrt{\pi} \int_0^x e^{-\frac{x^2}{4tk}} \, d\left(\frac{x}{2\sqrt{kt}}\right)
 \end{aligned}$$

(Refer Slide Time: 31:37)



Handwritten derivation on a whiteboard:

$$\begin{aligned}
 \int u &= \frac{1}{2} \sqrt{\frac{\pi}{t}} \cdot 2\sqrt{kt} \int_{\frac{x}{2\sqrt{kt}}}^{\frac{x}{2\sqrt{kt}}} e^{-u^2} \, du \quad u = \frac{x}{2\sqrt{kt}} \\
 &= \frac{\pi}{2} \operatorname{erf} \frac{x}{2\sqrt{kt}} \\
 v(x, t) &= v_0 \left[1 - \operatorname{erf} \frac{x}{2\sqrt{kt}} \right] \\
 &= v_0 \operatorname{erfc} \frac{x}{2\sqrt{kt}}
 \end{aligned}$$

Put

$$u = \frac{x}{2\sqrt{kt}} \quad \text{so that} \quad du = \frac{dx}{2\sqrt{kt}}$$

Therefore,

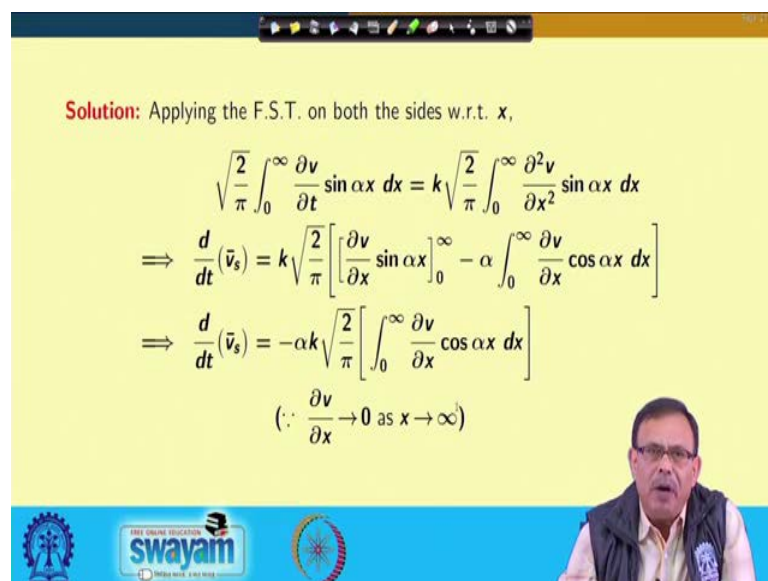
$$\begin{aligned} f(x) &= \sqrt{\pi} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2} du \\ \Rightarrow f(x) &= \sqrt{\pi} \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) \\ \Rightarrow f(x) &= \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} v(x, t) &= v_0 - \frac{2v_0 \pi}{\pi} \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) \\ \Rightarrow v(x, t) &= v_0 - v_0 \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) \\ \Rightarrow v(x, t) &= v_0 \left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right] = v_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right) \end{aligned}$$

which is the required solution.

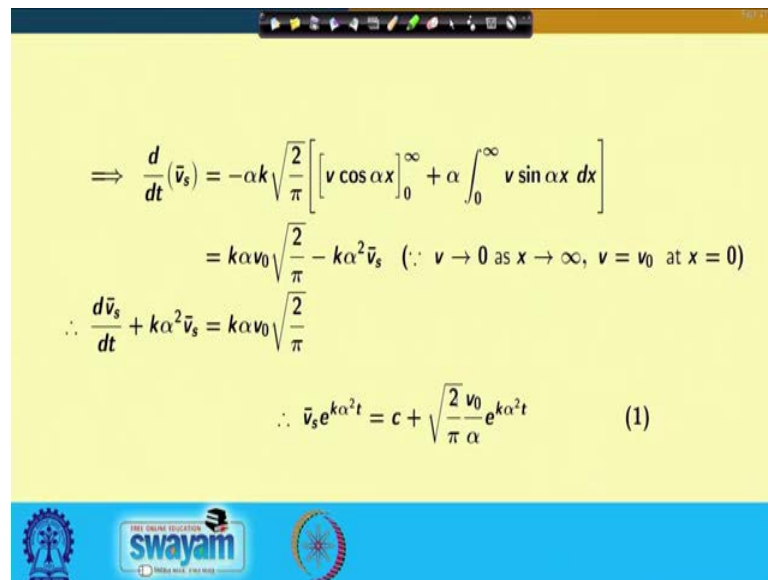
(Refer Slide Time: 33:03)



Solution: Applying the F.S.T. on both the sides w.r.t. x ,

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial v}{\partial t} \sin \alpha x \, dx &= k \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 v}{\partial x^2} \sin \alpha x \, dx \\ \Rightarrow \frac{d}{dt} (\bar{v}_s) &= k \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial v}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial v}{\partial x} \cos \alpha x \, dx \right] \\ \Rightarrow \frac{d}{dt} (\bar{v}_s) &= -\alpha k \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} \frac{\partial v}{\partial x} \cos \alpha x \, dx \right] \\ & \quad (\because \frac{\partial v}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty) \end{aligned}$$

(Refer Slide Time: 33:11)

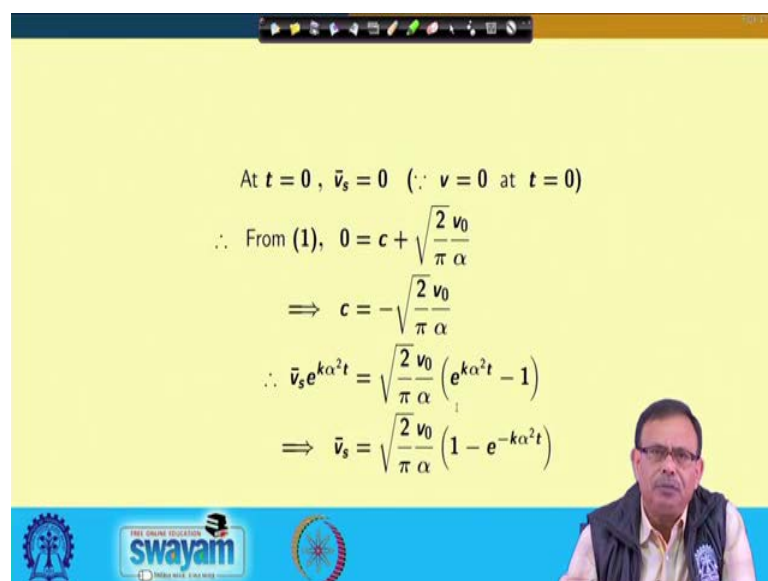


The slide shows the following mathematical derivation:

$$\begin{aligned}\Rightarrow \frac{d}{dt}(\bar{v}_s) &= -\alpha k \sqrt{\frac{2}{\pi}} \left[v \cos \alpha x \right]_0^\infty + \alpha \int_0^\infty v \sin \alpha x \, dx \\ &= k \alpha v_0 \sqrt{\frac{2}{\pi}} - k \alpha^2 \bar{v}_s \quad (\because v \rightarrow 0 \text{ as } x \rightarrow \infty, v = v_0 \text{ at } x = 0) \\ \therefore \frac{d\bar{v}_s}{dt} + k \alpha^2 \bar{v}_s &= k \alpha v_0 \sqrt{\frac{2}{\pi}} \\ \therefore \bar{v}_s e^{k \alpha^2 t} &= c + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} e^{k \alpha^2 t} \quad (1)\end{aligned}$$

The slide also features the Swamyam logo and a circular diagram at the bottom.

(Refer Slide Time: 33:21)

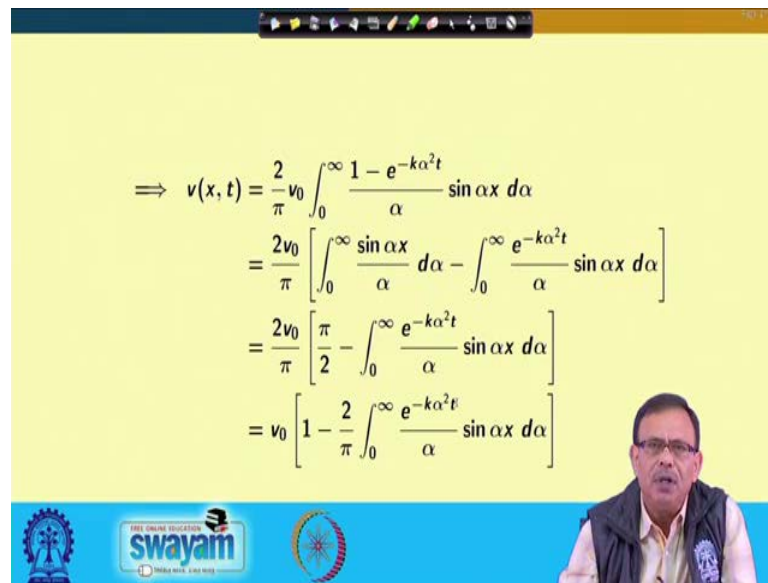


The slide shows the following mathematical derivation:

$$\begin{aligned}\text{At } t = 0, \bar{v}_s &= 0 \quad (\because v = 0 \text{ at } t = 0) \\ \therefore \text{From (1), } 0 &= c + \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} \\ \Rightarrow c &= -\sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} \\ \therefore \bar{v}_s e^{k \alpha^2 t} &= \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} (e^{k \alpha^2 t} - 1) \\ \Rightarrow \bar{v}_s &= \sqrt{\frac{2}{\pi}} \frac{v_0}{\alpha} (1 - e^{-k \alpha^2 t})\end{aligned}$$

The slide also features the Swamyam logo and a circular diagram at the bottom, along with a small inset image of a person in the bottom right corner.

(Refer Slide Time: 33:41)

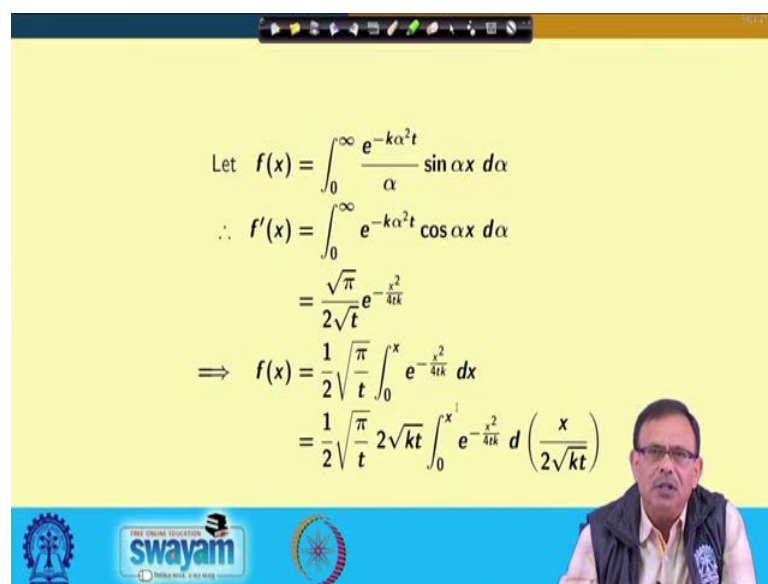


The slide displays a mathematical derivation for $v(x, t)$. The equations are as follows:

$$\begin{aligned}\Rightarrow v(x, t) &= \frac{2}{\pi} v_0 \int_0^{\infty} \frac{1 - e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \\ &= \frac{2v_0}{\pi} \left[\int_0^{\infty} \frac{\sin \alpha x}{\alpha} \, d\alpha - \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \right] \\ &= \frac{2v_0}{\pi} \left[\frac{\pi}{2} - \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \right] \\ &= v_0 \left[1 - \frac{2}{\pi} \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \right]\end{aligned}$$

The slide also features a video feed of a presenter in the bottom right corner and logos for Swamyam and other institutions at the bottom.

(Refer Slide Time: 33:57)

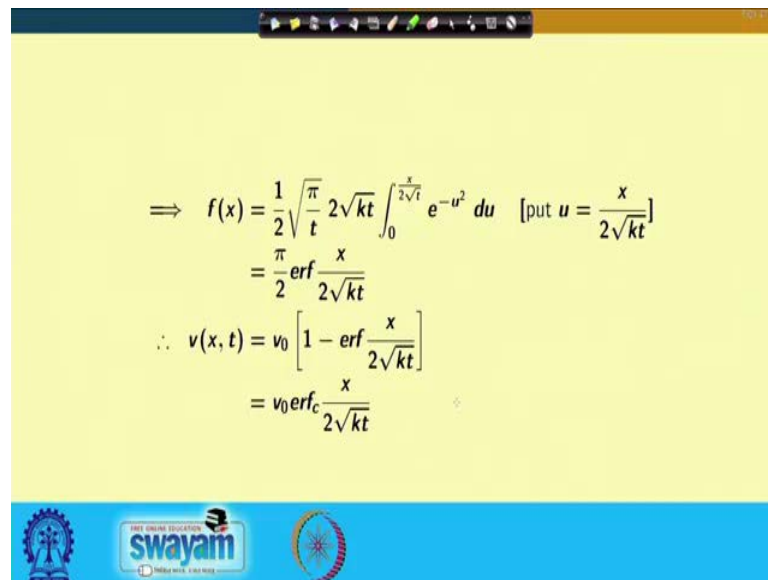


The slide displays a mathematical derivation for $f(x)$. The equations are as follows:

$$\begin{aligned}\text{Let } f(x) &= \int_0^{\infty} \frac{e^{-k\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha \\ \therefore f'(x) &= \int_0^{\infty} e^{-k\alpha^2 t} \cos \alpha x \, d\alpha \\ &= \frac{\sqrt{\pi}}{2\sqrt{t}} e^{-\frac{x^2}{4kt}} \\ \Rightarrow f(x) &= \frac{1}{2} \sqrt{\frac{\pi}{t}} \int_0^x e^{-\frac{x'^2}{4kt}} \, dx' \\ &= \frac{1}{2} \sqrt{\frac{\pi}{t}} 2\sqrt{kt} \int_0^{x'} e^{-\frac{x'^2}{4kt}} \, d\left(\frac{x'}{2\sqrt{kt}}\right)\end{aligned}$$

The slide also features a video feed of a presenter in the bottom right corner and logos for Swamyam and other institutions at the bottom.

(Refer Slide Time: 34:35)


$$\begin{aligned}\Rightarrow f(x) &= \frac{1}{2} \sqrt{\frac{\pi}{t}} \frac{1}{\sqrt{kt}} \int_0^{\frac{x}{\sqrt{kt}}} e^{-u^2} du \quad \left[\text{put } u = \frac{x}{\sqrt{kt}} \right] \\ &= \frac{\pi}{2} \operatorname{erf} \frac{x}{\sqrt{kt}} \\ \therefore v(x, t) &= v_0 \left[1 - \operatorname{erf} \frac{x}{\sqrt{kt}} \right] \\ &= v_0 \operatorname{erfc} \frac{x}{\sqrt{kt}}\end{aligned}$$

Thank you.