

Transform Calculus and its Applications in Differential Equations
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Lecture – 47
Solution of Partial Differential Equations using Fourier Cosine Transform and
Fourier Sine Transform

In the last lecture, we have initially talked about the application of Fourier transform for solving partial differential equations. We have provided the criteria under which we should use Fourier transform or Fourier cosine transform or Fourier sine transform accordingly and we have solved one problem using Fourier cosine transform also.

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Example
Solve the following PDE using F.C.T.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty$$

with $u(x, 0) = 0$ when $x > 0$, $\frac{\partial u}{\partial x} = -\mu$ (constant) when $x = 0$, and $u, \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty, t > 0$

Let us just quickly go through that problem again. So, we want to find out the solution of the following PDE using Fourier cosine transform:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty, t > 0$$

with, $u(x, 0) = 0$ when $x > 0$ and $\frac{\partial u}{\partial x} = -\mu$ (constant) when $x = 0$

$$u, \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

We apply Fourier cosine transform with respect to x on the given equation. Therefore, we obtain,

$$\begin{aligned}
\mathcal{F}_c \left[\frac{\partial u}{\partial t} \right] &= k \mathcal{F}_c \left[\frac{\partial^2 u}{\partial x^2} \right] \\
\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx &= k \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx \\
\Rightarrow \frac{d}{dt} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u \cos \alpha x \, dx \right] &= k \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \cos \alpha x \right]_0^\infty + \alpha \int_0^\infty \frac{\partial u}{\partial x} \sin \alpha x \, dx \right] \\
\Rightarrow \frac{d\bar{u}_c}{dt} &= k \sqrt{\frac{2}{\pi}} \left[\mu + \alpha \left\{ [u \sin \alpha x]_0^\infty - \alpha \int_0^\infty u \cos \alpha x \, dx \right\} \right] \\
&\quad \left[\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty, \quad u_x(0, t) = -\mu \right] \\
\Rightarrow \frac{d\bar{u}_c}{dt} &= k\mu \sqrt{\frac{2}{\pi}} - k\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u \cos \alpha x \, dx \quad [\because u \rightarrow 0 \text{ as } x \rightarrow \infty] \\
\Rightarrow \frac{d\bar{u}_c}{dt} &= k\mu \sqrt{\frac{2}{\pi}} - k\alpha^2 \bar{u}_c \quad \text{where } \bar{u}_c(\alpha, t) = \mathcal{F}_c[u(x, t)]
\end{aligned}$$

Thus, the given PDE is reduced to a first order ODE. The integrating factor for the ODE is $e^{k\alpha^2 t}$. Therefore, multiplying by the integrating factor and after integration, the obtained ODE can be easily solved to get the solution as

$$\begin{aligned}
\bar{u}_c e^{k\alpha^2 t} &= A + \frac{\mu}{\alpha^2} \sqrt{\frac{2}{\pi}} e^{k\alpha^2 t} \\
\Rightarrow \bar{u}_c(\alpha, t) &= A e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}
\end{aligned} \tag{1}$$

where, A is the constant of integration.

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Solution: Taking the F.C.T. on both the sides w.r.t. x ,

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \cos \alpha x \, dx = k \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$

$$\Rightarrow \frac{d}{dt}(\bar{u}_c) = k \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \cos \alpha x \right]_0^{\infty} + \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \sin \alpha x \, dx \right]$$

$$= k \sqrt{\frac{2}{\pi}} \left[\mu + \alpha \int_0^{\infty} \sin \alpha x \frac{\partial u}{\partial x} \, dx \right]$$

($\because \frac{\partial u}{\partial x} = -\mu$ when $x = 0$, $\frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$)



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$$\Rightarrow \frac{d}{dt}(\bar{u}_c) = k \sqrt{\frac{2}{\pi}} \left[\mu + \alpha \left[u \sin \alpha x \right]_0^{\infty} - \alpha^2 \int_0^{\infty} u \cos \alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} k \mu - k \alpha^2 \bar{u}_c \quad (\because u \rightarrow 0 \text{ as } x \rightarrow \infty)$$

$$\therefore \frac{d\bar{u}_c}{dt} + k \alpha^2 \bar{u}_c = \sqrt{\frac{2}{\pi}} k \mu$$

$$\therefore \text{I.F.} = e^{\int k \alpha^2 dt} = e^{k \alpha^2 t}$$

$$\therefore e^{k \alpha^2 t} \bar{u}_c = A + \sqrt{\frac{2}{\pi}} k \mu \int e^{k \alpha^2 t} dt$$

$$\Rightarrow \bar{u}_c = A e^{-k \alpha^2 t} + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$


Now, we are provided with the following initial condition that

$$u(x, 0) = 0 \text{ when } x > 0$$

which implies that

$$\bar{u}_c(\alpha, 0) = 0$$

Therefore, (1) implies

$$0 = A + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

$$\Rightarrow A = -\sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

Hence, the solution is obtained from (1) as

$$\bar{u}_c(\alpha, t) = -\sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

$$\Rightarrow \bar{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} (1 - e^{-k\alpha^2 t})$$

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At $t=0$, $\bar{u}_c = 0$ ($\because u = 0$ at $t = 0$)

$$\therefore A = -\sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} \Rightarrow \bar{u}_c = \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} (1 - e^{-k\alpha^2 t})$$

$$\therefore u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{u}_c \cos \alpha x \, d\alpha$$

$$= \frac{2\mu}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2} (1 - e^{-k\alpha^2 t}) \, d\alpha$$

Now, taking the inverse Fourier cosine transform, we have,

$$u(x, t) = \mathcal{F}_c^{-1}[\bar{u}_c(\alpha, t)]$$

$$= \mu \sqrt{\frac{2}{\pi}} \mathcal{F}_c^{-1} \left[\frac{1}{\alpha^2} (1 - e^{-k\alpha^2 t}) \right]$$

$$= \mu \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{\alpha^2} (1 - e^{-k\alpha^2 t}) \cos \alpha x \, d\alpha$$

$$\Rightarrow u(x, t) = \frac{2\mu}{\pi} \int_0^{\infty} \frac{1 - e^{-k\alpha^2 t}}{\alpha^2} \cos \alpha x \, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x, t)$.

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Example
Solve the following PDE using F.S.T.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0$$

with $u(0, t) = 0$ when $t > 0$, $u(x, 0) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x \geq 1 \end{cases}$

and $u, \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.

Now, we want to solve the next problem using Fourier sine transform:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0$$

with, $u(0, t) = 0$ when $t > 0$ and $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$

$$u, \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_s \left[\frac{\partial u}{\partial t} \right] = \mathcal{F}_s \left[\frac{\partial^2 u}{\partial x^2} \right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

$$\Rightarrow \frac{d}{dt} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} u \sin \alpha x \, dx \right] = \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x \, dx \right]$$

$$\begin{aligned} \Rightarrow \frac{d\bar{u}_s}{dt} &= -\alpha \sqrt{\frac{2}{\pi}} \left[[u \cos \alpha x]_0^\infty + \alpha \int_0^\infty u \sin \alpha x dx \right] \left[\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \right] \\ \Rightarrow \frac{d\bar{u}_s}{dt} &= -\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u \sin \alpha x dx \quad [\because u \rightarrow 0 \text{ as } x \rightarrow \infty, u(0, t) = 0] \\ \Rightarrow \frac{d\bar{u}_s}{dt} &= -\alpha^2 \bar{u}_s \quad \text{where } \bar{u}_s(\alpha, t) = \mathcal{F}_s[u(x, t)] \end{aligned}$$

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{u}_s(\alpha, t) = Ae^{-\alpha^2 t} \quad (2)$$

where, A is the constant of integration.

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Apply F.S.T. w.r.t. x $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin \alpha x dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x dx$$

$$\Rightarrow \frac{d}{dt} (\bar{u}_s) = \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \frac{\partial u}{\partial x} \sin \alpha x dx - \alpha \int_0^\infty \frac{\partial u}{\partial x} \cos \alpha x dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\alpha \int_0^\infty \cos \alpha x \frac{\partial u}{\partial x} dx \right] \quad \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$= \sqrt{\frac{2}{\pi}} \left[-\alpha [u \cos \alpha x]_0^\infty - \alpha^2 \int_0^\infty u \sin \alpha x dx \right] \quad u \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$= -\alpha^2 \bar{u}_s$$

We have the initial condition as $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$. Therefore,

$$\begin{aligned} \bar{u}_s(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \sin \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^1 \sin \alpha x dx \end{aligned}$$

$$\begin{aligned}\Rightarrow \bar{u}_s(\alpha, 0) &= -\sqrt{\frac{2}{\pi}} \left[\frac{\cos \alpha x}{\alpha} \right]_0^1 \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha}{\alpha} \right]\end{aligned}$$

Thus, (2) implies

$$A = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha}{\alpha} \right]$$

Therefore, from (2), we have,

$$\bar{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \alpha}{\alpha} \right] e^{-\alpha^2 t}$$

(Refer Slide Time: 12:05)

The image shows a handwritten derivation on a whiteboard. It starts with the partial differential equation $\frac{d\bar{u}_s}{dt} + \alpha^2 \bar{u}_s = 0$ and the boundary conditions $u(x,0) = 1, 0 < x < 1$ and $= 0, x > 1$. The solution for the transformed function is given as $\bar{u}_s = A e^{-\alpha^2 t}$. At $t=0$, the expression for $\bar{u}_s(\alpha, 0)$ is derived as $\sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x,0) \sin \alpha x dx$. This is split into two integrals: $\sqrt{\frac{2}{\pi}} \int_0^1 \sin \alpha x dx + \sqrt{\frac{2}{\pi}} \int_1^{\infty} 0 \cdot \sin \alpha x dx$. The first integral is evaluated as $\sqrt{\frac{2}{\pi}} \left[-\frac{\cos \alpha x}{\alpha} \right]_0^1 = \sqrt{\frac{2}{\pi}} \frac{1 - \cos \alpha}{\alpha}$. Finally, the coefficient A is determined as $A = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - \cos \alpha}{\alpha}$.

Taking the inverse Fourier sine transform, we have,

$$\begin{aligned}u(x, t) &= \mathcal{F}_s^{-1}[\bar{u}_s(\alpha, t)] \\ \Rightarrow u(x, t) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \left[\left(\frac{1 - \cos \alpha}{\alpha} \right) e^{-\alpha^2 t} \right]\end{aligned}$$

$$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1 - \cos \alpha x}{\alpha} \right) e^{-\alpha^2 t} \sin \alpha x \, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x, t)$.

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Handwritten mathematical derivation on a whiteboard:

$$\bar{u}_s = \sqrt{\frac{2}{\pi}} \frac{1 - \cos \alpha x}{\alpha} \cdot e^{-\alpha^2 t}$$

$$u(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \alpha x}{\alpha} \cdot e^{-\alpha^2 t} \sin \alpha x \, d\alpha$$

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Solution: Applying the F.S.T. on both the sides w.r.t. x ,

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

$$\Rightarrow \frac{d}{dt} (\bar{u}_s) = \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\alpha \int_0^{\infty} \cos \alpha x \frac{\partial u}{\partial x} \, dx \right] \quad (\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty)$$

$$= \sqrt{\frac{2}{\pi}} \left[-\alpha \left[u \cos \alpha x \right]_0^{\infty} - \alpha^2 \int_0^{\infty} u \sin \alpha x \, dx \right]$$

$$= -\alpha^2 \bar{u}_s \quad (\because u \rightarrow 0 \text{ as } x \rightarrow \infty)$$

Logos for IIT Bombay and Swamyam are visible at the bottom of the slide.

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The slide displays the following mathematical steps:

$$\therefore \frac{d\bar{u}_s}{dt} + \alpha^2 \bar{u}_s = 0$$
$$\therefore \bar{u}_s = Ae^{-\alpha^2 t}$$

At $t = 0$, $\bar{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \sin \alpha x \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin \alpha x \, dx + \sqrt{\frac{2}{\pi}} \int_1^\infty 0 \cdot \sin \alpha x \, dx$$
$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \alpha x}{\alpha} \right]_0^1 = \sqrt{\frac{2}{\pi}} \frac{1 - \cos \alpha}{\alpha}$$

The slide also features a video feed of a presenter in the bottom right corner and logos for Swamyam and other institutions at the bottom.

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The slide displays the following mathematical steps:

$$\therefore A = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - \cos \alpha}{\alpha}$$
$$\Rightarrow \bar{u}_s = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - \cos \alpha}{\alpha} \cdot e^{-\alpha^2 t}$$
$$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \alpha}{\alpha} e^{-\alpha^2 t} \sin \alpha x \, d\alpha$$

The slide also features a video feed of a presenter in the bottom right corner and logos for Swamyam and other institutions at the bottom.

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Example
Solve the following PDE using F.S.T.

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0$$

with $u(0, t) = 0$ when $t > 0$, $u(x, 0) = e^{-x}$ when $x > 0$, $u, \frac{\partial u}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$.

Now, let us solve another problem as follows:

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} \quad x > 0, t > 0$$

with, $u(0, t) = 0$ when $t > 0$ and $u(x, 0) = e^{-x}$ when $x > 0$

$$u, \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\begin{aligned} \mathcal{F}_s \left[\frac{\partial u}{\partial t} \right] &= 2 \mathcal{F}_s \left[\frac{\partial^2 u}{\partial x^2} \right] \\ \Rightarrow \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x \, dx &= 2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx \\ \Rightarrow \frac{d}{dt} \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} u \sin \alpha x \, dx \right] &= 2 \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x \, dx \right] \\ \Rightarrow \frac{d\bar{u}_s}{dt} &= -2\alpha \sqrt{\frac{2}{\pi}} \left[[u \cos \alpha x]_0^{\infty} + \alpha \int_0^{\infty} u \sin \alpha x \, dx \right] \quad \left[\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \right] \\ \Rightarrow \frac{d\bar{u}_s}{dt} &= -2\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} u \sin \alpha x \, dx \quad \left[\because u \rightarrow 0 \text{ as } x \rightarrow \infty, u(0, t) = 0 \right] \end{aligned}$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -2\alpha^2 \bar{u}_s \quad \text{where } \bar{u}_s(\alpha, t) = \mathcal{F}_s[u(x, t)]$$

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{u}_s(\alpha, t) = Ae^{-2\alpha^2 t} \quad (3)$$

where, A is the constant of integration.

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Apply F. s. T. w.r.t. x $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x \, dx = 2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

$$\frac{d}{dt}(\bar{u}_s) = 2 \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x \, dx \right]$$

$$= 2 \sqrt{\frac{2}{\pi}} \left[-\alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x \, dx \right] \quad \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$= -2\alpha^2 \bar{u}_s \quad \begin{matrix} u(0, t) = 0, \\ u \rightarrow 0 \text{ as } x \rightarrow \infty \end{matrix}$$

$$\frac{d\bar{u}_s}{dt} + 2\alpha^2 \bar{u}_s = 0 \Rightarrow \bar{u}_s = 2A e^{-2\alpha^2 t}$$

Now, we are provided with the following initial condition that

$$u(x, 0) = e^{-x} \text{ when } x > 0$$

Therefore,

$$\begin{aligned} \bar{u}_s(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, 0) \sin \alpha x \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin \alpha x \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1 + \alpha^2} (-\sin \alpha x - \alpha \cos \alpha x) \right]_0^{\infty} \end{aligned}$$

$$\Rightarrow \bar{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1 + \alpha^2}$$

Thus, (3) implies

$$A = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1 + \alpha^2}$$

Therefore, from (3), we have,

$$\bar{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1 + \alpha^2} e^{-2\alpha^2 t}$$

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At $t=0$, $\bar{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, 0) \sin \alpha x dx$
 $= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin \alpha x dx$
 $= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1 + \alpha^2} (-\sin \alpha x - \alpha \cos \alpha x) \right]_0^{\infty}$
 $= \frac{\alpha}{1 + \alpha^2} \sqrt{\frac{2}{\pi}}$
 $A = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + 1}$

Taking the inverse Fourier sine transform, we have,

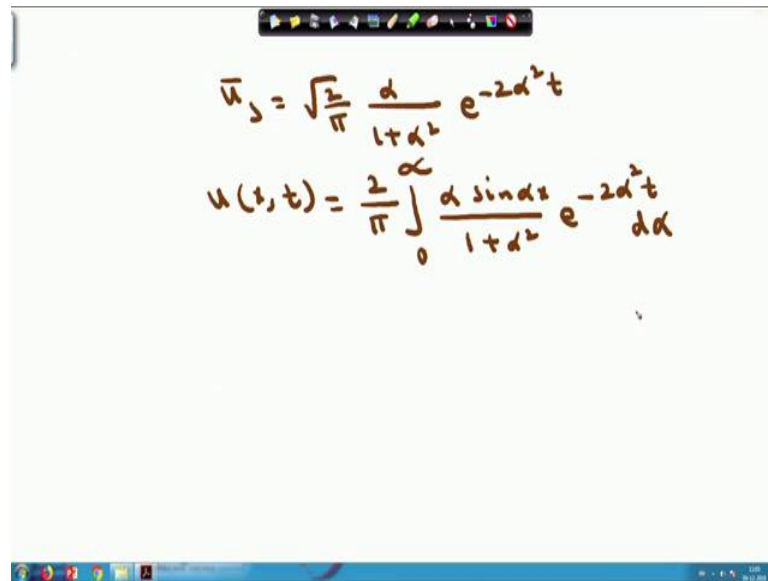
$$u(x, t) = \mathcal{F}_s^{-1}[\bar{u}_s(\alpha, t)]$$

$$\Rightarrow u(x, t) = \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \left[\frac{\alpha}{1 + \alpha^2} e^{-2\alpha^2 t} \right]$$

$$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha}{1 + \alpha^2} e^{-2\alpha^2 t} \sin \alpha x d\alpha$$

Evaluation of the above integral will give the required solution for $u(x, t)$.

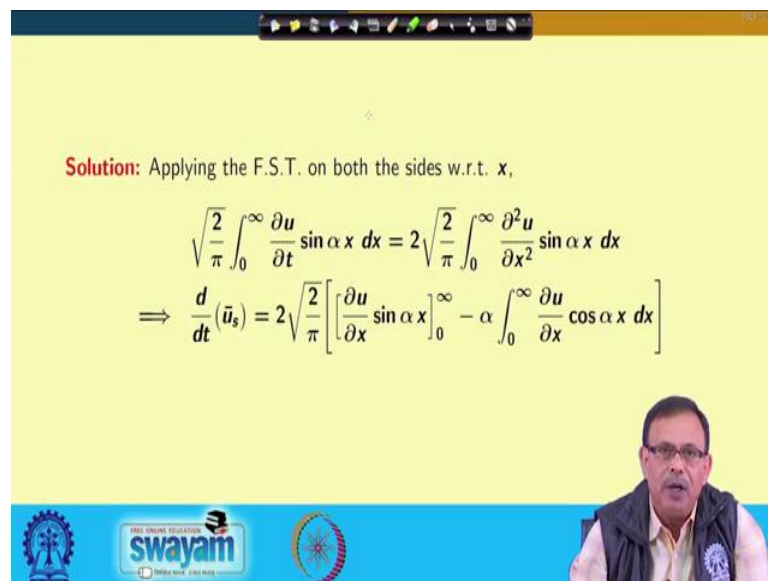
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The image shows a whiteboard with handwritten mathematical equations. The first equation is $\bar{u}_s = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2} e^{-2\alpha^2 t}$. The second equation is $u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha \sin \alpha x}{1+\alpha^2} e^{-2\alpha^2 t} d\alpha$. The whiteboard has a toolbar at the top and a Windows taskbar at the bottom.

$$\bar{u}_s = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2} e^{-2\alpha^2 t}$$
$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha \sin \alpha x}{1+\alpha^2} e^{-2\alpha^2 t} d\alpha$$

(Refer Slide Time: 25:13)



The image shows a slide with a yellow background. The text reads: "Solution: Applying the F.S.T. on both the sides w.r.t. x,". Below this, the following equations are shown: $\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x dx = 2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x dx$ and $\Rightarrow \frac{d}{dt} (\bar{u}_s) = 2 \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x dx \right]$. At the bottom right, there is a small video inset of a man speaking. At the bottom left, there are logos for "swayam" and "Free Online Education".

Solution: Applying the F.S.T. on both the sides w.r.t. x ,

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial u}{\partial t} \sin \alpha x dx = 2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x dx$$
$$\Rightarrow \frac{d}{dt} (\bar{u}_s) = 2 \sqrt{\frac{2}{\pi}} \left[\left[\frac{\partial u}{\partial x} \sin \alpha x \right]_0^{\infty} - \alpha \int_0^{\infty} \frac{\partial u}{\partial x} \cos \alpha x dx \right]$$

(Refer Slide Time: 25:39)

$$\begin{aligned}\Rightarrow \frac{d}{dt}(\bar{u}_s) &= 2\sqrt{\frac{2}{\pi}} \left[-\alpha \int_0^{\infty} \cos \alpha x \frac{\partial u}{\partial x} dx \right] \quad (\because \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty) \\ &= 2\sqrt{\frac{2}{\pi}} \left[-\alpha [u \cos \alpha x]_0^{\infty} - \alpha^2 \int_0^{\infty} u \sin \alpha x dx \right] \\ &= -2\alpha^2 \bar{u}_s \quad (\because u(0, t) = 0 \text{ and } u \rightarrow 0 \text{ as } x \rightarrow \infty)\end{aligned}$$
$$\therefore \frac{d\bar{u}_s}{dt} + 2\alpha^2 \bar{u}_s = 0$$
$$\therefore \bar{u}_s = Ae^{-2\alpha^2 t}$$

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$$\begin{aligned}\text{At } t = 0, \bar{u}_s(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, 0) \sin \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin \alpha x dx \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1 + \alpha^2} (-\sin \alpha x - \alpha \cos \alpha x) \right]_0^{\infty} \\ &= \frac{\alpha}{1 + \alpha^2} \sqrt{\frac{2}{\pi}}\end{aligned}$$

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$$\begin{aligned}\therefore A &= \sqrt{\frac{2}{\pi}} \frac{\alpha}{1 + \alpha^2} \\ \Rightarrow \bar{u}_s &= \sqrt{\frac{2}{\pi}} \frac{\alpha}{1 + \alpha^2} e^{-2\alpha^2 t} \\ \Rightarrow u(x, t) &= \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{1 + \alpha^2} e^{-2\alpha^2 t} d\alpha\end{aligned}$$

The slide features a yellow background with mathematical derivations. At the bottom, there is a blue banner with logos for 'swayam' and 'Free Online Education'. A small video inset in the bottom right corner shows a man with glasses and a dark vest over a light shirt.

Thank you.