Transform Calculus and its Applications in Differential Equations
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Lecture – 47
Solution of Partial Differential Equations using Fourier Cosine Transform and Fourier Sine Transform

In the last lecture, we have initially talked about the application of Fourier transform for solving partial differential equations. We have provided the criteria under which we should use Fourier transform or Fourier cosine transform or Fourier sine transform accordingly and we have solved one problem using Fourier cosine transform also.

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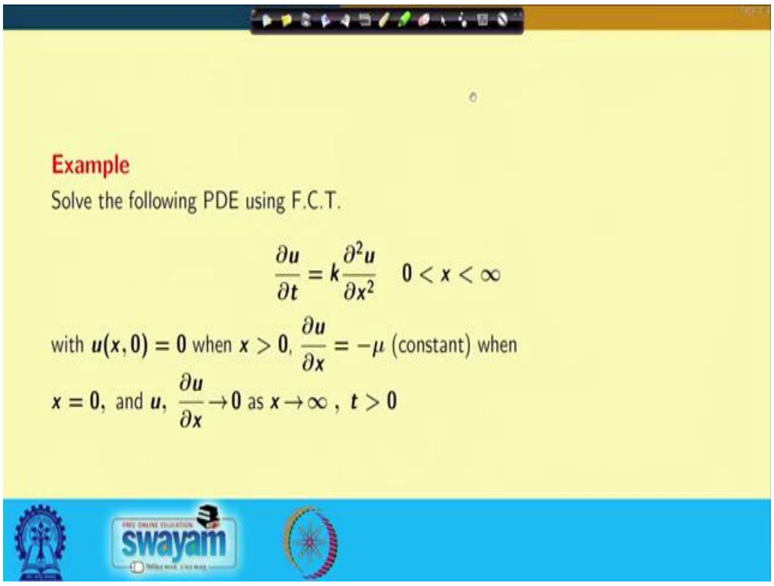

Let us just quickly go through that problem again. So, we want to find out the solution of the following PDE using Fourier cosine transform:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \infty, \;\; t > 0$$

$$\text{with, } u(x,0) = 0 \text{ when } x > 0 \text{ and } \frac{\partial u}{\partial x} = -\mu \text{ (constant) when } x = 0$$

$$u, \frac{\partial u}{\partial x} \to 0 \;\text{ as }\; x \to \infty$$

We apply Fourier cosine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_c\left[\frac{\partial u}{\partial t}\right] = k\mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x\,dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx\right] = k\sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty + \alpha\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x\,dx\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\sqrt{\frac{2}{\pi}}\left[\mu + \alpha\left\{[u\sin\alpha x]_0^\infty - \alpha\int_0^\infty u\cos\alpha x\,dx\right\}\right]$$

$$\left[\because \frac{\partial u}{\partial x}\to 0 \text{ as } x\to\infty, \qquad u_x(0,t) = -\mu\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\mu\sqrt{\frac{2}{\pi}} - k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx \qquad [\because u\to 0 \text{ as } x\to\infty]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\mu\sqrt{\frac{2}{\pi}} - k\alpha^2\bar{u}_c \qquad \text{where } \bar{u}_c(\alpha,t) = \mathcal{F}_c[u(x,t)]$$

Thus, the given PDE is reduced to a first order ODE. The integrating factor for the ODE is $e^{k\alpha^2 t}$. Therefore, multiplying by the integrating factor and after integration, the obtained ODE can be easily solved to get the solution as

$$\bar{u}_c e^{k\alpha^2 t} = A + \frac{\mu}{\alpha^2}\sqrt{\frac{2}{\pi}}\,e^{k\alpha^2 t}$$

$$\Rightarrow \bar{u}_c(\alpha,t) = A\,e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2} \qquad (1)$$

where, A is the constant of integration.

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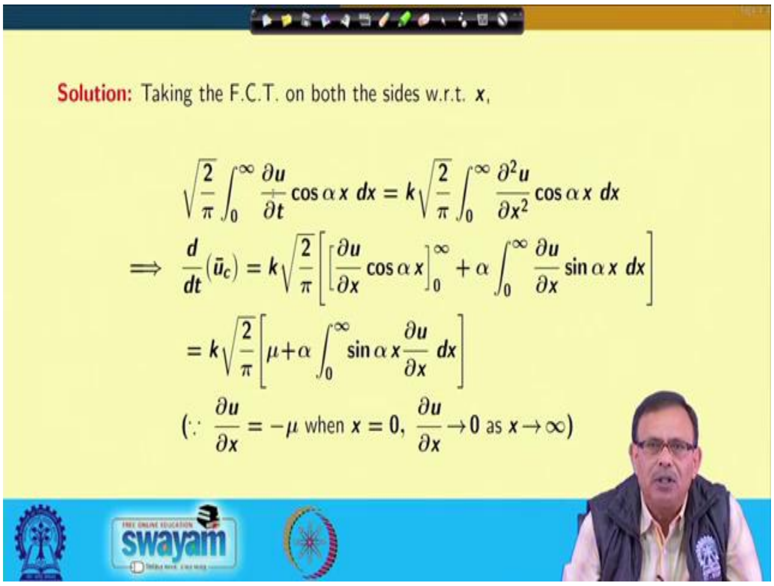

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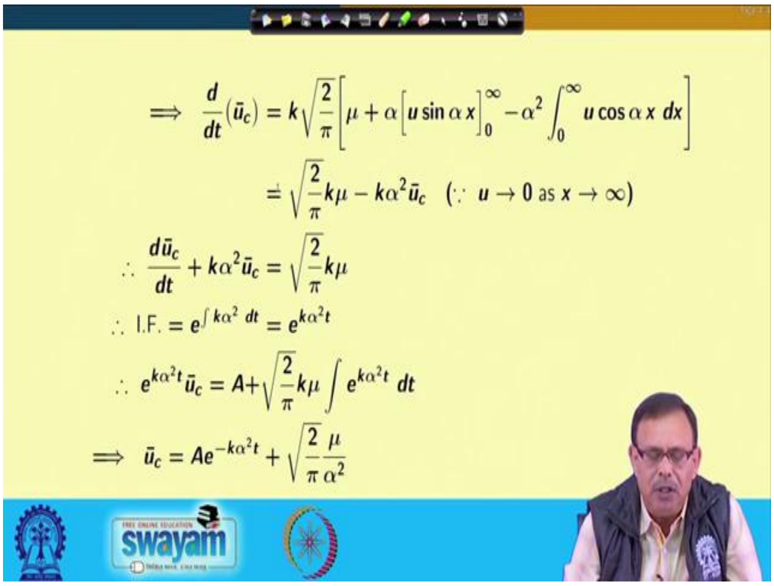

Now, we are provided with the following initial condition that

$$u(x,0) = 0 \text{ when } x > 0$$

which implies that

$$\bar{u}_c(\alpha,0) = 0$$

Therefore, (1) implies

$$0 = A + \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}$$

$$\Rightarrow A = -\sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}$$

Hence, the solution is obtained from (1) as

$$\bar{u}_c(\alpha, t) = -\sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}\, e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}$$

$$\Rightarrow \bar{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)$$

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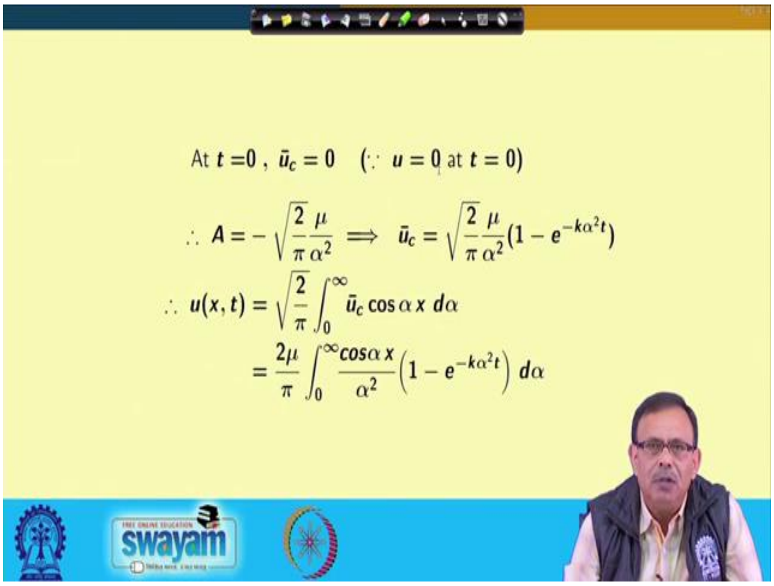

Now, taking the inverse Fourier cosine transform, we have,

$$u(x,t) = \mathcal{F}_c^{-1}[\bar{u}_c(\alpha, t)]$$

$$= \mu\sqrt{\frac{2}{\pi}}\,\mathcal{F}_c^{-1}\left[\frac{1}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)\right]$$

$$= \mu\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)\cos\alpha x \, d\alpha$$

$$\Rightarrow u(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{1- e^{-k\alpha^2 t}}{\alpha^2}\cos \alpha x\, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x,t)$.

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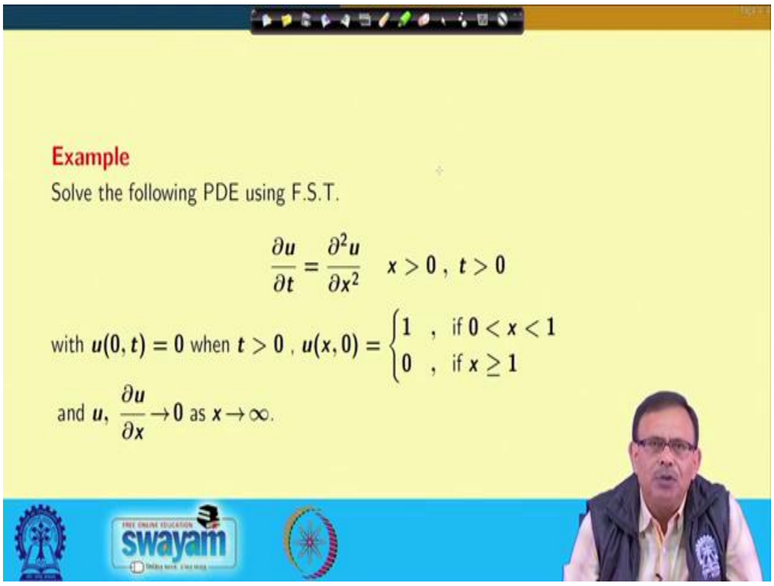

Now, we want to solve the next problem using Fourier sine transform:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \;\; t > 0$$

$$\text{with}, u(0,t) = 0 \text{ when } t > 0 \text{ and } u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

$$u, \frac{\partial u}{\partial x} \to 0 \;\text{ as }\; x \to \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin \alpha x\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin \alpha x\, dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u \sin \alpha x\, dx\right] = \sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\sin \alpha x\right]_0^\infty - \alpha\int_0^\infty \frac{\partial u}{\partial x}\cos \alpha x\, dx\right]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -\alpha\sqrt{\frac{2}{\pi}}\left[[u\cos\alpha x]_0^\infty + \alpha\int_0^\infty u\sin\alpha x\,dx\right] \quad \left[\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\sin\alpha x\,dx \quad [\because u \to 0 \text{ as } x \to \infty,\ u(0,t) = 0]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -\alpha^2\bar{u}_s \qquad \text{where } \bar{u}_s(\alpha,t) = \mathcal{F}_s[u(x,t)]$$

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{u}_s(\alpha,t) = Ae^{-\alpha^2 t} \tag{2}$$

where, A is the constant of integration.

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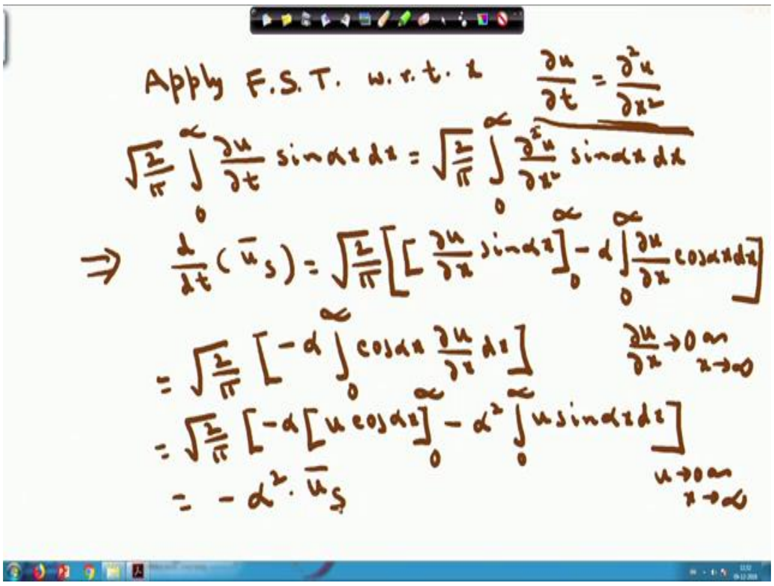

We have the initial condition as $u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$. Therefore,

$$\bar{u}_s(\alpha,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin\alpha x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \sin\alpha x\,dx$$

$$\Rightarrow \bar{u}_s(\alpha, 0) = -\sqrt{\frac{2}{\pi}}\left[\frac{\cos \alpha x}{\alpha}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos \alpha}{\alpha}\right]$$

Thus, (2) implies

$$A = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos \alpha}{\alpha}\right]$$

Therefore, from (2), we have,

$$\bar{u}_s(\alpha, t) = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos \alpha}{\alpha}\right] e^{-\alpha^2 t}$$

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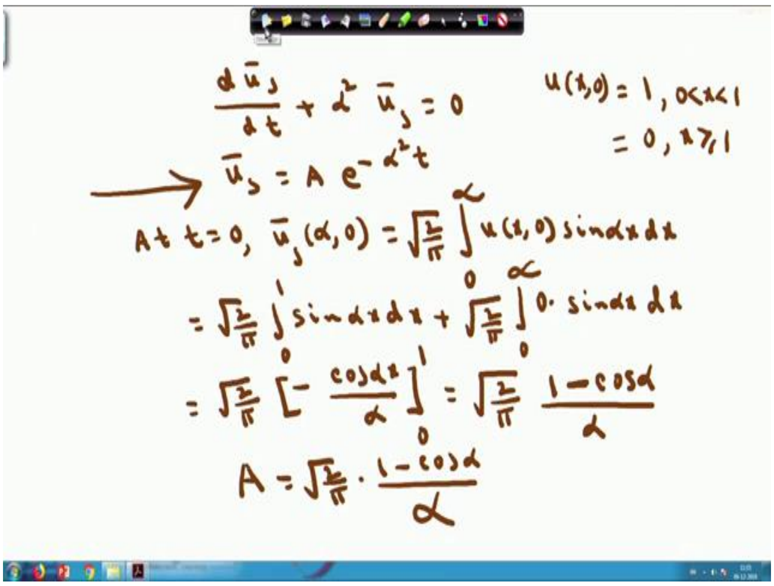

Taking the inverse Fourier sine transform, we have,

$$u(x,t) = \mathcal{F}_s^{-1}[\bar{u}_s(\alpha, t)]$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\mathcal{F}_s^{-1}\left[\left(\frac{1-\cos \alpha}{\alpha}\right) e^{-\alpha^2 t}\right]$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^{\infty} \left(\frac{1-\cos\alpha}{\alpha}\right) e^{-\alpha^2 t} \sin\alpha x \, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x,t)$.

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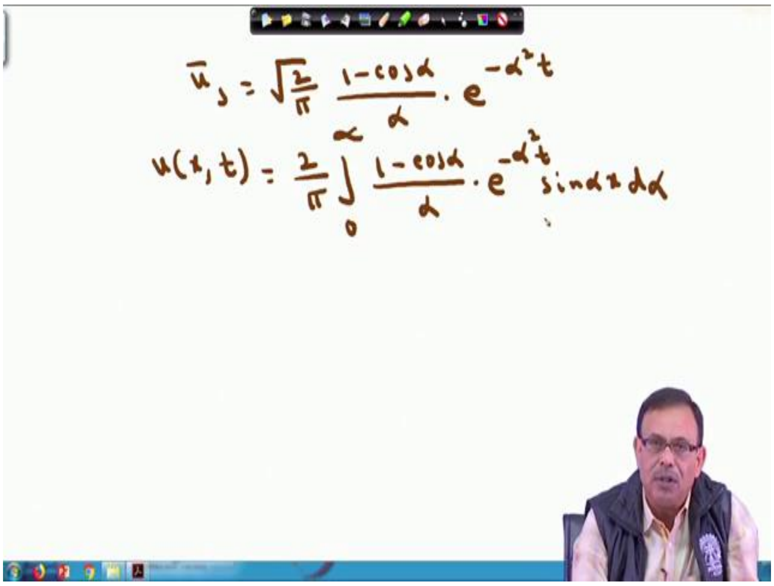

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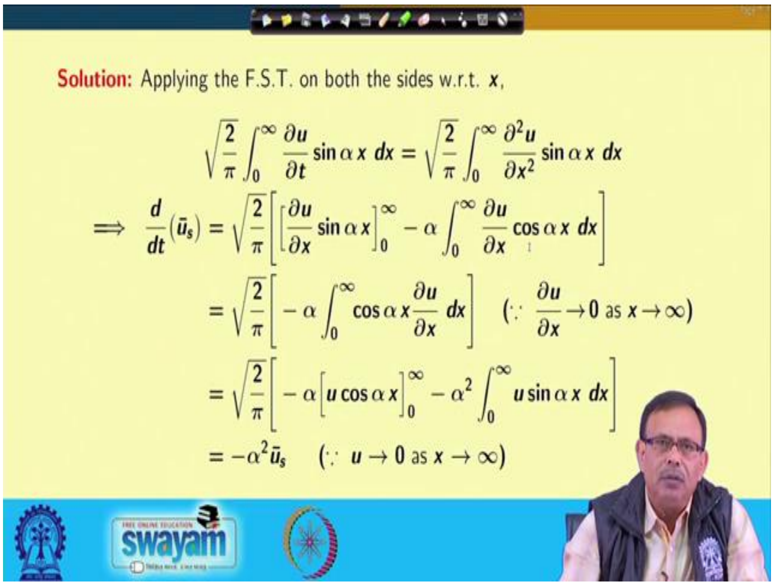

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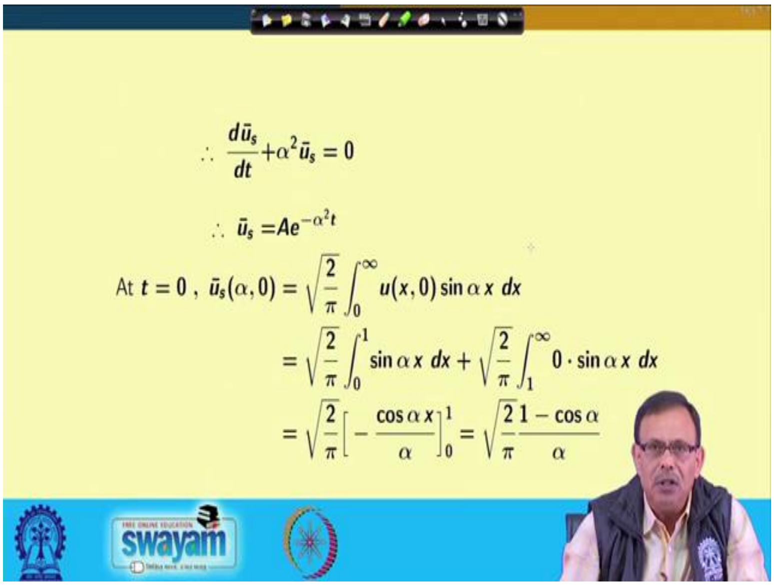

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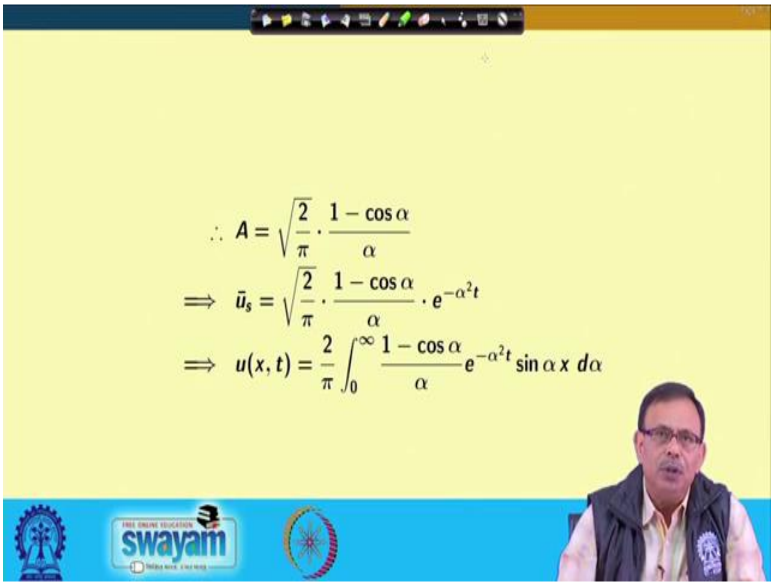

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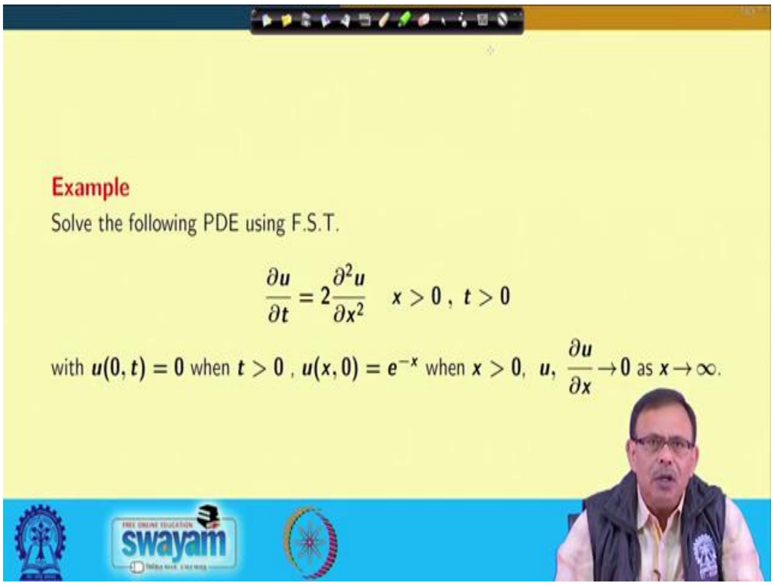

Now, let us solve another problem as follows:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad x > 0,\ \ t > 0$$

$$\text{with, } u(0,t) = 0 \text{ when } t > 0 \text{ and } u(x,0) = e^{-x} \text{ when } x > 0$$

$$u, \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

We apply Fourier sine transform with respect to x on the given equation. Therefore, we obtain,

$$\mathcal{F}_s\left[\frac{\partial u}{\partial t}\right] = 2\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin\alpha x\,dx = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin\alpha x\,dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u\sin\alpha x\,dx\right] = 2\sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\sin\alpha x\right]_0^\infty - \alpha\int_0^\infty \frac{\partial u}{\partial x}\cos\alpha x\,dx\right]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -2\alpha\sqrt{\frac{2}{\pi}}\left[[u\cos\alpha x]_0^\infty + \alpha\int_0^\infty u\sin\alpha x\,dx\right] \quad \left[\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -2\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\sin\alpha x\,dx \quad [\because u \to 0 \text{ as } x \to \infty,\ \ u(0,t) = 0]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -2\alpha^2\bar{u}_s \qquad \text{where } \bar{u}_s(\alpha, t) = \mathcal{F}_s[u(x, t)]$$

This reduces the given PDE to a first order ODE which can be easily solved and the solution is given as:

$$\bar{u}_s(\alpha, t) = Ae^{-2\alpha^2 t} \tag{3}$$

where, A is the constant of integration.

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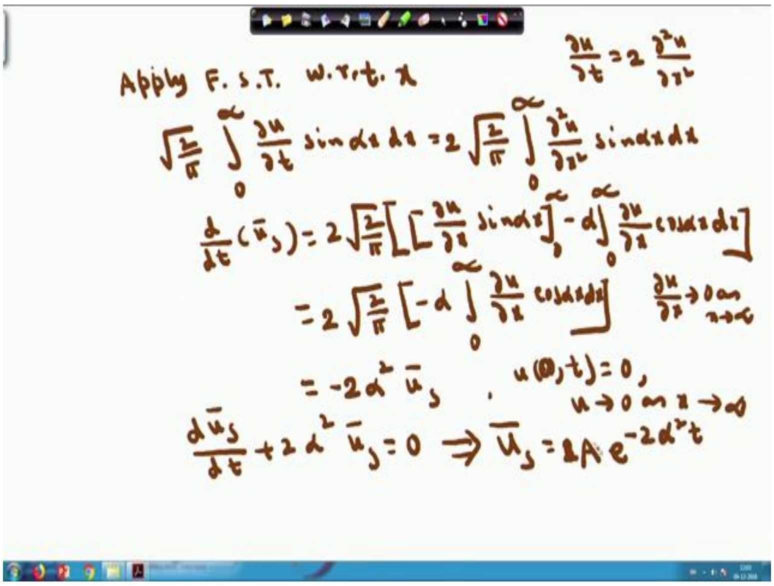

Now, we are provided with the following initial condition that

$$u(x, 0) = e^{-x} \text{ when } x > 0$$

Therefore,

$$\bar{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1 + \alpha^2} (-\sin \alpha x - \alpha \cos \alpha x) \right]_0^\infty$$

$$\Rightarrow \bar{u}_s(\alpha,0) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{1+\alpha^2}$$

Thus, (3) implies

$$A = \sqrt{\frac{2}{\pi}}\frac{\alpha}{1+\alpha^2}$$

Therefore, from (3), we have,

$$\bar{u}_s(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{1+\alpha^2}e^{-2\alpha^2 t}$$

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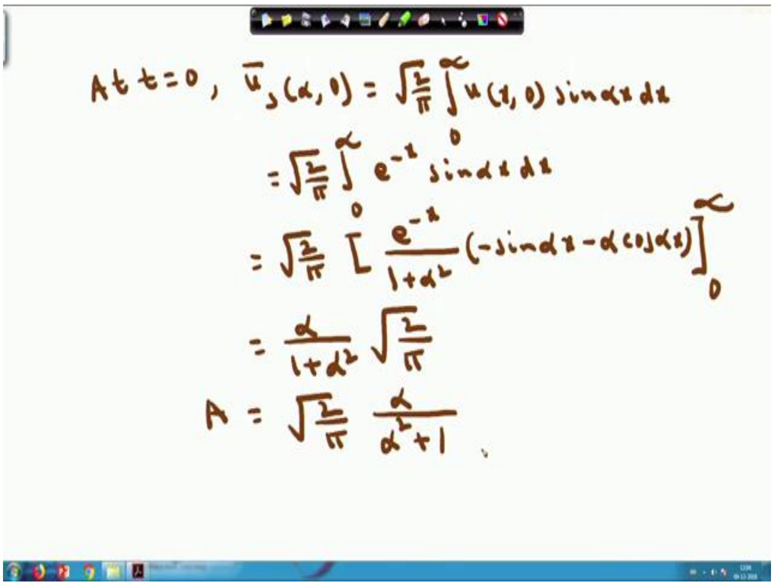

Taking the inverse Fourier sine transform, we have,

$$u(x,t) = \mathcal{F}_s^{-1}[\bar{u}_s(\alpha,t)]$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\mathcal{F}_s^{-1}\left[\frac{\alpha}{1+\alpha^2}e^{-2\alpha^2 t}\right]$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{1+\alpha^2}e^{-2\alpha^2 t}\sin\alpha x\, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x,t)$.

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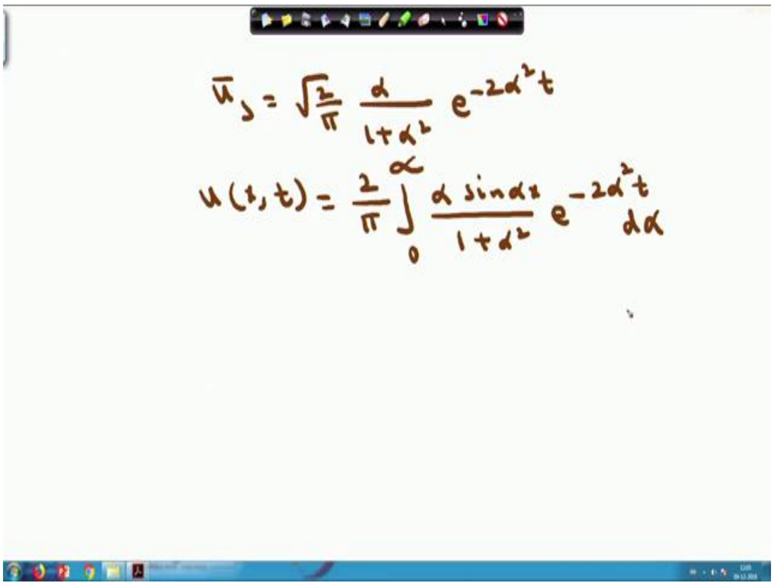

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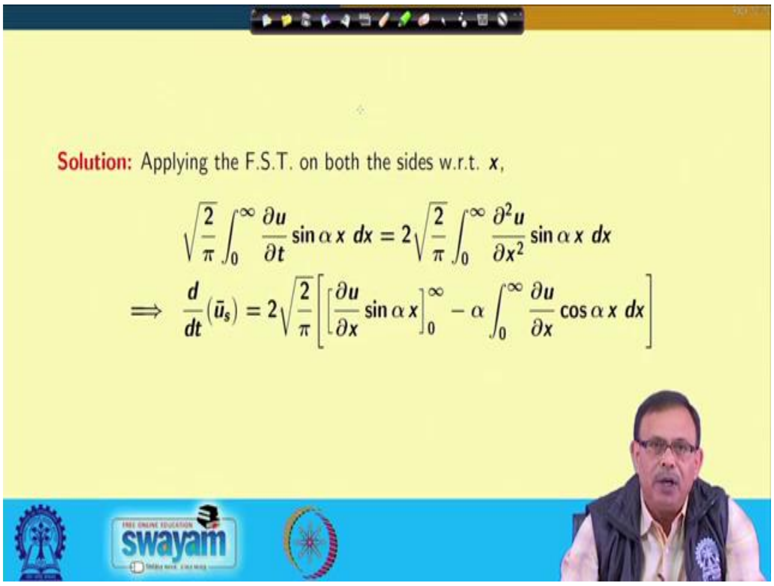

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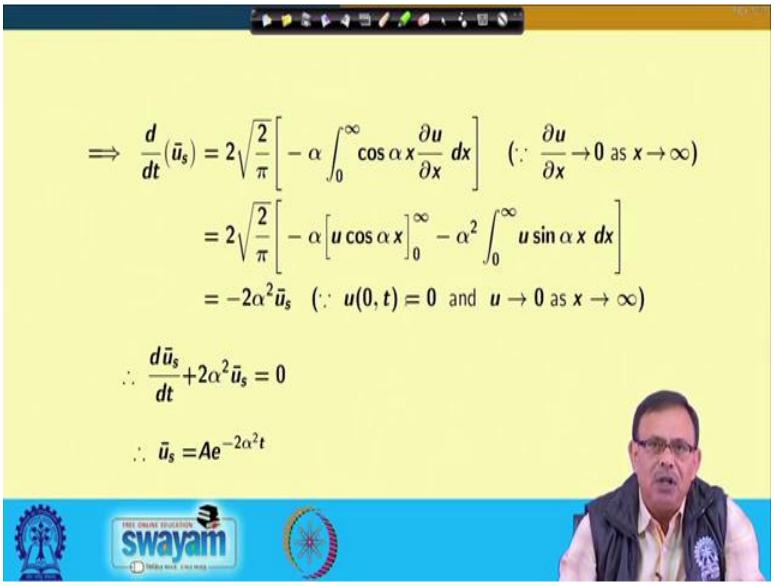

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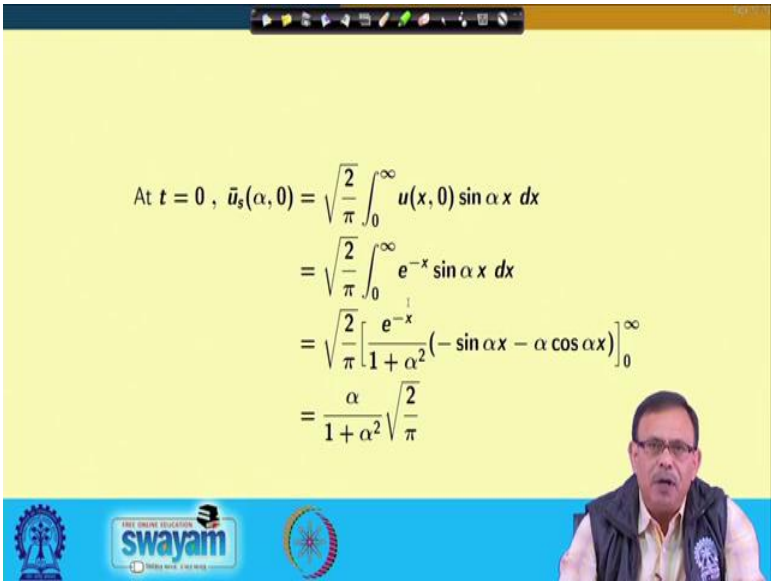

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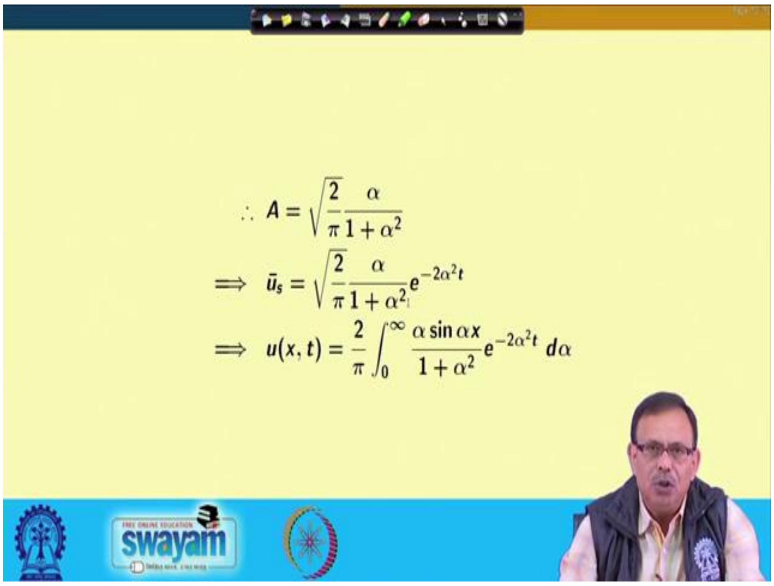

Thank you.