Transform Calculus and its Applications in Differential Equations

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Lecture – 46
Criteria for choosing Fourier Transform, Fourier Sine Transform, Fourier Cosine Transform in solving Partial Differential Equations

In this lecture, we are going to study the solution of Partial Differential Equations using Fourier Transform; that is application of Fourier transform in solving PDE. So, first thing what we have to study is, what are the conditions under which we should use Fourier transform or Fourier cosine transform or Fourier sine transform.

Like for Laplace transform, we gave the conditions i.e., we need to know certain values, then only we can find out the solution using the Laplace transform. Similarly, there are certain criteria and if those criteria or conditions are fulfilled, then only we can use either Fourier transform or Fourier cosine or Fourier sine transform. So, first we will study that part, that what are the conditions required to solve these problems.

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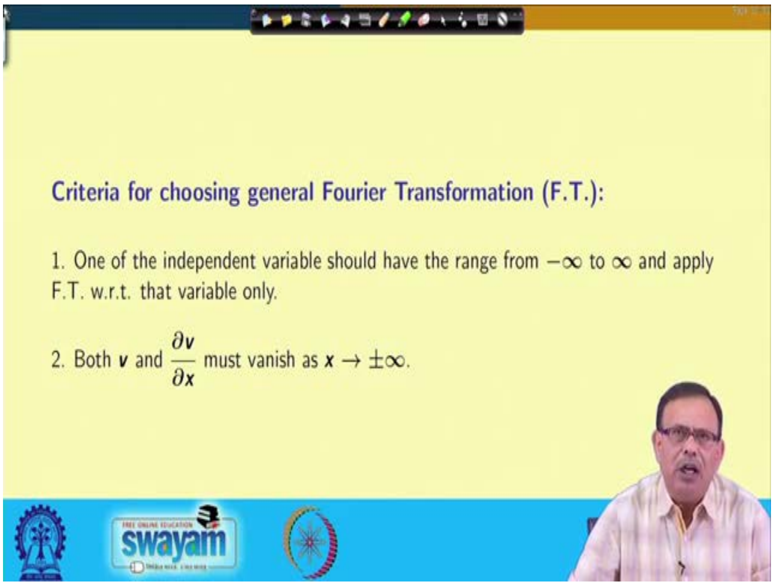

First, we are taking the criteria for choosing general Fourier transform. If we recall, Fourier transform has been defined from $-\infty$ to ∞. From the definition, we know that the Fourier transform with respect to x means x should vary from $-\infty$ to ∞. Therefore, the first condition should be one of the independent variables (because usually there will be two independent variables), at least one of the independent variables should have the range from $-\infty$ to ∞.

If one independent variable has the range from $-\infty$ to ∞, then we can apply Fourier transform with respect to that variable only. If the variable range is something different, then please note that we cannot use the Fourier transform. And the second point is both the dependent variable v and $\frac{\partial v}{\partial x}$ must vanish as x approaches $\pm\infty$.

We will explain why this is required, but please note that both v and $\frac{\partial v}{\partial x}$ must vanish as x approaches $\pm\infty$. So, two conditions are given: one is, we have to check among the independent variables, at least one independent variable must be there whose range or which varies from $-\infty$ to ∞, then only we can apply Fourier transform with respect to that variable.

Point number 2, both the functions v and $\frac{\partial v}{\partial x}$ must vanish as x approaches $\pm\infty$. Please note this one. If these two conditions are not satisfied, we cannot use the Fourier transform. Now the question arises, why the second condition is needed? Let us see why second condition is required over here. We will come to that, but before that let us just go through the criteria for sine and cosine also; then from there itself it will be clear.

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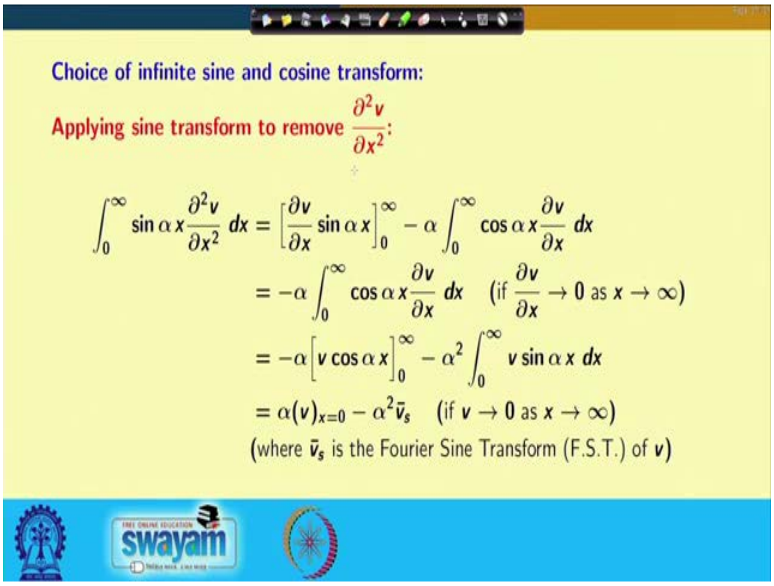

So, if we want to use sine transform, then to remove $\frac{\partial^2 v}{\partial x^2}$, what will happen?

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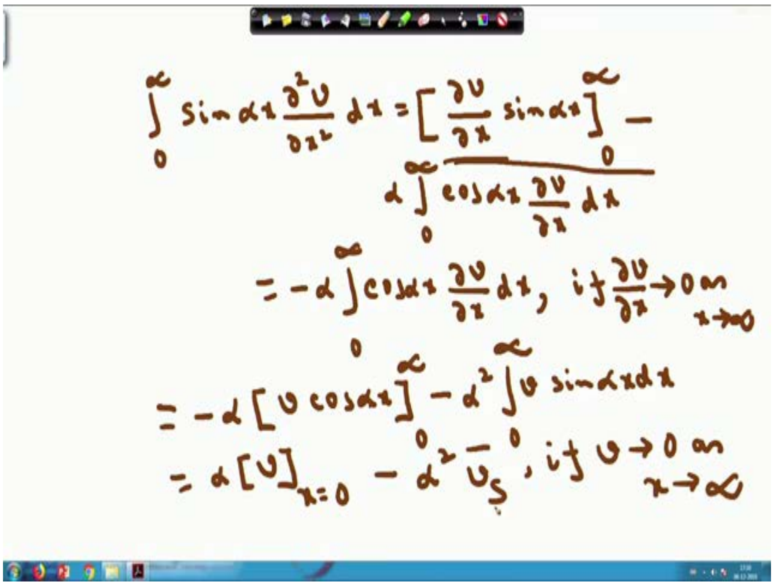

For this one, we have,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 v}{\partial x^2}\sin\alpha x\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\partial v}{\partial x}\sin\alpha x\right]_0^\infty - \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial v}{\partial x}\cos\alpha x\,dx$$

$$= -\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial v}{\partial x}\cos\alpha x\,dx \quad \text{if } \frac{\partial v}{\partial x} \to 0 \text{ as } x \to \infty$$

$$= -\alpha\sqrt{\frac{2}{\pi}}[v\cos\alpha x]_0^\infty - \alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty v\sin\alpha x\,dx$$

$$= \alpha\sqrt{\frac{2}{\pi}}[v]_{x=0} - \alpha^2\bar{v}_s \quad \text{if } v \to 0 \text{ as } x \to \infty$$

where, $\bar{v}_s$ denotes the Fourier sine transform of v w.r.t x. Therefore, whenever we want to take the Fourier sine transform with respect to x, say, then we must be provided with these conditions: $v, \frac{\partial v}{\partial x} \to 0$ as $x \to \infty$ and the value of v at $x = 0$ must be provided as well.

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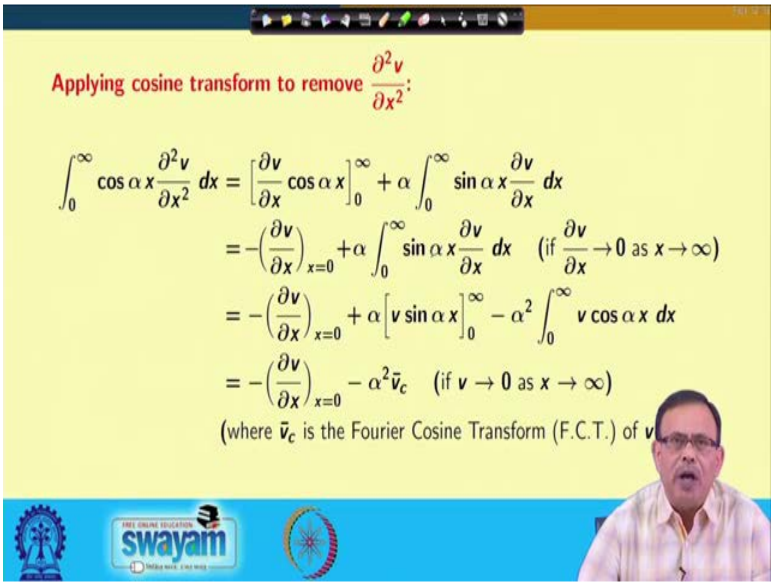

Similarly, if we have to apply the Fourier cosine transform with respect to the variable x, say, then,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 v}{\partial x^2}\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\partial v}{\partial x}\cos\alpha x\right]_0^\infty + \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial v}{\partial x}\sin\alpha x\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\partial v}{\partial x}\right]_{x=0} + \alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial v}{\partial x}\sin\alpha x\,dx \quad \text{if } \frac{\partial v}{\partial x} \to 0 \text{ as } x \to \infty$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\partial v}{\partial x}\right]_{x=0} + \alpha\sqrt{\frac{2}{\pi}}\left[v\sin\alpha x\right]_0^\infty - \alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty v\cos\alpha x\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\partial v}{\partial x}\right]_{x=0} - \alpha^2\bar{v}_c \quad \text{if } v \to 0 \text{ as } x \to \infty$$

where, $\bar{v}_c$ denotes the Fourier cosine transform of v w.r.t x. Therefore, whenever we want to take the Fourier cosine transform with respect to x, say, then we must be provided with these conditions: $v, \frac{\partial v}{\partial x} \to 0$ as $x \to \infty$ and the value of $\frac{\partial v}{\partial x}$ at $x = 0$ must be provided as well.

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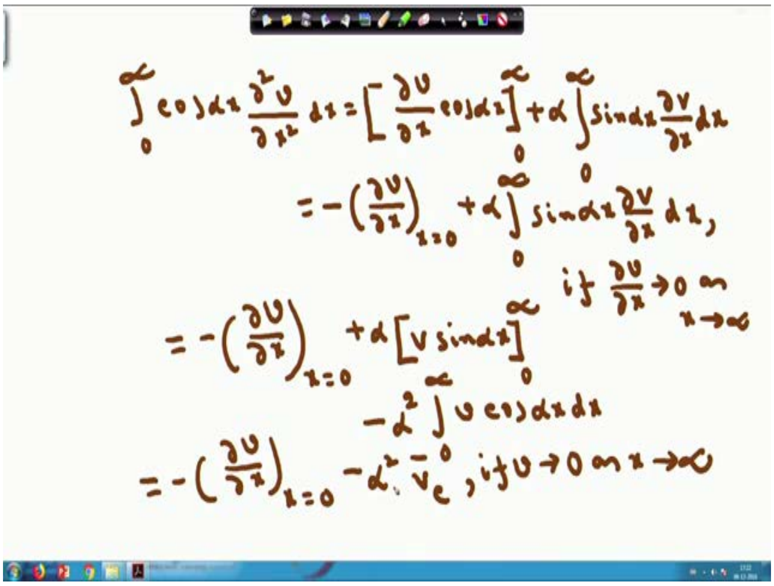

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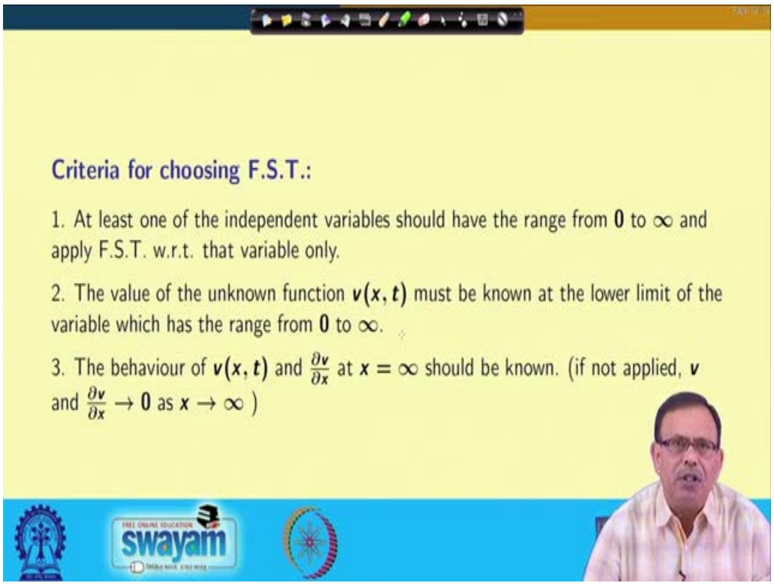

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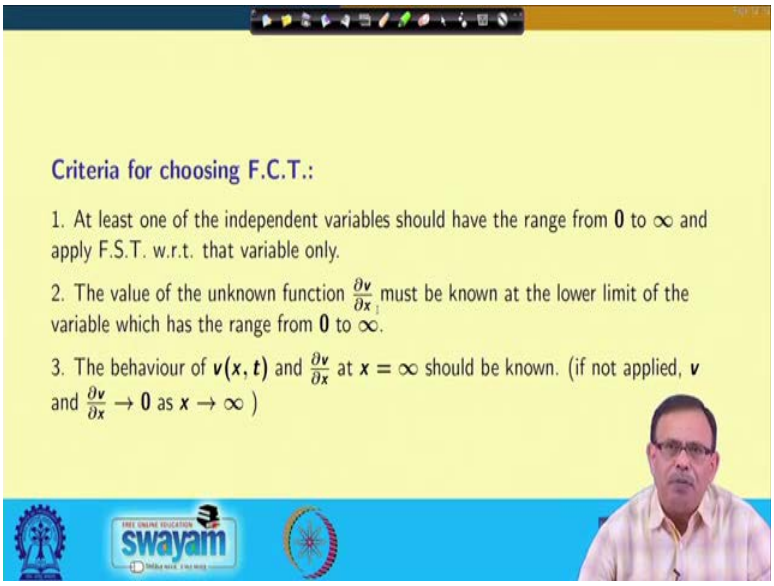

The criteria discussed so far are summarized in the above slides. Kindly refer to the above slides.

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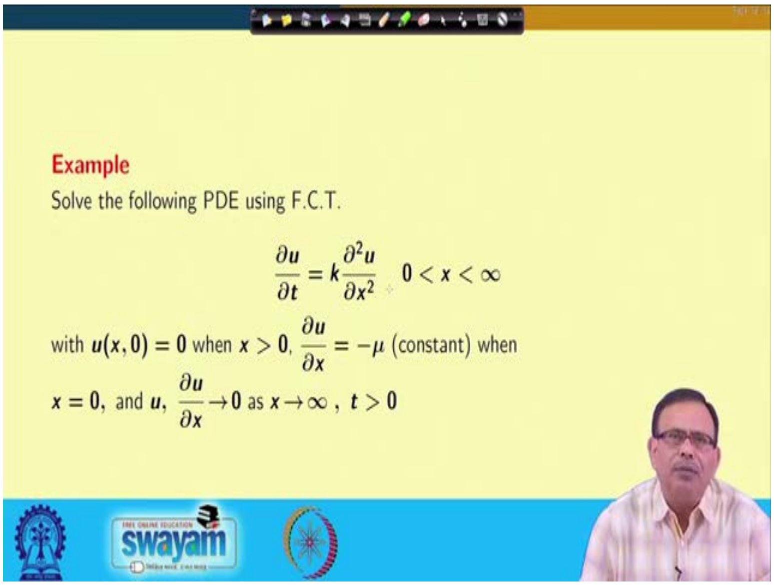

Let us discuss one problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0<x<\infty, \;\; t>0$$

$$\text{with, } u(x,0)=0 \text{ when } x>0 \text{ and } \frac{\partial u}{\partial x} = -\mu \text{ (constant) when } x=0$$

$$u, \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

So, here, both the variables x and t vary from 0 to ∞. Clearly, we cannot apply general Fourier transform to this problem as none of the two variables have the range from $-\infty$ to ∞. At the same time, we can see that $u, \frac{\partial u}{\partial x} \to 0$ as $x \to \infty$. This condition has been provided which means Fourier sine or cosine transform with respect to the variable x can be applied, and not with respect to t. But the value of u at $x = 0$ is not provided, so that it is not possible to apply Fourier sine transform. Instead, $\frac{\partial u}{\partial x}$ at $x = 0$ is given. Therefore, it is clear that we can apply Fourier cosine transform to this problem.

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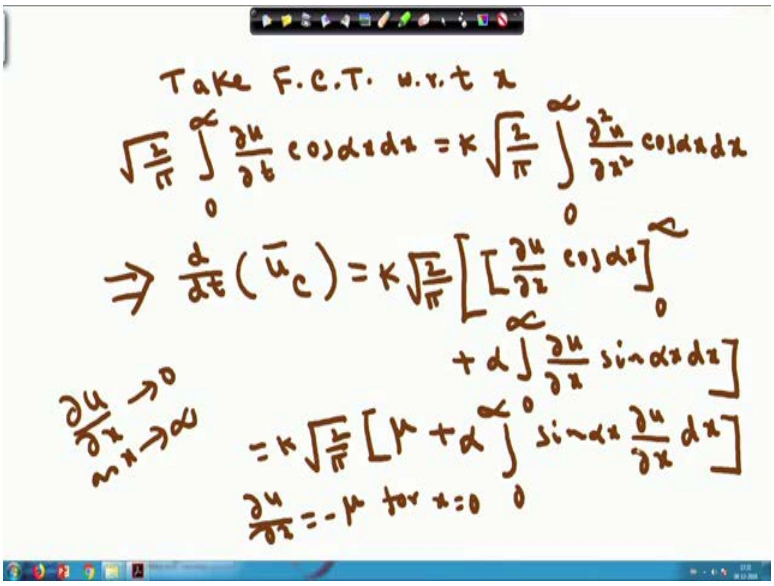

So, we take Fourier cosine transform on both sides of the given PDE with respect to x. Therefore, we obtain,

$$\mathcal{F}_c\left[\frac{\partial u}{\partial t}\right] = k\mathcal{F}_c\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x\,dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx\right] = k\sqrt{\frac{2}{\pi}}\left[\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty + \alpha\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x\,dx\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\sqrt{\frac{2}{\pi}}\left[\mu + \alpha\left\{[u\sin\alpha x]_0^\infty - \alpha\int_0^\infty u\cos\alpha x\,dx\right\}\right]$$

$$\left[\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty, \qquad u_x(0,t) = -\mu\right]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\mu\sqrt{\frac{2}{\pi}} - k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx \qquad [\because u \to 0 \text{ as } x \to \infty]$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = k\mu\sqrt{\frac{2}{\pi}} - k\alpha^2\bar{u}_c \qquad \text{where } \bar{u}_c(\alpha,t) = \mathcal{F}_c[u(x,t)]$$

Thus, the given PDE is reduced to a first order ODE. The integrating factor for the ODE is $e^{k\alpha^2 t}$. Therefore, multiplying by the integrating factor and after integration, the obtained ODE can be easily solved to get the solution as

$$\bar{u}_c e^{k\alpha^2 t} = A + \frac{\mu}{\alpha^2}\sqrt{\frac{2}{\pi}}\,e^{k\alpha^2 t}$$

$$\Rightarrow \bar{u}_c(\alpha,t) = A\,e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2} \tag{1}$$

where, A is the constant of integration.

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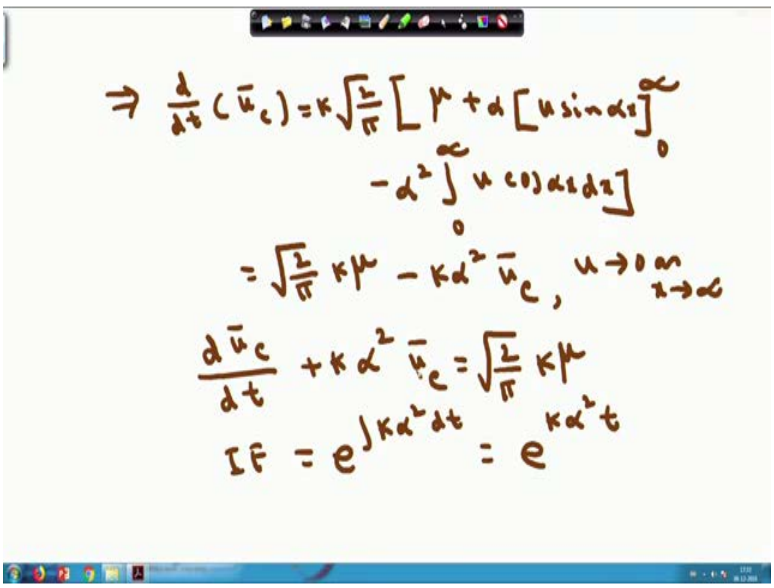

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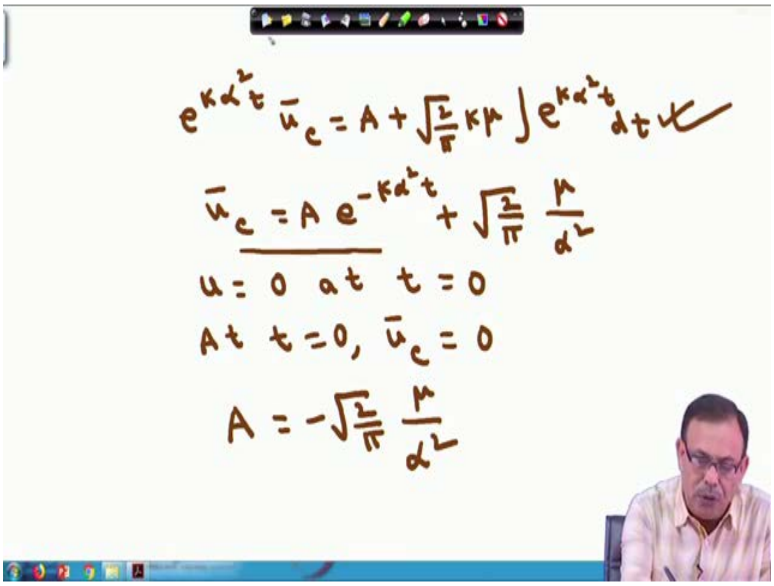

Now, we are provided with the following initial condition that

$$u(x, 0) = 0 \text{ when } x > 0$$

which implies that

$$\bar{u}_c(\alpha, 0) = 0$$

Therefore, (1) implies

$$0 = A + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

$$\Rightarrow A = -\sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

Hence, the solution is obtained from (1) as

$$\bar{u}_c(\alpha, t) = -\sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} e^{-k\alpha^2 t} + \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2}$$

$$\Rightarrow \bar{u}_c(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{\mu}{\alpha^2} \left(1 - e^{-k\alpha^2 t}\right)$$

Now, taking the inverse Fourier cosine transform, we have,

$$u(x,t) = \mathcal{F}_c^{-1}[\bar{u}_c(\alpha,t)]$$

$$= \mu\sqrt{\frac{2}{\pi}}\,\mathcal{F}_c^{-1}\left[\frac{1}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)\right]$$

$$= \mu\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)\cos\alpha x\, d\alpha$$

$$\Rightarrow u(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{1 - e^{-k\alpha^2 t}}{\alpha^2}\cos\alpha x\, d\alpha$$

Evaluation of the above integral will give the required solution for $u(x,t)$.

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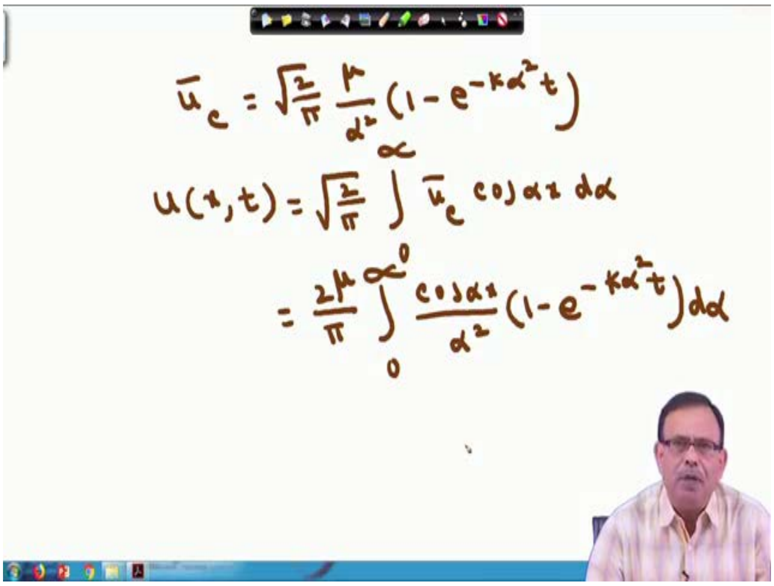

In the next lectures also, we will proceed with more examples on this, so that the procedure becomes clearer. Thank you.